Quantum kinematic theory of a point charge in a constant magnetic field

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A group-theoretic quantization method is applied to the ''complete symmetry group'' describing the motion of a point charge in a constant magnetic field. Within the regular ray representation, the Schrödinger operator is obtained as the Casimir operator of the extended Lie algebra. Configuration ray representations of the complete group cast the Schrödinger operator into the familiar space-time differential operator. Next, ''group quantization'' yields the superselection rules, which produce irreducible configuration ray representations. In this way, the Schrödinger operator becomes diagonalized, together with the angular momentum. Finally, the evaluation of an invariant integral, over the group manifold, gives rise to the Feynman propagation kernel $\langle t',\mathbf{x}'|t,\mathbf{x}\rangle$ of the system. Everything stems from the assumed symmetry group. Neither canonical quantization nor the path-integral method is used in the present analysis. [S1050-2947(96)04111-X]

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I. INTRODUCTION

In this paper quantum mechanics appears as a theory founded exclusively on the symmetries of a system, and formulated in the mathematical language of Lie groups. Our main purpose is to show that all one needs to know in order to produce a satisfactory quantum model (besides some given empirical parameters) is the *complete symmetry group* characterizing a system; for then one *quantizes* the group within the regular representation. These two notions, ''completeness'' of the given symmetries and ''quantization'' of the corresponding group, constitute the main points of our approach. The concept of the ''complete symmetry group of a system'' has been introduced in the literature recently [1], and shall be briefly discussed below. On the other hand, by ''quantization of a Lie group'' we mean that one substitutes the *parameters* of the group by a set of *generalized position operators* of the group manifold [2]. The generators of the regular representation are, of course, Hermitian operators by their own right, and may be interpreted as *generalized momenta*. In this manner, generalized position and momentum operators satisfy well-defined commutation relations [see, for instance, Eqs. (3.17), below]. So, if the group is physically relevant, such *generalized Heisenberg commutation relations* may be of potential value for physics, because they yield *new quantum kinematic foundations* of dynamics. Thus, let us examine quantum mechanics from this particular standpoint.

Here we illustrate the conceptual framework leading to such a group-theoretic formulation of quantum mechanics, by means of a concrete example. Specifically, this paper addresses the *quantum kinematic theory* of the *complete symmetry group* of a charged particle confined to move in a plane orthogonal to a constant magnetic field. The standard theory of our chosen example is well known and has been discussed for many years. It was first developed by Landau [3]. Even today, this example is of interest in itself. However, here we shall concentrate our attention exclusively on the *complete Landau group* (see below), in order to obtain a quantum model of the system.

As will be shown, the quantum kinematic treatment reproduces *all* the familiar features of the Landau solution. However, some features of the model do not appear so clearly in the canonical approach as in quantum kinematics. Indeed, in the usual treatment, there are some features that must be introduced ''by hand,'' since they stem from some intuitive quantum-mechanical ground and not from the canonical model itself. In our approach, they arise rather naturally, and one can interpret them properly. In this sense, the method leads to an understanding of the quantal structure and dynamical behavior of the Landau model. Since our approach is not known to most physicists, we will explain this particular example in some detail, so that the reader can gain an assessment for the possibilities contained in the ''quantization-through-the-symmetry'' method.

The organization of this paper follows the *deductive* procedure of quantum kinematic theory (as developed in Ref. [2]). Unfortunately, detailed examples of quantum kinematics are rather lengthy. For this reason, in Fig. 1 we provide the reader with a block diagram of the kinematic quantization argument. This leads to a synoptic setting, which can be contrasted with the three known quantization procedures used in quantum mechanics (as discussed, for instance, in [4]). We hope that for readers acquainted with the power and elegance of group theory in physics, the ''quantization diagram'' shown in Fig. 1 will look clear and appealing.

In Sec. II, we identify the *complete Landau group*, and we obtain the Casimir operator of the extended Lie algebra. In Sec. III, configuration ray representations are examined, as well as the generalized momenta of the group, which lead to the *Schrödinger operator* in configuration space-time. We then introduce *group quantization* and obtain *generalized* Heisenberg commutation relations. Section IV is devoted to the study of quantum dynamics, within the *left* kinematic formalism of the Landau group. *Superselection rules* are introduced, *which yield the irreducible configuration ray representations* carrying the physical states of the model. We then diagonalize (simultaneously) the Schrödinger and the angular-momentum operators, obtaining the expected *degeneracy* of their common eigenstates. Finally, we calculate the *propagation kernel*, in terms of an *invariant* integral over the group manifold. The physical interpretation of the model is

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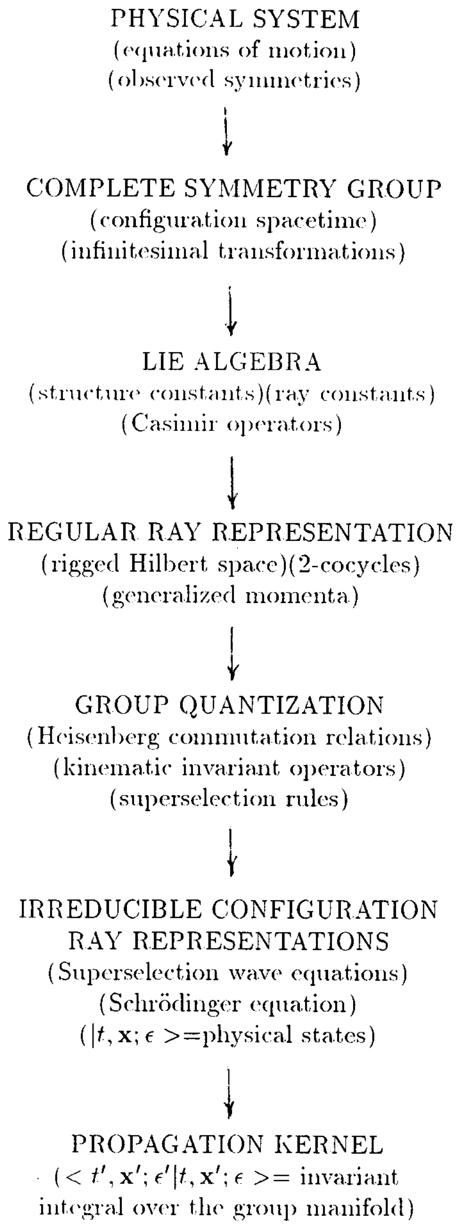

FIG. 1. Block diagram of the deductive kinematic-quantization argument (exhibiting the derived structures and main concepts, which play a key role at each step).

discussed in Sec. V, once all the pieces of the puzzle have been assembled. Appendixes A and B contain some basic group-theoretic tools used in this paper.

II. LANDAU GROUP

A. The complete symmetry group of the equations of motion

Our first task will be to identify the *complete symmetry group* characterizing the system. For the magnetic field we set $\mathbf{B}=B\hat{\mathbf{k}}$, where B is a constant and $\hat{\mathbf{k}}$ is a unit vector in the direction of the z axis. We ignore the free motion of the particle along the z axis. Thus, we write the equations of motion in the (x,y) plane as follows: $\ddot{\mathbf{x}}=\omega\boldsymbol{\mathcal{E}}\cdot\dot{\mathbf{x}}$, where ω is the cyclotron frequency, the 2×2 matrix $\boldsymbol{\mathcal{E}}=[\epsilon_{jk}]$ corresponds to the antisymmetric symbol, and $\mathbf{x}=(x,y)$ is a two-component column vector. Considering an orthogonal matrix $\mathbf{R}(\omega t)\in\mathrm{SO}(2)$, with entries $\cos(\omega t)$ and $\pm\sin(\omega t)$, since $\dot{\mathbf{R}}(\omega t)=\omega\boldsymbol{\mathcal{E}}\cdot\mathbf{R}(\omega t)$, one obtains the general solution to these equations of motion, which reads: $\mathbf{x}(t)=\mathbf{R}(\omega t)\cdot\mathbf{a}+\mathbf{b}$. Here, the two-component vectors $\mathbf{a}$ and $\mathbf{b}$ denote the constants of integration; their kinematic meaning in the (x,y) plane is immediate. The system has four *basic* constants of motion [5]: $\mathbf{a}=-\omega^{-1}\boldsymbol{\mathcal{E}}\cdot\mathbf{R}^T(\omega t)\cdot\dot{\mathbf{x}}(t)$ and $\mathbf{b}=\mathbf{x}(t)+\omega^{-1}\boldsymbol{\mathcal{E}}\cdot\dot{\mathbf{x}}(t)$. In this way, we settle our starting frame of work.

