

Time of arrival in quantum mechanics

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We study the problem of computing the probability for the time of arrival of a quantum particle at a given spatial position. We consider a solution to this problem based on the spectral decomposition of the particle's (Heisenberg) state into the eigenstates of a suitable operator, which we denote as the "time-of-arrival" operator. We discuss the general properties of this operator. We construct the operator explicitly in the simple case of a free nonrelativistic particle and compare the probabilities it yields with the ones estimated indirectly in terms of the flux of the Schrödinger current. We derive a well-defined uncertainty relation between time of arrival and energy; this result shows that the well-known arguments against the existence of such a relation can be circumvented. Finally, we define a "time representation" of the quantum mechanics of a free particle, in which the time of arrival is diagonal. Our results suggest that, contrary to what is commonly assumed, quantum mechanics exhibits a hidden equivalence between independent (time) and dependent (position) variables, analogous to the one revealed by the parametrized formalism in classical mechanics. [S1050-2947(96)03911-X]

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I. INTRODUCTION

Consider the following experimental arrangement. A particle moves in one dimension, along the x axis. A detector is placed in the position $x = X$. Let T be the time at which the particle is detected, which we denote as the "time of arrival" of the particle at X . Can we predict T from the knowledge of the initial state of the particle?

In classical mechanics, the answer is simple. Let $x(t; x_0, p_0)$ be the general solution of the equations of motion corresponding to initial position and momentum x_0 and p_0 at $t = 0$. We obtain the time of arrival T as follows. We invert the function $x = x(t; x_0, p_0)$ with respect to t , obtaining the function $t(x; x_0, p_0)$. The time of arrival T at X of a particle with initial data x_0 and p_0 is then

$$T = t(X; x_0, p_0). \quad (1)$$

Two remarks are in order. First, if $t(x; x_0, p_0)$ is multivalued, we are only interested in its lowest value, since the particle is detected the first time it gets to X . Second, for certain values of x_0 and p_0 , it may happen that X is outside the range of the function $x(t; x_0, p_0)$. This indicates that the detector in X will never detect a particle with that initial state. The time of arrival is a physical variable that, in a sense, can take two kinds of values: either a real number $T \in \mathbb{R}$ or the value "T=never." Notice that in the latter case, the quantity T formally computed from (1) turns out to be complex. Thus a complex T from (1) (for given X, x_0, p_0) means that the particle with initial data x_0, p_0 is never detected at X .

In quantum mechanics, the problem is surprisingly harder. In this case the time of arrival can be determined only proba-

bilistically. Let $\pi(T)dT$ be the probability that the particle is detected at detector location X within the time interval T and $T + dT$. Thus

$$\int_{T_1}^{T_2} \pi(T)dT \quad (2)$$

is the probability that the particle is detected between the time T_1 and the time T_2 . How can we compute $\pi(T)$ from the quantum state, e.g., from the particle's wave function $\psi(x)$ at $t = 0$?

The importance of this problem has been stressed by others. Mielnik [1] refers to it as "a dark spot in the perfection of quantum scheme." Smith [2] begins with the following: "It is surprising that the current apparatus of quantum mechanics does not include a simple representation for so eminently observable a quantity as the lifetime of metastable entities." In a similar vein, Werner [3] states, "There is a strange discrepancy between many theoretical accounts of measurement in quantum mechanics and the typical measurement carried out in a laboratory. On the one hand every counter clicks at a certain time, and frequently these times are quite important for the evaluation of an experiment, e.g., a correlation experiment. On the other hand, operators describing the probability that a counter responds during a given time interval are rarely discussed in quantum mechanics textbooks."

To the best of our knowledge, this question has not received a complete treatment in the standard literature on quantum mechanics. In this literature, the problem has been variously referred to as the "tunneling time problem," the "time of arrival problem," or the "screen problem." The existing literature appears to be vast, and we do not intend to give a complete review.

Allcock [4] has given a proof of the nonexistence of a time-of-arrival operator on the usual Hilbert space of quantum mechanics. The problem of computing the time of detection of a particle is usually treated in a very indirect man-

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ner. For instance, the probability (in time) of detecting a decay product, say a particle escaping the potential of a nucleus, can be obtained from the time evolution of the probability that the particle is still within the confining potential. Alternatively, one can treat the detector that measures the time of arrival quantum mechanically and compute the probabilities for the positions of the detector pointer at a later time; in this way one can trade a measurement of the time of arrival for a measurement of position at a late fixed time.