In the present case, a simple glance at the equations of motion yields the following group of point-symmetry transformations:

$$t\to t'=t+q^0, \quad \mathbf{x}\to\mathbf{x}'=\mathbf{R}(q^5)\cdot\mathbf{x}+\mathbf{R}(\omega t)\cdot\mathbf{q}_{(2)}+\mathbf{q}_{(1)}, \tag{2.1}$$

where q^0, $\mathbf{q}_{(1)}=(q^1,q^2)$, and $\mathbf{q}_{(2)}=(q^3,q^4)$ are Cartesian parameters of the group manifold, while $-\pi\leqslant q^5\leqslant+\pi$. These transformations define a realization of a six-dimensional Lie group G_L, acting freely and transitively in $X=\{(t,x,y)\}$. We call it the *Landau group*, for brief. The generators of the infinitesimal transformations read:

$$Z_0=\partial_t, \quad \mathbf{Z}_{(1)}=\boldsymbol{\nabla}, \quad \mathbf{Z}_{(2)}=\mathbf{R}^T(\omega t)\cdot\boldsymbol{\nabla}, \quad Z_5=-\mathbf{x}^T\cdot\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\nabla}, \tag{2.2}$$

where $\mathbf{Z}_{(1)}=(Z_1,Z_2)$ and $\mathbf{Z}_{(2)}=(Z_3,Z_4)$. They satisfy the Lie algebra of G_L, with the nonvanishing structure constants $f^1_{\ 25}=f^2_{\ 51}=f^3_{\ 45}=f^4_{\ 53}=1$ and $f^3_{\ 04}=f^4_{\ 30}=\omega$; all other structure constants vanish. The main properties of the Landau group are examined in Appendix A.

It can be shown that G_L is indeed *complete* [1]. This means that Eqs. (2.1) keep invariant the equations of motion $\ddot{x}=\omega\dot{y}$ and $\ddot{y}=-\omega\dot{x}$; and, furthermore, these are the *only* second-order ordinary differential equations in the plane that remain invariant under the point transformations defined in Eqs. (2.1). Hence, G_L corresponds to the (dynamically complete) *special relativity theory* of the Landau system, and of *no other* system evolving in the same space-time arena; i.e., G_L is a faithful theoretical representative of the system.

B. Casimir operator of the extended Lie algebra

Unitary *ray* representations should be used in quantum theory, in general. Let us here recall that the ''extended Lie algebra'' of a Lie group is the algebra obeyed by the generators of a ray (i.e., *projective*) representation. We call ''ray constants'' the multiples of the identity that figure in the commutation relations defining the extended Lie algebra. [See, for instance, Eq. (3.10d) below.] Bargmann emphasized the importance of ray constants k_{ab} giving rise to an extended Lie algebra, which have to satisfy the following constraints: $k_{ab}+k_{ba}=0$ and $f^d_{\ ab}k_{dc}+f^d_{\ ca}k_{db}+f^d_{\ bc}k_{da}=0$, as follows from the ray representation property [6]. These constants are the starting point of two-cocycle calculus [7]. Note that *genuine* ray constants are necessarily of the form $k_{ab}\neq f^c_{\ ab}k_c$. Only genuine ray constants (if any) can bear some physical meaning, for they can be changed but *not eliminated* by a gauge transformation of the ray representation. Trivial ray constants (i.e., $k_{ab}=f^c_{\ ab}k_c$, with k_a arbitrary) generate two-cocycles that are just *coboundaries* devoid of any physical meaning. (Cf., reference given in [7], for these details.)

In this way, given the structure constants of G_L, it follows that we can set $k_{15}=k_{25}=k_{35}=k_{45}=k_{03}=k_{04}=0$, while the genuine constants are obtained by solving the Bargmann constraints. This implies that k_{12}, k_{05}, and k_{34} are the only nontrivial Bargmann coefficients that can differ from zero.

With the aim of finding the Casimir operators of the extended Lie algebra, one uses the operators of the *associate-adjoint* representation [4]; i.e., one defines the following operators $D_a^{(k)}(p)=(f^c_{ab}p_c+k_{ab})(\partial/\partial p_b)$, which act in an auxiliary space $\{p\}$. It can be shown that these operators satisfy the (*nonextended*) Lie algebra $[D_a^{(k)}(p),D_b^{(k)}(p)]=f^c_{ab}D_c^{(k)}(p)$, even if $k_{ab}\neq 0$. Nevertheless, in order to obtain the *invariant operators of the extended Lie algebra* one solves the system of equations $D_a^{(k)}(p)S(p)=0$. In this fashion, for the Landau group, one has to solve the following system of coupled differential equations:

$$(p_1\partial_5-k_{12}\partial_1)S(p)=0,\quad (p_2\partial_5-k_{12}\partial_2)S(p)=0,$$

$$[p_3(\omega\partial_0-\partial_5)+k_{34}\partial_3]S(p)=0,$$

$$[p_4(\omega\partial_0-\partial_5)+k_{34}\partial_4]S(p)=0,$$

$$[\omega(p_3\partial_4-p_4\partial_3)+k_{05}\partial_5]S(p)=0,$$

$$[(p_2\partial_1-p_1\partial_2)+(p_4\partial_3-p_3\partial_4)-k_{05}\partial_0]S(p)=0, \quad (2.3)$$

where $\partial_a=\partial/\partial p_a$ (momentarily). One notices that if $k_{05}\neq 0$ there is no solution, besides the trivial one: $S=$const. Thus, on physical ground, we set $k_{05}=0$. But then we get only *one* solution to Eqs. (2.3), which is given by

$$S_0(p)=p_0-\frac{\omega}{2k_{12}}(p_1^2+p_2^2)-\frac{\omega}{k_{34}}(p_3^2+p_4^2)-\omega p_5. \quad (2.4)$$

This function yields the Casimir operator of the extended group, if one substitutes for the p's the generators of the diverse ray representations considered in this paper. As we shall see presently, this function leads to the Schrödinger equation of the system, without recourse to a classical canonical analog.

III. QUANTUM KINEMATICS

A. Configuration ray representations

The regular representation of a Lie group is the paramount structure in our quantization method. (Appendix B contains a sketchy review of the regular ray representation of G_L.) We now give up the usual quantization procedures [4], since here we follow a different path leading to quantum dynamics (see Fig. 1).

Once the regular representation has been introduced, the first task is to obtain a *configuration model* that manifests the complete action of G_L on X. To this end, one poses the problem of finding kets $|t,x,y\rangle$ in the rigged Hilbert space $\widetilde{\mathcal{H}}(G_L)$ (cf., Appendix B), which are in one-to-one correspondence with the events $(t,x,y)\in X$ and which under the action of the *left* unitary operators of G_L [as defined in Eq. (B2)] transform in a covariant manner with respect to Eqs. (2.1); that is, one requires

$$U_L^{(k)}(q)|t,\mathbf{x}\rangle=e^{(i/\hbar)\varphi_k(t,\mathbf{x};q)}|t',\mathbf{x}'\rangle, \quad (3.1)$$

where $(t,\mathbf{x})\rightarrow(t',\mathbf{x}')$ is given in (2.1), and where the exponent $\varphi_k(t,\mathbf{x};q)$ is a real-valued phase function. Such vectors carry a *configuration ray representation* of the action of G_L on X. Necessary and sufficient conditions for the existence of such configuration representations have been discussed in our previous work (cf., Ref. [2]). Henceforth, we adopt the *left regular representation* as our exclusive working frame.

We next summarize this procedure. Let us first briefly examine the phase function. If one defines a set of *phase generators*:

$$\sigma_a^{(k)}(t,\mathbf{x})=\lim_{q\rightarrow c}\partial_a\varphi_k(t,\mathbf{x};q) \quad (3.2)$$

($a=0,1,\ldots,5$), where e^a is the identity point, one proves that they *necessarily* satisfy the following *inhomogeneous* non-Abelian curl equations:

$$Z_a\sigma_b^{(k)}(t,\mathbf{x})-Z_b\sigma_a^{(k)}(t,\mathbf{x})-f^c_{ab}\sigma_c^{(k)}(t,\mathbf{x})=-k_{ab}, \quad (3.3)$$

where the Z's are given in Eq. (2.2). One then solves the following set of differential equations for obtaining the phase function $\varphi_k(t,\mathbf{x};q)$:

$$X_a(q)\varphi_k(t,\mathbf{x};q)=\sigma_a^{(k)}(t',\mathbf{x}')-r_a^{(k)}(q) \quad (3.4)$$

[where q generates $(t,\mathbf{x})\rightarrow(t',\mathbf{x}')$]. Here, $r_a^{(k)}(q)$ denotes the *right exponent generators* introduced in Eq. (B6). Using this method, for the Landau group one finds that Eqs. (3.3) have *no* solution, unless one takes $k_{12}=-k_{34}=k$; in which case, a set of phase generators is given by

$$\sigma_{(1)}^{(k)}(\mathbf{x})=\tfrac{1}{2}k\boldsymbol{\mathcal{E}}\cdot\mathbf{x},\quad \sigma_{(2)}^{(k)}(t,\mathbf{x})=-\tfrac{1}{2}k\boldsymbol{\mathcal{E}}\cdot\mathbf{R}^T(\omega t)\cdot\mathbf{x}, \quad (3.5)$$

and $\sigma_0^{(k)}=\sigma_5^{(k)}=0$. Then, using this result and solving for Eqs. (3.4) yields

$$\begin{aligned}\varphi_k(t,\mathbf{x};\mathbf{q}_{(1)},\mathbf{q}_{(2)},q^5)&\\ =\tfrac{1}{2}k\{&\mathbf{q}_{(1)}^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{R}(\omega t)\cdot\mathbf{q}_{(2)}\\ &+[\mathbf{q}_{(1)}^T-\mathbf{q}_{(2)}^T\cdot\mathbf{R}^T(\omega t)]\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{R}(q^5)\cdot\mathbf{x}\}. \quad (3.6)\end{aligned}$$

This phase function is consistent with the μ gauge [cf., (B5)]; i.e., it satisfies

$$\varphi_k(t,\mathbf{x};q)+\varphi_k(t',\mathbf{x}';q)=\mu_k(q)=0, \quad (3.7)$$

where q produces the transformation $(t,\mathbf{x})\rightarrow(t',\mathbf{x}')$, and q produces the inverse transformation $(t',\mathbf{x}')\rightarrow(t,\mathbf{x})$, according to Eq. (2.1).

Furthermore, once $\varphi_k(t,\mathbf{x};q)$ has been found, one still has much freedom in settling a configuration ray representation within $\widetilde{\mathcal{H}}(G_L)$. In fact, one may consider *any* function $\psi(t,\mathbf{x})$ defined in X, provided the vector given by

$$|t,\mathbf{x}\rangle=\mu_0\int d^6q\,\psi^*[t+q^{-0},\mathbf{R}(q^{-5})\cdot\mathbf{x}+\mathbf{R}(\omega t)\cdot\overline{\mathbf{q}}_{(2)}+\overline{\mathbf{q}}_{(1)}]e^{(i/h)\varphi_k(t,\mathbf{x};\overline{q})}|q\rangle \tag{3.8}$$

belongs to the rigged Hilbert space $\widetilde{\mathcal{H}}(G_L)$, where $\overline{q}$ denotes the "inverse" parameters defined in Eqs. (A3), and we assume the μ gauge (3.7). In fact, every vector $|t,\mathbf{x}\rangle$, as defined in Eq. (3.8), transforms covariantly according to Eq. (3.1). So, in order to obtain dynamically meaningful configuration ray representations one must choose a *generating wave function* $\psi(t,\mathbf{x})$ on some physical ground. To this end, we shall use the *superselection rules* dictated by the structure of G_L itself (cf. Sec. IV B).

B. Generalized momentum operators and the extended Lie algebra

We now consider the generators $L_a^{(k)}$ as *generalized momentum operators*. According to Eqs. (A6) and (B6), the infinitesimal transformations corresponding to Eqs. (B2) yield the following realizations of these operators, in the "Q representation" of the formalism:

$$L_0^{(k)}|q\rangle=i\hbar\partial_0|q\rangle,$$

$$L_5^{(k)}|q\rangle=-i\hbar[\mathbf{q}_{(1)}^T\cdot\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\partial}_{(1)}+\mathbf{q}_{(2)}^T\cdot\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\partial}_{(2)}-\boldsymbol{\partial}_5]|q\rangle, \tag{3.9a}$$

$$\mathbf{L}_{(1)}^{(k)}|q\rangle=[i\hbar\boldsymbol{\partial}_{(1)}-\tfrac{1}{2}k\boldsymbol{\mathcal{E}}\cdot\mathbf{q}_{(1)}]|q\rangle,$$

$$\mathbf{L}_{(2)}^{(k)}|q\rangle=\mathbf{R}^T(\omega q^0)\cdot[i\hbar\boldsymbol{\partial}_{(2)}+\tfrac{1}{2}k\boldsymbol{\mathcal{E}}\cdot\mathbf{q}_{(2)}]|q\rangle. \tag{3.9b}$$

Thus we get the *extended Lie algebra* obeyed by the left momenta; i.e.,

$$[L_0^{(k)},\mathbf{L}_{(1)}^{(k)}]=\mathbf{0},\quad[\mathbf{L}_{(1)}^{(k)},\mathbf{L}_{(2)}^{(k)}]=\mathbf{O}, \tag{3.10a}$$

$$[L_0^{(k)},\mathbf{L}_{(2)}^{(k)}]=i\hbar\omega\boldsymbol{\mathcal{E}}\cdot\mathbf{L}_{(2)}^{(k)},\quad[\mathbf{L}_{(1)}^{(k)},L_5^{(k)}]=i\hbar\boldsymbol{\mathcal{E}}\cdot\mathbf{L}_{(1)}^{(k)}, \tag{3.10b}$$

$$[L_0^{(k)},L_5^{(k)}]=0,\quad[\mathbf{L}_{(2)}^{(k)},L_5^{(k)}]=i\hbar\boldsymbol{\mathcal{E}}\cdot\mathbf{L}_{(2)}^{(k)}, \tag{3.10c}$$

$$[L_1^{(k)},L_2^{(k)}]=-i\hbar k,\quad[L_3^{(k)},L_4^{(k)}]=i\hbar k. \tag{3.10d}$$

The commutation relations (3.10d) are noteworthy, for they contain the only *physically meaningful* ray constant k of the system. So, according to Eq. (2.4), in terms of the generalized momenta the Casimir operator reads

$$S_0^{(k)}=L_0^{(k)}-\frac{\omega}{2k}(\mathbf{L}_{(1)}^{(k)})^2+\frac{\omega}{k}(\mathbf{L}_{(2)}^{(k)})^2-\omega L_5^{(k)}. \tag{3.11}$$

In this way, a straightforward calculation yields $S_0^{(k)}$ in the "Q representation."

C. The Schrödinger operator

Following the same approach, let us now consider Eq. (3.1) in a neighborhood of the identity. A typical group-theoretic calculation yields the general formula [2]

$$L_a^{(k)}|t,\mathbf{x}\rangle=[i\hbar Z_a(t,\mathbf{x})-\sigma_a^{(k)}(t,\mathbf{x})]|t,\mathbf{x}\rangle, \tag{3.12}$$

where the phase generators $\sigma_a^{(k)}$ are given in Eq. (3.5). Hence the expressions for the left-momentum operators follow:

$$L_0^{(k)}|t,\mathbf{x}\rangle=i\hbar\partial_t|t,\mathbf{x}\rangle, \tag{3.13a}$$

$$\mathbf{L}_{(1)}^{(k)}|t,\mathbf{x}\rangle=\left(i\hbar\boldsymbol{\nabla}-\frac{k}{2}\boldsymbol{\mathcal{E}}\cdot\mathbf{x}\right)|t,\mathbf{x}\rangle. \tag{3.13b}$$

$$\mathbf{L}_{(2)}^{(k)}|t,\mathbf{x}\rangle=\mathbf{R}^T(\omega t)\cdot\left(i\hbar\boldsymbol{\nabla}+\frac{k}{2}\boldsymbol{\mathcal{E}}\cdot\mathbf{x}\right)|t,\mathbf{x}\rangle, \tag{3.13c}$$

$$L_5^{(k)}|t,\mathbf{x}\rangle=-i\hbar\mathbf{x}^T\cdot\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\nabla}|t,\mathbf{x}\rangle. \tag{3.13d}$$

These are purely *kinematic* realizations of the momentum operators when acting on space-time vectors belonging to any given configuration ray representation. As we see, in quantum kinematics the exponent generators appear as "compensating gauge potentials," although we are not using a Lagrangian formulation at all.

In this fashion, by means of Eqs. (3.11) and (3.13), we obtain the expression for $S_0^{(k)}$ within *any* given configuration ray representation of G_L, which takes the particular form

$$\langle t,\mathbf{x}|S_0^{(k)}|\psi\rangle=\left(-i\hbar\partial_t-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\frac{i\hbar\omega}{2}\mathbf{x}^T\cdot\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\nabla}+\frac{m\omega^2}{8}\mathbf{x}^2\right)\psi(t,\mathbf{x}). \tag{3.14}$$

Here $\psi(t,\mathbf{x})=\langle t,\mathbf{x}|\psi\rangle$ is any wave function, and we adjust the ray constant to the *physical* value $k=m\omega$. This expression is precisely the *time-dependent Schrödinger differential operator*, as introduced in Hamiltonian quantum mechanics. We shall call $S_0^{(k)}$ the *Schrödinger operator* of the Landau problem.