Apart from the above apparently undeveloped ideas, various concrete approaches have been taken. One of the earliest is that of the time operator as the energy derivative operator. Wigner considered the problem of relating the energy derivative of the wave function's phase shift to the scattering delay of a particle [5]. This approach, later developed by Smith [2] and others (see, for instance, [6]), gives the average delay, but fails to provide the full probability distribution of the time of arrival. Kijowski [7] obtained a probability distribution, but not on the usual Hilbert space; thus the interpretation of the wave function in terms of familiar quantities is obscure. Piron discussed the problem [8], sketching ideas related to the ones developed here. From the Wigner distribution, Werner [9] obtained a quasiprobability, which, however, is not positive definite. Ideas related to the ones presented here were also explored by Olkhovsky *et al.* [10], but in this case too only the average time of arrival was obtained and not its full probability distribution. More recently, Marinov and Segev [11] have applied this approach to calculate tunnelling times. Muga *et al.* [12], as well as Kijowski [7], estimated the time-of-arrival probability density from the Schrödinger current density. This density is not positive definite and the approach has been otherwise criticized (see, e.g., [1]). Werner took a mathematical approach [3,13] to the problem of the construction of “screen observables,” i.e., observables on timelike hypersurfaces [1]. Among more “space-time” approaches, Marolf [14] has calculated probabilities for particle detection in space-time regions, but (not unreasonably) finds them to be very sensitive to the model of detection. Kumar [15] studied the quantum first-passage problem in a path-integral approach, but did not obtain a positive probability density. The problem has been studied in the framework of Hartle's generalized quantum mechanics [16] by using sum over histories methods. Various attempts in this direction and discussions of difficulties can be found in Ref. [17]. See also the recent paper Ref. [18] for a discussion of the problem and for references; in particular, Ref. [18] discusses the difficulties one has to face in trying to compute *sequences* of times of arrival: an important problem that, however, we do not address here. As stressed by Hartle in [18], generalized quantum mechanics *generalizes* “usual quantum mechanics”; here, on the other hand, we are interested in the question whether $\pi(T)$ can be computed within the mathematical framework of conventional *Hamiltonian* quantum mechanics.

We see two reasons of interest for discussing the problem of computing time of arrival in quantum mechanics. First, it is a well-posed problem in simple quantum theory and there must be a solution. Echoing Smith [2], we do not expect that quantum mechanics could fail to predict a probability distribution that can be experimentally measured by simply placing a detector at a fixed position and noting the time at which

it “clicks.” The problem is not just academic: It is related to the problem of computing the full probability distribution (as opposed to the expectation value) for the tunneling time through a potential barrier. This problem has relevance, for instance, in computing rates of chemical reactions (see, for example, [15]). Second, the problem bears directly on the interpretation of quantum theories without Newtonian time [19,20] and thus on quantum gravity; we shall briefly comment on this issue in closing.

This paper is the first of a sequence of two. Here we develop a general theory for the time-of-arrival operator and study the free nonrelativistic particle case in detail. In a companion paper [21], we investigate a technique for the explicit construction of the time-of-arrival operator in more general cases, we extend our formalism to parametrized systems, and we study some less trivial models: a particle in an exponential potential and a cosmological model.

In Sec. II we give a general argument, based on the superposition principle, for the existence of an operator \hat{T} (the time-of-arrival operator) such that $\pi(T)$ can be obtained from the spectral decomposition of $\psi(x)$ in eigenstates of \hat{T} , in the usual manner in which probability distribution are obtained in quantum theory. \hat{T} has peculiar properties that distinguish it from conventional quantum observables. We give a general argument based on the correspondence principle indicating that \hat{T} can be expressed in terms of position and momentum operators by the inverse of the classical equations of motion Eq. (1). This does not suffice in fixing the operator, since factor-ordering ambiguities can be serious. The problem of the actual construction of the operator \hat{T} in more general systems will be addressed in [21]. In Sec. III we study an explicit form of the operator in the case of a free nonrelativistic particle. We diagonalize the operator, providing a general expression for the time-of-arrival probability density $\pi(T)$. In particular, we calculate $\pi(T)$ explicitly for a Gaussian wave packet. In Sec. IV we discuss some consequences of our construction. We notice that the existence of the operator implies that the quantum mechanics of a free particle can be expressed in a “time-representation” basis. We derive time-energy uncertainty relations. We conclude in Sec. V with a general comment on the equivalence between time and position variables suggested by our results.

In the Appendix we study whether the probability distribution we computed is reasonable, by comparing it with the one estimated indirectly using the Schrödinger current. We find that the two agree within second order in the de Broglie wavelength of the particle. The probability computed from the Schrödinger current cannot be physically correct to all orders because it is not positive definite; whether or not the probability distribution computed with \hat{T} is physically correct to all orders is a question that can perhaps be decided experimentally.

II. TIME OF ARRIVAL: GENERAL THEORY

A. Incomplete spectral family $P(T)$

Consider the quantum analog of the experimental situation sketched at the beginning of the paper: A particle is in

an initial state ψ at $t=0$ and a particle detector is placed at $x=X$. Let T be the time at which the particle is detected. Let $\pi_\psi(T)dT$ be the probability that the particle is detected between times T and $T+dT$. Let ψ and ϕ be two quantum states such that both $\pi_\psi(T)$ and $\pi_\phi(T)$ have support in the interval $I=(T, T+\Delta T)$. Consider the state formed as the linear combination $a\psi+b\phi$ (where a and b are any two complex numbers with $|a|^2+|b|^2=1$).

According to the superposition principle, if a measurable quantity has a definite value λ_ψ when the system is in the state ψ and value λ_ϕ when the system is in the state ϕ , then a measurement of such a quantity in the state $a\psi+b\phi$ will yield either λ_ψ or λ_ϕ (with respective probabilities $|a|^2$ and $|b|^2$) [22]. If we assume the general validity of the superposition principle, we must then expect that the probability distribution $\pi_{a\psi+b\phi}(T)$ will have support in the interval $(T, T+\Delta T)$ as well. Therefore, the states ψ such that $\pi_\psi(T)$ has support on a given interval $I=(T, T+\Delta T)$ form a linear subspace of the state space. We can therefore define a projection operator P_I as the projector on such a subspace.

The superposition principle could fail for the time of arrival. However, we would be surprised if it did. Notice that the question can (probably easily) be decided experimentally. Perhaps an experiment testing the validity of the superposition principle in this contest could have some interest. Here we assume that the principle holds, and thus the projectors $P_{T, T+\Delta T}$ are well defined.

By their very definition, the projectors satisfy $P_I P_J = P_J$ if the interval J is contained in the interval I and $P_I P_J = 0$ if the two intervals are disjoint. Then one can show that $P_{I \cup J} = P_I + P_J$, and it follows that the operators P_I can be written in terms of a family of (spectral) projectors $P(T)$ as

$$P_{T, T+\Delta T} = \int_T^{T+\Delta T} P(T') dT'. \quad (3)$$

The (spectral) family $P(T)$ contains all the information needed to compute $\pi(T)$. Indeed, from the definition given and using again the superposition principle, we have easily

$$\pi_\psi(T) = \langle \psi | P(T) | \psi \rangle. \quad (4)$$

Thus the probability distribution $\pi(T)$ can be obtained in terms of the spectral family $P(T)$ in the same way in which all probability distributions are obtained in quantum mechanics. Indeed, recall that if \hat{A} is the self-adjoint quantum operator corresponding to the observable quantity A , then the probability distribution $\pi_\psi(A)$ of measuring the value A on the state ψ is $\langle \psi | P(A) | \psi \rangle$, where $P(A)$ is the spectral family associated to \hat{A} , namely,

$$\hat{A} = \int A P(A) dA. \quad (5)$$

B. Operators \hat{T} and $\hat{\mathcal{P}}$

One may be immediately tempted to define a ‘‘time-of-arrival operator’’ in analogy with (5) as

$$\hat{T} = \int_{-\infty}^{+\infty} T P(T) dT \quad (6)$$

so that an eigenstate of this operator with eigenvalue T would be a (generalized) state detected precisely at time T . However, there is an important difference from usual self-adjoint quantum-mechanical observables that must be addressed before doing so. If $P(A)$ is the spectral family of a self-adjoint operator \hat{A} , then

$$\int_{-\infty}^{+\infty} P(A) dA = \hat{1}, \quad (7)$$

where $\hat{1}$ is the identity operator. On the other hand, define $\hat{\mathcal{P}}$ by

$$\hat{\mathcal{P}} := \int_{-\infty}^{+\infty} P(T) dT; \quad (8)$$

there is no reason for $\hat{\mathcal{P}}$ to be the identity. If it is not, we say that the spectral family $P(T)$ is ‘‘incomplete.’’

Incompleteness occurs because it is not true that any state is certainly detected at some time. Most likely, there are states that are never detected, given that such states exist in the classical theory as well. Thus $\hat{\mathcal{P}}$ projects on the subspace $\mathcal{H}_{\text{detected}}$ formed by the states in which the particle is detected at some time at X and $(\hat{1} - \hat{\mathcal{P}})$ is the projector on the subspace $\mathcal{H}_{\text{never detected}}$ of states in which the particle is never detected at X . The fact that those two classes of states form orthogonal linear subspaces follows from the superposition principle again.

Thus \hat{T} is properly defined by (6) on $\mathcal{H}_{\text{detected}}$ only. If we define the time-of-arrival operator by (6) on the entire state space, then we have the awkward consequence that the states in the range of $P(T=0)$ and the ones in $\mathcal{H}_{\text{never detected}}$ are both annihilated by \hat{T} . Namely, \hat{T} does not distinguish the states in which the particle is detected at $T=0$ from the ones in which it is never detected.