This result is very reassuring for the present approach to quantum mechanics. Note that one is able to *deduce* the correct form of the *Hamiltonian* $H_0^{(k)}$, which appears in $S_0^{(k)}=i\hbar\partial_t+H_0^{(k)}$, by means of a group-theoretic analysis of the complete configuration symmetries of the system, without recourse to the *prequantized* canonical analog. Moreover, this is not the whole story, because in order to further analyze the physical contents of Eq. (3.14) (and end up by solving it in terms of *a propagation kernel*) we need to understand the *quantal structure* of the symmetry group itself.

D. Group quantization

We now turn to the concept of *generalized position operators* of the group manifold; i.e., we shall *quantize* G_L. Let us define a set of Hermitian operators Q^a in $\widetilde{\mathcal{H}}(G_L)$ by means of the following spectral integrals:

$$Q^a=\mu_0\int d^6q|q\rangle q^a\langle q|,\quad a=0,1,\ldots,5, \tag{3.15}$$

which we borrow from standard quantum mechanics. These operators are such that $Q^a|q\rangle=q^a|q\rangle$ and $[Q^a,Q^b]=0$ hold.

Furthermore, it can be shown that the Q's transform according to the following covariant law under the group operators [cf., Eq. (B2)]:

$$U_L^{(k)\dagger}(q)Q^a U_L^{(k)}(q) = g^a(q;Q), \tag{3.16}$$

where the operators $g^a(q;Q)$ are defined by means of the corresponding spectral integrals using Eqs. (A2).

Therefore, the infinitesimal version of these unitary transformations yields *generalized Heisenberg commutation relations* for the position operators with the non-Abelian momenta of the group [2]. In general, these commutation relations read: $[Q^b, L_a^{(k)}] = i\hbar R_a^b(Q)$. In this manner, from Eqs. (A4), it follows that

$$[Q^b, L_a^{(k)}] = i\hbar \begin{bmatrix} I & 0^T & 0^T & 0 \\ 0 & \mathbf{I} & \mathbf{O} & 0 \\ 0 & \mathbf{O} & \mathbf{R}^T(\omega Q^0) & 0 \\ 0 & -\mathbf{Q}_{(1)}^T \cdot \mathcal{E} & -\mathbf{Q}_{(2)}^T \cdot \mathcal{E} & I \end{bmatrix}, \tag{3.17}$$

from where one gets the desired commutator immediately. (This structure has no classical canonical analog if the group is non-Abelian.) A great deal of quantum kinematic descriptions stem from this equation.

At this point, one needs to observe that the angle q^5 is *not a faithful* parameter of $SO(2) \subset G_L$. Actually, we shall not use the "angle-operator" Q^5, because $[Q^5, L_5^{(k)}] = i\hbar$ would follow, which leads to a well-known paradox [8]. Hence, instead of Q^5 itself, it is better to define the following two *bounded* operators on $M(G_L)$: $\cos Q^5$ and $\sin Q^5$. These obey the correct Heisenberg commutation relations for SO(2); i.e., $[\cos Q^5, L_5] = -i\hbar \sin Q^5$, and $[\sin Q^5, L_5] = i\hbar \cos Q^5$, which are consistent with the periodic eigenvectors of the angular-momentum operator L_5 and satisfy the constraint $\cos^2 Q^5 + \sin^2 Q^5 = I$ [8]. We assume this scheme in what follows.

The Heisenberg commutation relations (3.17) do not close to form a finite algebra. However, one easily obtains a closed algebra pertaining to the quantized group [9,10]. In fact, given any regular function $F(q)$ defined in $M(G_L)$, one defines operators $F(Q)$, so that one obtains $[F(Q), L_a^{(k)}] = i\hbar X_a(Q) F(Q)$, since $X_a(q) F(q) = R_a^b(q) F_{,b}(q)$. Therefore, Eq. (A8) immediately yields the following *closed* commutation relations:

$$[A_b^c(Q), L_a^{(k)}] = i\hbar f_{ab}^d A_d^c(Q). \tag{3.18}$$

Having now a *larger* closed algebra, we can look for *new* invariant operators, besides the Casimir operator $S_0^{(k)}$ (which is all one gets if one does *not* quantize the group). Indeed, it can be shown that the operators given by the general formula [10]

$$R_a^{(k)}(Q,L) = A_a^b(Q)[L_b^{(k)} + r_b^{(k)}(Q)] - l_a^{(k)}(Q) - \frac{i}{2}\hbar f_{ab}^b, \tag{3.19}$$

are Hermitian and commute with *all* the generators of the *left* regular ray representation. Here, $\overline{A}_a^b(q)$ is the matrix of the antiadjoint representation (cf., Appendix A); $r_a^{(k)}(q)$ and $l_a^{(k)}(q)$ denote the right and left exponent generators of the ray representation, respectively (cf., Appendix B). Hence, inverting the matrix (A8) of the adjoint representation, we get the following six *basic quantum-kinematic invariants* for the Landau group:

$$R_0^{(k)}(Q,L) = L_0^{(k)} - \omega \mathbf{Q}_{(2)}^T \cdot \boldsymbol{\mathcal{E}} \cdot \mathbf{R}(\omega Q^0) \cdot \mathbf{L}_{(2)}^{(k)}, \tag{3.20a}$$

$$\mathbf{R}_{(1)}^{(k)}(Q,L) = \mathbf{R}^T(Q^5) \cdot [\mathbf{L}_{(1)}^{(k)} + k\boldsymbol{\mathcal{E}} \cdot \mathbf{Q}_{(1)}], \tag{3.20b}$$

$$\mathbf{R}_{(2)}^{(k)}(Q,L) = \mathbf{R}^T(Q^5) \cdot [\mathbf{R}(\omega Q^0) \cdot \mathbf{L}_{(2)}^{(k)} - k\boldsymbol{\mathcal{E}} \cdot \mathbf{Q}_{(2)}], \tag{3.20c}$$

$$R_5^{(k)}(Q,L) = \mathbf{Q}_{(1)}^T \cdot \boldsymbol{\mathcal{E}} \cdot \mathbf{L}_{(1)}^{(k)} + \mathbf{Q}_{(2)}^T \cdot \boldsymbol{\mathcal{E}} \cdot \mathbf{R}(\omega Q^0) \cdot \mathbf{L}_{(2)}^{(k)} + L_5^{(k)} - \tfrac{1}{2}k\mathbf{Q}_{(1)}^T \cdot \mathbf{Q}_{(1)} + \tfrac{1}{2}k\mathbf{Q}_{(2)}^T \cdot \mathbf{R}(\omega Q^0 - Q^5) \cdot \mathbf{Q}_{(2)}. \tag{3.20d}$$

These are, in fact, the generators of the *right* representation, acting as invariant operators within the left representation [10]. Hence, they satisfy the *right* extended Lie algebra, which is obtained by substituting $L_a^{(k)} \to R_a^{(k)}$, $f_{bc}^a \to f_{cb}^a$, and $k_{ab} \to k_{ba}$, in Eqs. (3.10). In particular, let us note that

$$[R_1^{(k)}, R_2^{(k)}] = i\hbar k, \quad [R_3^{(k)}, R_4^{(k)}] = -i\hbar k. \tag{3.21}$$

One has always $S_0^{(k)} \equiv S_0(L^{(k)}) = S_0(R^{(k)})$ for a Casimir operator [9]. Therefore, the Schrödinger operator can be also written in terms of the kinematic invariants, as follows:

$$S_0^{(k)} = R_0^{(k)} - \frac{\omega}{2k}(\mathbf{R}_{(1)}^{(k)})^2 + \frac{\omega}{k}(\mathbf{R}_{(2)}^{(k)})^2 - \omega R_5^{(k)}. \tag{3.22}$$

IV. QUANTUM DYNAMICS

A. Invariant ladder operators

Already we have enough information to look at the connection between group theory and quantum mechanics [12]. Let us derive some consequences of the previous formalism. First (and most importantly), from Eqs. (3.21) one obtains

$$[a_k, a_k^\dagger] = I, \quad [b_k, b_k^\dagger] = I, \quad [a_k, b_k] = 0, \quad [a_k, b_k^\dagger] = 0, \tag{4.1}$$

where one defines $a_k = (2\hbar k)^{-1/2}(R_1^{(k)} + iR_2^{(k)})$ and $b_k = (2\hbar k)^{-1/2}(R_3^{(k)} - iR_4^{(k)})$, which are *invariant* operators indeed. Thus, we introduce the following Hermitian operators:

$$H_1^{(k)} = \tfrac{1}{2}\hbar\omega(a_k a_k^\dagger + a_k^\dagger a_k), \quad H_2^{(k)} = \tfrac{1}{2}\hbar\omega(b_k b_k^\dagger + b_k^\dagger b_k), \tag{4.2}$$

where $H_1^{(k)} = (1/2m)(\mathbf{R}_{(1)}^{(k)})^2$ and $H_2^{(k)} = (1/2m)(\mathbf{R}_{(2)}^{(k)})^2$. These operators commute, and have the form of two Hamiltonians describing *uncoupled* harmonic oscillators. Thus, the Schrödinger operator reads [cf., Eq. (3.22)]

$$S_0^{(k)} = R_0^{(k)} - \omega R_5^{(k)} - H_1^{(k)} + 2H_2^{(k)}. \tag{4.3}$$

Hence, from the extended Lie algebra it follows that the Schrödinger operator $S_0^{(k)}$ can be decomposed as a linear combination of four *commuting* "partial Hamiltonians": $R_0^{(k)}$, $\omega R_5^{(k)}$, $H_1^{(k)}$, and $H_2^{(k)}$. (Indeed, one obtains the required commutation relations of $R_0^{(k)}$ and $R_5^{(k)}$ with the annihilation operators a_k and b_k that imply this fact.) We also observe that the ''partial Hamiltonians'' of Eq. (4.3) are linearly independent. Of course, if one diagonalizes simultaneously the commuting ''partial Hamiltonians,'' one solves the complete eigenvalue problem of $S_0^{(k)}$.

B. Superselection rules: Irreducible configuration ray representations

Henceforth we consider the following set of *four* linearly independent operators:

$$\mathcal{S}^{(k)}[G_L]=\{R_0^{(k)},\omega R_5^{(k)},H_1^{(k)},H_2^{(k)}\}. \tag{4.4}$$

It can be proved that this is a *maximal set* of compatible invariant operators. We shall use them as *superselection rules*. Now, with the aim of arriving at a reasonable physical interpretation of the model, let us examine the heuristic postulate [2]:

The physical pure states of the system correspond to simultaneous eigenvectors of the invariant ''partial Hamiltonians'' contained in $\mathcal{S}^{(k)}[G_L]$.

In this way one solves the superselection rule associated with the Schrödinger operator $S_0^{(k)}$, which corresponds to the *law of conservation of total energy* of an isolated system consisting of *four noninteracting parts*, whose physical meanings remain to be discussed.

Thus one ''diagonalizes'' the Hilbert space of the regular representation, into *invariant* subspaces: $\mathcal{H}_{(E_0,n_5,n_1,n_2)}(G_L)$ (with $n_5=\cdots, -1,0,1,\ldots$, and $n_1,n_2=0,1,2,\ldots$), each carrying an *irreducible ray representation* of the group, since the superselection rules are *maximal*. The invariant subspaces are orthogonal and carry the physical states of the model.

Here we shall attain a quantum-mechanical description of the system by means of a *configuration ray representation*. To this end, let us look for wave functions $\psi(t,\mathbf{x})$ generating configuration-state vectors $|t,\mathbf{x};\psi\rangle$ that satisfy the four superselection rules. Namely, we require

$$R_0^{(k)}|t,\mathbf{x};\psi\rangle=E_0|t,\mathbf{x};\psi\rangle,\quad R_5^{(k)}|t,\mathbf{x};\psi\rangle=n_5\hbar|t,\mathbf{x};\psi\rangle, \tag{4.5a}$$

$$H_1^{(k)}|t,\mathbf{x};\psi\rangle=\hbar\omega(n_1+\tfrac{1}{2})|t,\mathbf{x};\psi\rangle,$$

$$H_2^{(k)}|t,\mathbf{x};\psi\rangle=\hbar\omega(n_2+\tfrac{1}{2})|t,\mathbf{x};\psi\rangle. \tag{4.5b}$$

Admissible *generating wave functions* for the Landau system are defined by means of the following limit at the identity point $e\in M(G_L)$ (cf. Ref. [2]):

$$\psi_{E_0,n_5,n_1,n_2}(t,\mathbf{x})=\lim_{q\to e}\langle t,\mathbf{x};\psi|q\rangle=\langle t,\mathbf{x};\psi|e\rangle. \tag{4.6}$$

For instance, in this manner one obtains

$$\lim_{q\to e}\langle t,\mathbf{x};\psi|R_0^{(k)}|q\rangle=-i\hbar\partial_t\psi(t,\mathbf{x}), \tag{4.7}$$

as expected. To get the realizations of a_k, b_k, and $R_5^{(k)}$ in X, it is convenient to introduce the complex variable $z=x+iy$, which yields

$$\langle t,\mathbf{x};\psi|a_k|e\rangle=-i\sqrt{\frac{2\hbar}{k}}\left[\left(\frac{\partial}{\partial z^*}\right)-\frac{k}{4\hbar}z\right]\psi(t,\mathbf{x}), \tag{4.8a}$$

$$\langle t,\mathbf{x};\psi|a_k^\dagger|e\rangle=-i\sqrt{\frac{2\hbar}{k}}\left[\left(\frac{\partial}{\partial z}\right)+\frac{k}{4\hbar}z^*\right]\psi(t,\mathbf{x}), \tag{4.8b}$$

$$\langle t,\mathbf{x};\psi|b_k|e\rangle=-i\sqrt{\frac{2\hbar}{k}}e^{-i\omega t}\left[\left(\frac{\partial}{\partial z}\right)-\frac{k}{4\hbar}z^*\right]\psi(t,\mathbf{x}), \tag{4.8c}$$

$$\langle t,\mathbf{x};\psi|b_k^\dagger|e\rangle=-i\sqrt{\frac{2\hbar}{k}}e^{i\omega t}\left[\left(\frac{\partial}{\partial z^*}\right)+\frac{k}{4\hbar}z\right]\psi(t,\mathbf{x}), \tag{4.8d}$$

$$\langle t,\mathbf{x};\psi|R_5^{(k)}|e\rangle=\hbar\left[z^*\left(\frac{\partial}{\partial z^*}\right)-z(\partial/\partial z)\right]\psi(t,\mathbf{x}). \tag{4.9}$$

So we are ready to tackle the problem stated in Eqs. (4.5). One first looks for the ground state of $H_1^{(k)}$ and $H_2^{(k)}$. The generating wave function for this state reads

$$\psi_{E_0,0_1,0_2}(t,\mathbf{x})=\exp[(i/\hbar)E_0t-(m\omega/4\hbar)|z|^2]$$

(a normalization constant remains at our disposal). Then one sets

$$|t,\mathbf{x};E_0,n_1,n_2\rangle=(n_1!n_2!)^{-1/2}(a_k^\dagger)^{n_1}(b_k^\dagger)^{n_2}|t,\mathbf{x};E_0,0_1,0_2\rangle.$$

The detailed analysis of this procedure is rather lengthy and yields the final answer:

$$\begin{aligned}\psi_{E_0,n_1,n_2}(t,\mathbf{x})&=\langle t,\mathbf{x};E_0,n_1,n_2|e\rangle\\&=(i\sqrt{m\omega/2\hbar})^{n_1+n_2}\sqrt{n_1!n_2!}\\&\quad\times e^{(i/\hbar)(E_0-n_2\hbar\omega)t}e^{-(m\omega/4\hbar)|z|^2}z^{n_1}z^{*n_2}\\&\quad\times F_{\{n_1,n_2\}}[-(m\omega/2\hbar)|z|^2],\end{aligned} \tag{4.10}$$

where we define the function

$$F_{\{n_1,n_2\}}(x)=\sum_{j=0}^{n_<}\frac{x^{-j}}{(n_1-j)!(n_2-j)!j!}, \tag{4.11}$$

with $n_<$ denoting the smallest number in $\{n_1,n_2\}$.

In summary, since $[z^*(\partial/\partial z^*)-z(\partial/\partial z)]|z|^2=0$ and $[z^*(\partial/\partial z^*-z(\partial/\partial z)]z^{n_1}z^{*n_2}=(n_2-n_1)z^{n_1}z^{*n_2}$, one finds the physically admissible *pure states* of the Landau system, which satisfy the superselection rules, with the following proviso:

$$S_0^{(k)}|t,\mathbf{x};E_0,n_1,n_2\rangle=[E_0+\hbar\omega(n_2+\tfrac{1}{2})]|t,\mathbf{x};E_0,n_1,n_2\rangle, \tag{4.12}$$

$$R_5^{(k)}|t,\mathbf{x};E_0,n_1,n_2\rangle=(n_2-n_1)\hbar|t,\mathbf{x};E_0,n_1,n_2\rangle, \tag{4.13}$$

(i.e., $n_5=n_2-n_1$). The *degeneracy* shown in these eigenvalue equations raises an interesting question (cf., Sec. V).