The full information that we need in order to compute $\pi(T)$ is contained in the incomplete spectral family $P(T)$ or, equivalently, in the two mutually commuting operators $\hat{\mathcal{P}}$ and \hat{T} , where \hat{T} is a self-adjoint operator on the Hilbert space $\mathcal{H}_{\text{detected}} = \text{Range}(\hat{\mathcal{P}})$. Notice that

$$\langle \hat{T} \rangle_\psi = \frac{\langle \psi | \hat{\mathcal{P}} \hat{T} \hat{\mathcal{P}} | \psi \rangle}{\langle \psi | \hat{\mathcal{P}} | \psi \rangle} \quad (9)$$

is the expected time of arrival in those states that are detected at all and is thus a *conditional* expectation value.

As defined in (6), \hat{T} annihilates all states in $\mathcal{H}_{\text{never detected}}$. It is useful to replace this definition by fixing the following convention for the action of \hat{T} on $\mathcal{H}_{\text{never detected}}$. We define \hat{T} on the *entire* state space, by mimicking what happens in classical mechanics: We choose (arbitrarily, at this stage) a (diagonalizable) action of \hat{T} on $\mathcal{H}_{\text{never detected}}$ with a complex (nonreal) spectrum, with the understanding that any complex eigenvalue be interpreted as ‘‘the particle is never detected.’’ If we use this convention,

the operator \hat{T} is not self-adjoint, but it may still have a complete and orthogonal basis of eigenstates (perhaps after a regularization; see later). The reason for such a convention (which we postulate from now on) will be given below; its utility will be particularly clear in [21].

C. Partial characterization of \hat{T} from its classical limit

In the preceding subsection we have argued on general grounds that a time-of-arrival operator \hat{T} giving the time-of-arrival probability distribution should exist. How can we construct the operator \hat{T} from the knowledge of the dynamics of the system? Let us work in the Heisenberg picture. The quantum theory is defined by the Heisenberg state space, in which states do not evolve. Let ψ be a Heisenberg state. The elementary operators are Heisenberg position operator \hat{x}_0 and the momentum operator \hat{p}_0 , representing position and momentum at $t=0$. Since all operators can be constructed in terms of \hat{x}_0 and \hat{p}_0 , we expect to be able to express \hat{T} as an operator function of \hat{x}_0 and \hat{p}_0 . A key requirement on \hat{T} is that it yield the correct results in the classical limit (Bohr's correspondence principle). If so, the dependence of \hat{T} on \hat{x}_0 and \hat{p}_0 should reduce to the classical dependence of T on x_0 and p_0 in the classical limit. This indicates that the dependence of the operator \hat{T} on \hat{x}_0 and \hat{p}_0 is given by some ordering of the function (1), which, we recall, was obtained by inverting the solutions $x(t; x_0, p_0)$ of the classical equations of motion. Thus we should have that

$$\hat{T} = t(X; \hat{x}_0, \hat{p}_0), \quad (10)$$

where an ordering has to be chosen. The c number X , we recall, is the position of the detector. Notice at this point the usefulness of the convention that complex eigenvalues represent nondetection: This can go through naturally in the classical limit. Equation (10) does not suffice in general for characterizing \hat{T} uniquely because the correct physical ordering of the operator function can be highly nontrivial. In the companion paper [21], we investigate a technique for constructing the operator $\hat{T}(X)$ and fixing ordering ambiguities.

In order to emphasize the dependence of the time-of-arrival operator \hat{T} on the position X of the detector, we will, from now on, write the operator as $\hat{T}(X)$. Analogously, we will denote the spectral family of projectors associated to $\hat{T}(X)$ as $P(T; X)$ and the probability distribution of the time of arrival at X as $\pi(T; X)$.

The construction above can be easily generalized to systems with n degrees of freedom. A classical state of such a system is described by a point in the $2n$ -dimensional phase space Γ , with coordinates $\{z^i, i=1, \dots, 2n\}$. The dynamics is generated by the Hamiltonian $H(z^i)$. The Hamilton equations of motion are $dz^i(t)/dt = \{z^i, H\}$. The general solutions to these equations can be written as

$$z^i(t) = z^i(z_0^i, t), \quad i=1, \dots, 2n, \quad (11)$$

where z_0^i are $2n$ independent integration constants. In particular, we may choose as integration constants z_0^i the values of z^i at $t=0$, $z^i(0)$. The above equations enable us to compute the state of the system at any time t . We are interested

in the time of arrival of the particle at a given value of *one* of the phase space coordinates, say z^1 . (Since the coordinates z^i are arbitrary, z^1 can be any combination of dynamical variables.) To compute the time of arrival $T(Z)$ at $z^1=Z$, we solve the first equation of the system (11)

$$z^1(z_0^i, t) = Z \quad (12)$$

with respect to t , obtaining $t(Z; z_0^i)$. The time of arrival $T(Z)$ is then given by

$$T(Z) = t(Z; z_0^i). \quad (13)$$

Now, in the quantum theory, the integration constants z_0^i correspond to Heisenberg operators \hat{z}_0^i . Equation (10) is immediately generalized by "quantizing" (13) as

$$\hat{T}(Z) = t(Z; \hat{z}_0^i) \quad (14)$$

where again the operator $\hat{T}(Z)$ is given only up to the ordering.