So far we have completed the group-theoretic discussion leading to the structure of pure-state vectors carrying *irreducible configuration ray representations* of the model. We note that the fundamental wave function $\psi_{E_0,0_1,0_2}(t,\mathbf{x})$ that generates all these states is the familiar ground-state wave function of two uncoupled harmonic oscillators, moving in an ''environment'' that has basic energy E_0.

C. Quantum-kinematic propagation kernel

Finally, let us study the Feynman propagation kernel of the Landau system from the standpoint of quantum kinematics. Perhaps, this is the most interesting achievement of the present group-theoretic formulation of quantum mechanics.

With this aim, let us recall the well-known formula

$$\langle \psi';n_1',n_2'|\psi;n_1,n_2\rangle=\delta_{n_1'n_1}\delta_{n_2'n_2}\langle \psi';0_1,0_2|\psi;0_1,0_2\rangle, \tag{4.14}$$

which holds for the ''excited'' states of two uncoupled harmonic oscillators. Hence, for our purpose, it is enough to consider the ground state. To find the ground-state vector $|t,\mathbf{x};E_0,0_1,0_2\rangle$, we shall proceed as follows. One has $U_L^{(k)}(q)|e\rangle=|q\rangle$ and therefore, using Eq. (3.1), a straightforward calculation yields

$$\begin{aligned}\psi^*_{E_0,0_1,0_2}(t,\mathbf{x})&=e^{-(i/\hbar)E_0 t}e^{-(m\omega/4\hbar)|z|^2}\\&=\langle q|t_q',x_q';E_0,0_1,0_2\rangle e^{(i/\hbar)\varphi_k(t,\mathbf{x};q)},\end{aligned} \tag{4.15}$$

where q produces the change of variables $(t,\mathbf{x})\rightarrow(t',\mathbf{x}')=(t_q',\mathbf{x}_q')$. Since $\overline{q}$ produces the inverse transformation $(t_q',\mathbf{x}_q')\rightarrow(t,\mathbf{x})=(t'_{\overline{q}},\mathbf{x}'_{\overline{q}})$, we obtain the following expression for the ground-state vector:

$$|t,\mathbf{x};E_0,0_1,0_2\rangle=\mu_0\int d^6q\ \exp\left[-\frac{i}{\hbar}E_0(t+\overline{q^0})-\frac{m\omega}{4\hbar}|z_{\overline{q}}|^2\right]e^{(i/\hbar)\varphi_k(t,\mathbf{x};\overline{q})}|q\rangle, \tag{4.16}$$

where

$$\begin{aligned}|z_{\overline{q}}|^2=&\mathbf{x}^2+\mathbf{q}_{(1)}^2+\mathbf{q}_{(2)}^2-2\mathbf{x}^T\cdot\mathbf{q}_{(1)}\\&-2(\mathbf{x}-\mathbf{q}_{(1)})^T\cdot\mathbf{R}[\omega(t-q^0)]\cdot\mathbf{q}_{(2)},\end{aligned}$$

as well as

$$\begin{aligned}\varphi_k(t,\mathbf{x};\overline{q})=&(m\omega/2)\{\mathbf{x}^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{q}_{(1)}\\&-(\mathbf{x}-\mathbf{q}_{(1)})^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{R}[\omega(t-q^0)]\cdot\mathbf{q}_{(2)}\}\end{aligned}$$

can be calculated easily. Of course, applying the creation operators $a_k^\dagger$ and $b_k^\dagger$ yields the desired general states $|t,\mathbf{x};E_0,n_1,n_2\rangle$. We then *normalize* the ground state. To this end, one considers transition amplitudes with $t'=t$. Thus, one gets

$$\begin{aligned}&\langle t,\mathbf{x}';E_0',0_1,0_2|t,\mathbf{x};E_0,0_1,0_2\rangle\\&=\mu_0 e^{(i/\hbar)(E_0'-E_0)t}e^{-(m\omega/4\hbar)(\mathbf{x}'^2+\mathbf{x}^2)}\int d^6q\, e^{-(i/\hbar)(E_0'-E_0)q^0}\exp\left(-\frac{m\omega}{2\hbar}\{\mathbf{q}_{(1)}^2+\mathbf{q}_{(2)}^2-(\mathbf{x}'+\mathbf{x})^T\cdot\mathbf{q}_{(1)}\right.\\&\quad\left.+[2\mathbf{q}_{(1)}-(\mathbf{x}'+\mathbf{x})]^T\cdot\mathbf{R}[\omega(t-q^0)]\cdot\mathbf{q}_{(2)}\}\right)\exp\left\{-\frac{im\omega}{2\hbar}\{(\mathbf{x}'+\mathbf{x})^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{q}_{(1)}-(\mathbf{x}'-\mathbf{x})^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{R}[\omega(t-q^0)]\cdot\mathbf{q}_{(2)}\}\right\}\\&=\delta(E_0'-E_0)\,\delta^{(2)}(\mathbf{x}'-\mathbf{x}),\end{aligned} \tag{4.17}$$

where one defines the appropriate normalizing constant: $\mu_0=(m\omega)^3[2\pi(2\pi\hbar)]^{-4}$. This result assures one that, in general, one has

$$\langle t,\mathbf{x}';E_0',n_1',n_2'|t,\mathbf{x};E_0,n_1,n_2\rangle=\delta_{n_1'n_1}\delta_{n_2'n_2}\delta(E_0'-E_0)\,\delta^{(2)}(\mathbf{x}'-\mathbf{x}). \tag{4.18}$$

Let us now calculate the *transition amplitude* between two normalized irreducible configuration ground states, defined at the events $(t,\mathbf{x})$ and $(t',\mathbf{x}')$. According to Eq. (4.16), after some substitutions, one obtains the following integral:

$$\langle t',\mathbf{x}';E_0',0_1,0_2|t,\mathbf{x};E_0,0_1,0_2\rangle=\mu_0 2\pi(2\pi\hbar)\,\delta(E_0'-E_0)e^{-(i/\hbar)E_0(t'-t)}e^{-(m\omega/4\hbar)(x'^2+x^2)}\int\int d^2q_{(1)}$$
$$\times\exp\left\{-\frac{m\omega}{2\hbar}\left[\mathbf{q}_{(1)}{}^2-(\mathbf{x}'+\mathbf{x})^T\cdot\mathbf{q}_{(1)}-i(\mathbf{x}'-\mathbf{x})^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{q}_{(1)}\right]\right\}\int\int d^2q_{(2)}$$
$$\times\exp\left(-\frac{m\omega}{2\hbar}\{\mathbf{q}_{(2)}{}^2+[(\mathbf{q}_{(1)}-\mathbf{x}')^T\cdot\mathbf{R}(\omega t')+(\mathbf{q}_{(1)}-\mathbf{x})^T\cdot\mathbf{R}(\omega t)]\cdot\mathbf{q}_{(2)}\}\right)$$
$$\times\exp\left\{-\frac{im\omega}{2\hbar}[(\mathbf{q}_{(1)}-\mathbf{x}')^T\cdot\mathbf{R}(\omega t')-(\mathbf{q}_{(1)}-\mathbf{x})^T\cdot\mathbf{R}(\omega t)]\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{q}_{(2)}\right\}. \quad (4.19)$$

In this way, one arrives at the desired *propagation kernel*, which can be finally written in the closed form

$$\langle t',\mathbf{x}';E_0',n_1',n_2'|t,\mathbf{x};E_0,n_1,n_2\rangle$$
$$=\delta(E_0'-E_0)\,\delta_{n_1'n_1}\delta_{n_2'n_2}e^{-(i/\hbar)E_0\Delta t}\left(\frac{m}{2\pi i\hbar}\right)$$
$$\times\left(\frac{\omega\Delta t/2}{\sin\,\omega\Delta t/2}\right)e^{i\omega\Delta t/2}\exp\left\{\frac{im\omega}{4\hbar}[\cot(\omega\Delta t/2)(\mathbf{x}'-\mathbf{x})^2\right.$$
$$\left.-2i\mathbf{x}'^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{x}]\right\}, \quad (4.20)$$

where $\Delta t=t'-t$. This result corresponds exactly with the propagation kernel of the two-dimensional Landau system [in the (x,y) plane], as obtained in the *path integral* approach [11]. However, we emphasize the fact that the integral one evaluates in Eq. (4.19) is a *Hurwitz invariant integral*, defined over the Landau group manifold [2].

V. CONCLUDING REMARKS: PHYSICAL INTERPRETATION

In conclusion, we briefly turn our attention to the physical interpretation of the attained model. Notwithstanding the heuristic character of all interpretations, this task is unavoidable in theoretical physics, even when, as in the present case, one faces the assessment of a reformulation of an old, well established, theory. (Another important subject would be to use the *model* with the aim of making some *physical predictions*. Lack of space prevents us to dwell on this issue now.)