Before concluding this section, we add an important comment on the seemingly puzzling case $T < 0$, namely, when the detection time is *earlier* than $t=0$. In the classical case, the particle can be detected without being disturbed, but not so in quantum mechanics; therefore one might wonder about the meaning of a detection at $T < 0$ for a particle that has a certain state at time $t=0$. The difficulty is avoided by choosing the definition of "state" appropriate to the present context. Consider the classical case first. x_0 and p_0 fix a unique solution of the equations of motion. This solution could be characterized by the values of x and p at any other time or by any two other constants of the motion. The definition of "time of arrival" that avoids the problem of detection before preparation is the following. We are interested in the arrival time of a particle that is moving according to the (unique) solution of the equations of motion characterized by the fact that at $t=0$ the particle is at x_0 and p_0 if not disturbed. Analogously, in quantum mechanics T is the time of arrival of a particle that at an earlier time t (arbitrarily in the past) was in the (Schrödinger) state $\psi(t)$ uniquely characterized by the fact that, if not disturbed, it would evolve to the state $\psi(0)$ at $t=0$. In Sec. V we shall describe a general way of dealing with this situation.

To summarize, in this section we have put forth two physical hypotheses.

(i) The probability for the time of arrival $\pi(T; X)$, an experimentally measurable quantity, can be computed by

$$\pi_\psi(T; X) = \langle \psi | P(T; X) | \psi \rangle, \quad (15)$$

where $P(T; X)$ are the projectors on the real component of the spectrum of a diagonalizable operator $\hat{T}(X)$. The states in the span of the nonreal component of the spectrum of $\hat{T}(X)$ are never detected at X .

(ii) The operator $\hat{T}(X)$ is given by a suitable choice of ordering from the equation

$$\hat{T}(X) = t(X; \hat{x}_0, \hat{p}_0) \quad (16)$$

or, in general, Eq. (14).

The first hypothesis is motivated by our confidence in the general validity of the superposition principle. The second hypothesis is motivated by our confidence in the correspondence principle. In the next section we investigate some of the implications of these hypotheses and we illustrate the construction and the use of the operator $\hat{T}(X)$ in a simple case. More interesting models, with nontrivial \hat{P} operator, will be presented in the companion paper [21].

III. TIME OF ARRIVAL OF A FREE PARTICLE

Consider a nonrelativistic free particle in one dimension. Dynamics is generated by the Hamiltonian $H := p^2/2m$. The solutions of the classical equations of motion are

$$x(t; x_0, p_0) = \frac{p_0}{m}t + x_0. \quad (17)$$

The inversion of these yields the time at which a particle that at $t=0$ has initial position and momentum x_0, p_0 is detected at the position X [as in Eq. (1)],

$$T(X) = t(X; x_0, p_0) = \frac{m(X - x_0)}{p_0}. \quad (18)$$

Notice that up to problems at the point $p_0=0$ (problems with which we shall deal extensively later) the particle is always detected. In particular, $T(X)$ is never complex. This greatly simplifies the setting, since we may disregard the complications arising from the existence of (finite) regions of phase space in which the particle is not detected.

Let us consider the usual quantum theory of a free particle. We work in the Heisenberg picture. We have Heisenberg (nonevolving) states ψ and time-dependent Heisenberg position and momentum operators $\hat{x}(t), \hat{p}(t)$, expressed in terms of \hat{x}_0 and \hat{p}_0 . Following the ideas of Sec. II, we explore the hypothesis that the quantum probability distribution $\pi(T; X)$ of the time of arrival at X of the particle can be computed in terms of an operator $\hat{T}(X)$ defined by a suitable ordering of the (formal) operator function

$$\hat{T}(X) = \frac{m(X - \hat{x}_0)}{\hat{p}_0}. \quad (19)$$

Notice that the Heisenberg position operator is

$$\hat{x}(t) := e^{i\hat{H}t/\hbar} \hat{x}_0 e^{-i\hat{H}t/\hbar} = \frac{\hat{p}_0}{m}t + \hat{x}_0, \quad (20)$$

to be compared with (17): Though rarely emphasized, classical and quantum dynamics are generically related by the equation

$$\hat{x}(t) = x(t; \hat{x}_0, \hat{p}_0), \quad (21)$$

where the right-hand side is an operator function corresponding to an ordering of the solution $x(t; x_0, p_0)$ of the classical equations of motion. In general, Eq. (21) is of scarce use for solving the quantum dynamics, since the associated ordering

problem is serious; but in simple cases such as the free particle, we see from (20) that the natural ordering suffices.

We explore here the possibility that in the case of a free particle a natural ordering suffices for the time-of-arrival operator as well. Namely, we study the choice of a symmetric ordering for the operator (19). We thus define, tentatively,

$$\hat{T}(X) := \frac{mX}{\hat{p}_0} - m \frac{1}{\sqrt{\hat{p}_0}} \hat{x}_0 \frac{1}{\sqrt{\hat{p}_0}}. \quad (22)$$

In order to study this operator, let us choose a concrete representation for the Hilbert space, namely, a basis. It is convenient to use the momentum basis that diagonalizes \hat{p}_0 , because this basis makes the definition of $1/\sqrt{\hat{p}_0}$ simpler. Thus we work in the Heisenberg-picture momentum basis. The states are represented as functions $\psi(k) \in L^2(\mathbb{R})$ and the elementary operators \hat{x}_0 and \hat{p}_0 are given by

$$\hat{x}_0 \circ \psi(k) = i \frac{d}{dk} \psi(k), \quad \hat{p}_0 \circ \psi(k) = \hbar k \psi(k). \quad (23)$$

In terms of the above operators, we have, for example, the Heisenberg position operator (20)

$$\hat{x}(t) = \frac{\hbar k}{m}t + i \frac{d}{dk}. \quad (24)$$

In this representation, the operator $\hat{T}(X)$ given in (22) is

$$\hat{T}(X) \circ \psi(k) = \left[-i \frac{m}{\hbar} \frac{1}{\sqrt{k}} \frac{d}{dk} \frac{1}{\sqrt{k}} + \frac{mX}{\hbar k} \right] \psi(k). \quad (25)$$

(We always take the principal value of the square root: $\sqrt{k} = i\sqrt{|k|}$ for $k < 0$.)

Notice that the one-parameter family of operators $\hat{T}(X)$ can be generated unitarily via translations

$$\hat{T}(X) = e^{-ikX} \hat{T}(0) e^{ikX}. \quad (26)$$

Therefore it is sufficient to study the operator $\hat{T}(0)$, namely, we do not lose generality by assuming the detector to be at the origin. We thus set $X=0$ from now on and drop the explicit X dependence

$$\hat{T} := \hat{T}(0) = -i \frac{m}{\hbar} \frac{1}{\sqrt{k}} \frac{d}{dk} \frac{1}{\sqrt{k}}. \quad (27)$$

We will be interested in the operators corresponding to other positions of the detector later on.

In the momentum representation, the eigenvalue equation for \hat{T}

$$\hat{T}|T\rangle = T|T\rangle \quad (28)$$

becomes

$$\hat{T} \circ g_T(k) \equiv \left[-i \frac{m}{\hbar} \frac{1}{\sqrt{k}} \frac{d}{dk} \frac{1}{\sqrt{k}} \right] g_T(k) = T g_T(k), \quad (29)$$

where we have introduced the notation

$$g_T(k) \equiv \langle k|T \rangle \quad (30)$$

for the momentum representation of the eigenstate of \hat{T} . The eigenvalue equation is easily solved [in each half of the real line ($k \neq 0$)] by

$$g_T(k) = [\alpha_+ \theta(k) + \alpha_- \theta(-k)] \sqrt{\frac{\hbar}{2\pi m}} \sqrt{k} \exp\left(\frac{i\hbar T k^2}{2m}\right), \quad (31)$$

where $\theta(k)$ is the characteristic function of the positive half of the real line and α_{\pm} are constants independent of k . In order to fix the relation between α_+ and α_- , let us act on $g_T(k)$ by \hat{T} . A simple calculation shows

$$\hat{T} \circ g_T(k) = T g_T(k) - i \sqrt{\frac{m}{2\pi\hbar}} \frac{\delta(k)}{\sqrt{|k|}} (\alpha_+ + i\alpha_-). \quad (32)$$

Thus, in order to satisfy the eigenvalue equation, it is necessary that¹

$$\alpha_- = i\alpha_+. \quad (33)$$

At this point, we encounter a difficulty. The operator we have constructed does *not* have a basis of orthogonal eigenstates. This pathology destroys the possibility of giving \hat{T} the interpretation we want. In the next subsection we show that the eigenstates of \hat{T} are not orthogonal and we discuss a way out from this difficulty.