Many variables (i.e., q numbers) appear in quantum kinematics, which play no role in the canonical approach to quantum mechanics. The Q's are the *quantized parameters* of that very special group, whose action on the configurations of the system is indeed *complete*. As c numbers, the parameters of the group command the specific symmetry transformations of the allowed world lines, and of *no other* class of motions evolving in the same configuration spacetime. They must be *quantized*, for they play an essential mechanical role and are endowed with a clear physical meaning.

For instance, q^0 corresponds to the fact that for both (classical and quantal) descriptions of the system it does not matter when we fix the initial state. Therefore, one introduces an *initial time* operator Q^0 related to this physical degree of freedom. The conjugate operators are the corresponding generators of G_L, either $L_0^{(k)}$ or $R_0^{(k)}$. The spectrum is $-\infty<E_0<+\infty$, which is just an arbitrary constant. We interpret these generators as purely *kinematic* ''partial Hamiltonians'' devoid of any dynamical meaning. The whole dynamic of the system is contained in $S_0^{(k)}$. Indeed, the *total energy of the system* is given by the spectrum of the Schrödinger operator, which is physically defined only up to an arbitrary additive constant. So, in quantum kinematics, one solves the vexed problem of a ''time operator'' in a very peculiar (albeit natural) way. It must be emphasized that, in the present theory, the *total energy operator* $S_0^{(k)}$ appears as the *dynamical* conjugate momentum of the initial time operator Q^0, in a ''canonical'' sense. In fact, one has

$$[Q^0,S_0^{(k)}]=i\hbar I, \quad (5.1)$$

as *can be proved* rather easily.

One quantizes the other parameters of the group in the same manner, and for the same reasons. These parameters are related with mechanical symmetry operations that can be performed on the system. If we look back at Eq. (2.1) we see that the complete symmetry transformation $\mathbf{x}\rightarrow\mathbf{x}'$ consists in a *rotation* of the system with respect to the Cartesian axes, through an angle q^5 about the origin O, plus the addition of a *general motion* of the system, parametrized by $\mathbf{q}_{(1)}$ and $\mathbf{q}_{(2)}$.

Let us first consider $\mathbf{Q}_{(1)}$. Classically, the effect of the constant magnetic field on the moving point charge is to pull it towards a (fixed) guiding center $\mathbf{b}$, with a constant centripetal force. On an intuitive ground, one sees that one has to quantize the fixed center of motion $\mathbf{b}$ somehow, since this degree of freedom suffers a kind of *zitterbewegung*, due to the quantum fluctuations of the particle, which ''jumps'' from one world line to another in a permanent and random fashion. Hence, the quantization rule $\mathbf{b}\equiv\mathbf{q}_{(1)}\rightarrow\mathbf{Q}_{(1)}$ seems to be in order. In this manner, one interprets the corresponding *partial Hamiltonian* $H_1^{(k)}=(1/2m)(\mathbf{R}_{(1)}^{(k)})^2$ as a dynamical observable, which describes quantal harmonic oscillations (i.e., *zitterbewegung*) of frequency ω, suffered by the mechanical variable $\mathbf{b}$. No wonder, n_1 makes no contribution to the *total energy of the particle*, since this quantity is independent of the position of the fixed orbit center in the (x,y) plane. Therefore the degeneracy exhibited in Eq. (4.12) follows. Let us also remark that in the current approach to this problem one quantizes the orbit center in a rather bizarre *ad hoc* manner, since $\mathbf{b}$ is not a canonical variable.

Similar considerations apply to $\mathbf{Q}_{(2)}$. Within the classical theory, initially one has $\mathbf{x}(0)=\mathbf{a}+\mathbf{b}$ and $\dot{\mathbf{x}}(0)^T\cdot\mathbf{a}=0$; therefore, one expects the quantization rule $\mathbf{a}\equiv\mathbf{q}_{(2)}\rightarrow\mathbf{Q}_{(2)}$. Indeed, it would be contrary to the concepts of quantum mechanics to think of $\mathbf{a}$ as a static vector connecting two randomly fluctuating points. In this way, one interprets the associated *partial Hamiltonian* $H_2^{(k)}=(1/2m)(\mathbf{R}_{(2)}^{(k)})^2$ dynamically, as describing the simple harmonic oscillations, of frequency ω, due to the *zitterbewegung* of $\mathbf{a}$'s degrees of freedom. From what we know about the classical analog of the system, the *total energy* of the particle in the field must be given by $\hbar\omega(n_2+1/2)$, within an arbitrary additive constant E_0; i.e., Eq. (4.12) appears, as expected.

In the same spirit, let us comment on the rotational symmetry of the system *about the origin* O. The *classical angular momentum* J_O *of the particle about* O is given by $J_O=m\mathbf{x}(t)^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{x}(t)$, which yields $J_O=J_B-m\omega\mathbf{a}^T\cdot\mathbf{R}^T(\omega t)\cdot\mathbf{b}$, where J_B denotes the angular momentum about the orbit center. According to this elementary rule, the kinematic projection $\mathbf{a}^T\cdot\mathbf{R}^T(\omega t)\cdot\mathbf{b}$ describes the simple harmonic oscillator (of frequency ω) corresponding to the *uniform circular motion* $\mathbf{R}^T(\omega t)\cdot\mathbf{b}$ *relative to* $\mathbf{a}$. From this intuitive picture, we understand immediately the meaning of the degeneracy $n_5=n_2-n_1$ found in Eq. (4.13) (i.e., $J_O=J_B-J_A$), as an unavoidable feature of the model, since $R_5^{(k)}$ is an angular-momentum operator *about the origin*.

The fact that $R_0^{(k)}$, $R_5^{(k)}$, $H_1^{(k)}$, and $H_2^{(k)}$ are all *invariant observables* of the system is also worth noting. This means that *pure states* are indeed *objective*. Although these states describe only *potentialities* of the system (Heisenberg), they are always the same for the whole *class of equivalent preferred observers*. Otherwise, it would be absurd to think that the same, equally prepared, system (even in an ensemble) could be found in the *ground state* by one observer and in an *excited state* by another, if both observers are physically equivalent.

The last point we would like to make is that by a systematic application of general quantum-kinematic tools we have been *able to deduce* the Schrödinger equation, as well as to *calculate* the corresponding propagation kernel of the system, without recourse to the traditional approaches to quantum mechanics. *Group-quantization* analysis of the corresponding complete symmetry group has been enough to this end.

This paper presents but just another successful application of a group-theoretic formalism of quantum mechanics [12]. In light of these results, one cannot avoid wondering whether a sounder formulation of quantum mechanics is possible, stemming *directly* from the symmetries of a system. If this is really so, it should be uncovered as completely as possible.

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APPENDIX A: STRUCTURE OF THE LANDAU GROUP

It is our purpose here to provide a brief survey of the main properties of the Landau group. First, let us note that G_L is isomorphic to the following group structure:

$$G_L\sim[\mathcal{T}^{(1)}(2)\circledS\mathrm{SO}^{(5)}(2)]\otimes\{\mathcal{R}_+^{(0)}\circledS[\mathcal{T}^{(2)}(2)\circledS\mathrm{SO}^{(5)}(2)]\},\tag{A1}$$

where each factor $\mathcal{T}(2)\circledS\mathrm{SO}(2)$ denotes the Euclidean group $\mathcal{E}(2)$ in the plane, each $\mathcal{T}(2)$ is the group of rigid translations in two dimensions, $\mathcal{R}_+$ is the additive group of real numbers, and we write $\circledS$ for the semidirect product. Hence, the group space $M(G_L)$ is a noncompact connected manifold, which is *not* simply connected. The *group-multiplication law* $q''^a=g^a(q';q)$ is given by the following combinations of the parameters:

$$q''^0=g^0(q':q)=q'^0+q^0,\tag{A2.1}$$

$$\mathbf{q}''_{(1)}=\mathbf{g}_{(1)}(q';q)=\mathbf{q}'_{(1)}+\mathbf{R}(q'^5)\cdot\mathbf{q}_{(1)},\tag{A2.2}$$

$$\mathbf{q}''_{(2)}=\mathbf{g}_{(2)}(q';q)=\mathbf{R}(\omega q^0)\cdot\mathbf{q}'_{(2)}+\mathbf{R}(q'^5)\cdot\mathbf{q}_{(2)},\tag{A2.3}$$

$$q''^5=g^5(q';q)=(q'^5+q^5)_{2\pi},\tag{A2.4}$$

with $\mathbf{q}_{(1)}=(q^1,q^2)$ and $\mathbf{q}_{(2)}=(q^3,q^4)$. Since the identity point $e\in M(G_L)$ is at the origin, the *inversion law* for the parameters follows:

$$\overline{q^0}=-q^0,\quad \overline{\mathbf{q}_{(1)}}=-\mathbf{R}^T(q^5)\cdot\mathbf{q}_{(1)},$$
$$\overline{\mathbf{q}_{(2)}}=-\mathbf{R}^T(q^5+\omega q^0)\cdot\mathbf{q}_{(2)},\quad \overline{q^5}=-q^5;\tag{A3}$$

i.e., one has $g^a(q;\overline{q})=g^a(\overline{q};q)=0$, for $a=0,1,\ldots,5$, and for all $q\in M(G_L)$.