A. Difficulties with \hat{T} and a regulation

A simple calculation shows that for any two eigenstates of \hat{T} with eigenvalues T and T'

$$\begin{aligned} \langle T|T' \rangle &\equiv \int_{-\infty}^{\infty} dk \overline{g_T(k)} g_{T'}(k) = \frac{\hbar}{2\pi m} \int_0^{\infty} dk^2 e^{(i\hbar/2m)k^2(T'-T)} \\ &= \delta(T-T') - \frac{i}{\pi} \frac{1}{(T-T')}. \end{aligned} \quad (34)$$

The eigenstates fail to be orthogonal. One can also see that \hat{T} as defined above has no self-adjoint extensions by noticing that its deficiency indices are unequal.

This difficulty stalled us for some time and various attempts to circumvent the problem failed. A way out was then suggested by Marolf [23]. The idea is to seek an operator that in the classical limit would not reproduce the time of arrival exactly, but would rather reproduce a quantity arbitrary close to the time of arrival. Namely, we want to approximate the time of arrival with a different quantity, free from pathologies. It is easy to trace the above pathology to the singular behavior of $1/k$ at $k=0$. Even classically, a state with $k=0$ is physically disturbing: Either the particle is never detected or the particle may stably sit over the detector. Therefore, we seek a small modification of (18) such that no divergences

occur at $p_0=0$. The modified time of arrival can perhaps be interpreted as the outcome of a measurement by an apparatus arbitrarily similar to a perfect detector, but which does not allow the particle to stand still.

Let us introduce an arbitrary small positive number ϵ . Consider a one-parameter family of real bounded continuous odd functions $f_{\epsilon}(k)$ that approach $1/k$ pointwise. More precisely, we require

$$\begin{aligned} f_{\epsilon}(k) &= \frac{1}{k} \quad \text{for } |k| > \epsilon \\ &\neq 0 \quad \text{for all } k \neq 0. \end{aligned} \quad (35)$$

For instance, we may choose

$$f_{\epsilon}(k) = \begin{cases} \frac{1}{k} & \text{for } |k| > \epsilon \\ \epsilon^{-2}k & \text{for } |k| < \epsilon. \end{cases} \quad (36)$$

Using this, we define the regulated time-of-arrival operator as

$$\hat{T}_{\epsilon} = -i \frac{m}{\hbar} \sqrt{f_{\epsilon}(k)} \frac{d}{dk} \sqrt{f_{\epsilon}(k)}, \quad (37)$$

to be compared with the unregulated operator (27). Notice that on any state with support on $|k| > \epsilon$ the operators \hat{T}_{ϵ} and \hat{T} are equal. Their action differs only on the component of a state with arbitrary low momentum. As we shall see, the probability distribution for the time of arrival $\pi(T)$ computed from \hat{T}_{ϵ} will turn out to be *independent of ϵ* for states with support away from $k=0$, reinforcing the credibility of the regulation procedure we are using.

Let us study the operator \hat{T}_{ϵ} . A key point is that \hat{T}_{ϵ} commutes with $\Theta(k) = \text{sgn}(k) = k/|k|$. Thus we can choose a basis of solutions of the eigenvalue equation for \hat{T}_{ϵ} formed by functions of k that have support on positive or negative k only. Now \hat{T}_{ϵ} is a linear differential operator and since $f_{\epsilon}(k) \rightarrow 0$ as $k \rightarrow 0$, there is no continuity condition on its eigenstates at $k=0$. These two related properties lead to a degeneracy in the spectrum. For each eigenvalue T , there are two eigenstates, which we choose as having support in the $k > 0$ and $k < 0$ regions, respectively, namely

$$\begin{aligned} \hat{T}_{\epsilon} |T, + \rangle_{\epsilon} &= T |T, + \rangle_{\epsilon}, \\ \hat{T}_{\epsilon} |T, - \rangle_{\epsilon} &= T |T, - \rangle_{\epsilon}, \end{aligned} \quad (38)$$

where

$$\begin{aligned} \langle k|T, + \rangle_{\epsilon} &= 0 \quad \text{for } k < 0, \\ \langle k|T, - \rangle_{\epsilon} &= 0 \quad \text{for } k > 0. \end{aligned} \quad (39)$$

We introduce the notation

$${}_{\epsilon} g_T^{\pm}(k) \equiv \langle k|T, \pm \rangle_{\epsilon} \quad (40)$$

for the momentum representation of the eigenstates of \hat{T}_{ϵ} . A simple calculation shows that these are

¹Another approach to obtaining this result is to integrate the eigenvalue equation in a small region around $k=0$. One then obtains the same continuity condition (33) on $g_T(k)$.

$${}_{\epsilon}g_T^{\pm}(k) = \theta(\pm k) \sqrt{\frac{\hbar}{2\pi m}} \frac{1}{\sqrt{f_{\epsilon}}} \exp\left(i \frac{\hbar T}{m} \int_{\pm\epsilon}^k dk' [f_{\epsilon}(k')]^{-1}\right). \quad (41)$$