We next consider the *right transport matrix* for contravariant vectors in the group manifold. This matrix is defined as $R_a^b(q)=\lim_{q'\to e}\partial'_a g^b(q';q)$. Therefore, from Eqs. (A2), one obtains

$$R_a^b(q)=\begin{bmatrix}1 & 0^T & 0^T & 0\\ 0 & \mathbf{I} & \mathbf{O} & 0\\ 0 & \mathbf{O} & \mathbf{R}^T(\omega q^0) & 0\\ 0 & -\mathbf{q}_{(1)}^T\cdot\mathcal{E} & -\mathbf{q}_{(2)}^T\cdot\mathcal{E} & 1\end{bmatrix},\tag{A4}$$

where a labels the rows and b labels the columns. This matrix yields the *left* Lie vector fields acting in $M(G_L)$; i.e., $X_a(q)=R_a^{\,b}(q)\partial_b$. Thus one has the operators

$$X_0=\partial_0,\quad \mathbf{X}_{(1)}=\partial_{(1)},\quad \mathbf{X}_{(2)}=\mathbf{R}^T(\omega q^0)\cdot\partial_{(2)},$$
$$X_5=-\mathbf{q}_{(1)}^T\cdot\boldsymbol{\mathcal{E}}\cdot\partial_{(1)}-\mathbf{q}_{(2)}^T\cdot\boldsymbol{\mathcal{E}}\cdot\partial_{(2)}+\partial_5.\tag{A5}$$

Another useful feature concerns the *adjoint representation* of the group. For G_L, the matrix of the adjoint representation reads

$$A_a^b(q)=\begin{bmatrix}1 & 0^T & \omega\mathbf{q}_{(2)}^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{R}(q^5) & 0\\ 0 & \mathbf{R}(q^5) & \mathbf{O} & 0\\ 0 & \mathbf{O} & \mathbf{R}(q^5-\omega q^0) & 0\\ 0 & -\mathbf{q}_{(1)}^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{R}(q^5) & -\mathbf{q}_{(2)}^T\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{R}(q^5) & 1\end{bmatrix},\tag{A6}$$

wherefrom the inverse matrix $\overline{A_a^{\,b}}(q)$ of the *antiadjoint* representation follows. The usual defining property of this representation [that is, $A_a^{\,b}(e+\delta q)=\delta_a^b+\delta q^c f_{ca}^{\,b}$] can be checked directly from the following basic formula:

$$X_a(q)A_b^c(q)=f_{ab}^d A_d^c(q). \quad \text{(A7)}$$

The Landau group is *unimodular*; i.e., $A(q)=\det[A_a^{\,b}(q)]=1$. The features summarized in this appendix are not just minutiae, since they are fundamental tools in quantum kinematics. (For details, see [10].)

APPENDIX B: REGULAR RAY REPRESENTATION OF THE LANDAU GROUP

As we have seen in Sec. II B, the Landau group admits only two physically meaningful ray extensions, generated by the ray constants k_{12} and k_{34}. In this appendix we briefly consider the *regular ray representation* of G_L associated with these constants.

Since G_L is unimodular, the Hurwitz invariant measure is given simply by $d\mu(q)=\mu_0 d^6q=\mu_0 dq^0 dq^1 \ldots dq^5$, where μ_0 is an arbitrary normalization constant. In consequence, the Hilbert space $\mathcal{H}(G_L)$ that carries the *regular representation* [13] is defined as the set $\mathcal{L}^2(G_L)$ of *square-integrable* wave functions $\psi(q)=\psi(q^0,\mathbf{q}_{(1)},\mathbf{q}_{(2)},q^5)$ on $M(G_L)$, which are *periodic* on the unit circle $S_1\subset M(G_L)$; i.e., one defines $\langle\psi|\psi\rangle=\mu_0\int d^6q|\psi(q)|^2<\infty$ if, and only if, $|\psi\rangle\in\mathcal{H}(G_L)$. In quantum kinematics, one also needs to consider the *rigged* Hilbert space $\widetilde{\mathcal{H}}(G_L)$, attached with $\mathcal{H}(G_L)$, for this permits the definition of wave functions $\psi(q)$ on $M(G_L)$ in the usual manner: $\psi(q)=\langle q|\psi\rangle$, for all $|\psi\rangle\in\mathcal{H}(G_L)$ and all $q\in M(G_L)$ [14]. So we introduce a complete continuous orthogonal basis $\{|q\rangle=|q^0, \mathbf{q}_{(1)}, \mathbf{q}_{(2)}, q^5\rangle\}$ on the rigged Hilbert space $\widetilde{\mathcal{H}}$:

$$\langle q|q'\rangle=\mu_0^{-1}\delta^{(6)}(q-q'), \quad \mu_0\int d^6q|q\rangle\langle q|=I, \quad \text{(B1)}$$

where I denotes the identity operator, and $|q\rangle$ is periodic in q^5.

Thus we consider the *unitary operators* that carry the regular *ray* representations of G_L within $\widetilde{\mathcal{H}}$. Since the $|q\rangle$'s form a complete basis, the group ray operators may be defined as follows:

$$U_L^{(k)}(q)|q'\rangle=e^{-(i/\hbar)q^aL_a^{(k)}}|q'\rangle=e^{(i/\hbar)\phi_k(q;q')}|g(q;q')\rangle. \quad \text{(B2)}$$

for the *left* regular ray representation, with generators $L_a^{(k)}$, and where the exponent function $\phi_k(q;q')$ is a *two-cocycle* obtained from the set of genuine ray constants $k=\{k_{12},k_{34}\}$. In fact, these operators are unitary and satisfy the ray representation property

$$U_L^{(k)}(q')U_L^{(k)}(q)=e^{(i/\hbar)\phi_k(q';q)}U_L^{(k)}[g(q';q)]. \quad \text{(B3)}$$

In this paper we use the following two-cocycle of the Landau group:

$$\phi_k(q';q)=\tfrac{1}{2}k_{12}\mathbf{q}'^{T}_{(1)}\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{R}(q'^5)\cdot\mathbf{q}_{(1)} + \tfrac{1}{2}k_{34}\mathbf{q}'^{T}_{(2)}\cdot\boldsymbol{\mathcal{E}}\cdot\mathbf{R}(q'^5)\cdot\mathbf{R}^T(\omega q^0)\cdot\mathbf{q}_{(2)}. \quad \text{(B4)}$$

This exponent has been calculated by means of non-Abelian analytic techniques (as developed by Krause [7]). This is a *completely gauge reduced two-cocycle* and belongs to the μ gauge [7]; that is, one has

$$\mu_k(q)=\phi_k(\overline{q;q})=\phi_k(q;\overline{q})=0, \quad \text{(B5)}$$

which yields $U_L^{(k)\dagger}(q)=U_L^{(k)}(\overline{q})$. Next, we need to recall that the *right exponent generators* are defined as follows: $r_a^{(k)}(q)=\lim_{q'\to e}\partial'_a\phi_k(q';q)$. Therefore, the corresponding exponent generators of the Landau group are given by

$$\mathbf{r}_{(1)}^{(k)}(q)=\tfrac{1}{2}k_{12}\boldsymbol{\mathcal{E}}\cdot\mathbf{q}_{(1)}, \quad \mathbf{r}_{(2)}^{(k)}(q)=\tfrac{1}{2}k_{34}\boldsymbol{\mathcal{E}}\cdot\mathbf{R}^T(\omega q^0)\cdot\mathbf{q}_{(2)}, \quad \text{(B6)}$$

while $r_0^{(k)}(q)=r_5^{(k)}(q)=0$. Finally, we need to recall also that in the μ gauge the *left* exponent generators $l_a^{(k)}(q)$ are given by $l_a^{(k)}(q)=r_a^{(k)}(\overline{q})$. We have omitted the lengthy calculations leading to these results [7].

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[14] See, e.g., A. Bohm and M. Gadella, *Dirac Kets, Gamow Vectors and Gel'fand Triplets* (Springer-Verlag, Berlin, 1989), for a very readable introduction to the rigged Hilbert-space formulation of quantum mechanics.