Explicitly, for $|k| \geq \epsilon$ we have

$${}_{\epsilon}g_T^{\pm}(k) = \theta(\pm k) \sqrt{\frac{\hbar}{2\pi m}} \sqrt{k} \exp\left(\frac{i\hbar T}{2m}(k^2 - \epsilon^2)\right). \quad (42)$$

In order to derive various properties of these eigenstates, it is convenient to introduce new coordinates on the right- and left-hand sides of the real line as

$$Z^{\pm}(k) = \int_{\pm\epsilon}^k dk' \frac{1}{f_{\epsilon}(k')}. \quad (43)$$

In the region $|k| > \epsilon$, $Z^{\pm}(k) = (k^2 - \epsilon^2)/2$. In what follows we do not need the specific form of Z^{\pm} in the region $|k| < \epsilon$. Note first that on each side, the Jacobian of the coordinate transformation is nonvanishing for all $k \neq 0$ and thus the new coordinates are strictly monotonic. At the points $|k| = \epsilon$, $Z^{\pm}(k = \pm\epsilon) = 0$, respectively. Furthermore, since $|f_{\epsilon}(k)| \rightarrow 0$ rapidly enough as $|k| \rightarrow 0$, we see that both $Z^{\pm} \in (-\infty, \infty)$. In terms of these coordinates, the eigenstates are

$${}_{\epsilon}g_T^{\pm}(Z^{\pm}) = \sqrt{\frac{\hbar}{2\pi m}} \frac{1}{\sqrt{f_{\epsilon}(k(Z^{\pm}))}} e^{i(\hbar/m)TZ^{\pm}} \quad (44)$$

and the Hermitian inner product between two states ψ, ϕ is

$$\langle \psi | \phi \rangle = \sum_{\eta=+,-} \int_{-\infty}^{\infty} dZ^{\eta} f_{\epsilon}(k(Z^{\eta})) \overline{\psi(k(Z^{\eta}))} \phi(k(Z^{\eta})). \quad (45)$$

From (44) and (45), the orthogonality of the eigenstates is manifest:

$$\langle T, \eta | T', \eta' \rangle_{\epsilon} = \delta_{\eta, \eta'} \delta(T - T'). \quad (46)$$

In the new coordinates, completeness too is manifest. We can get the same result in the k representation with a little work:

$$\begin{aligned} & \sum_{\eta=+,-} \int_{-\infty}^{\infty} dT \langle k | T, \eta \rangle_{\epsilon} \langle T, \eta | k' \rangle \\ &= \frac{\hbar}{2\pi m} \sum_{\eta} \int_{-\infty}^{\infty} dT \frac{1}{\sqrt{f_{\epsilon}(k)}} \frac{1}{\sqrt{f_{\epsilon}(k')}} e^{i(\hbar/m)T(Z^{\eta}(k) - Z^{\eta}(k'))} \\ &= \sum_{\eta} \frac{\delta(Z^{\eta}(k) - Z^{\eta}(k'))}{\sqrt{f_{\epsilon}(k)} \sqrt{f_{\epsilon}(k')}} \\ &= \sum_{\eta} \frac{\delta(k - k') \theta(\eta k)}{|\partial Z^{\eta} / \partial k| \sqrt{f_{\epsilon}(k)} \sqrt{f_{\epsilon}(k')}} = \delta(k - k'). \end{aligned} \quad (47)$$

Since it has a complete orthogonal basis of (generalized) eigenstates with real eigenvalues, \hat{T}_{ϵ} is self-adjoint.

B. Time-of-arrival probability density

Following the general theory of Sec. II, if the particle is in the Heisenberg state $\psi(k)$, the probability density $\pi(T)$ of the time of arrival is the modulus square of the projection of the state on the T eigenstates of the time-of-arrival operator. Since these are doubly degenerate, we have in the present case

$$\pi(T) = |{}_{\epsilon}\langle T, + | \psi \rangle|^2 + |{}_{\epsilon}\langle T, - | \psi \rangle|^2. \quad (48)$$

If we assume that the support of $\psi(k)$ does not contain (an arbitrarily small finite region $|k| < \delta$ around) the origin, we can choose $\epsilon < \delta$ and, using the explicit form (42) of the eigenstates, we obtain the following expression for $\pi(T)$:

$$\begin{aligned} \pi(T) &= \frac{\hbar}{2\pi m} \left[\left| \int_0^{\infty} dk \sqrt{k} \exp\left(\frac{iT\hbar(k^2 - \epsilon^2)}{2m}\right) \psi(k) \right|^2 \right. \\ &\quad \left. + \left| \int_{-\infty}^0 dk \sqrt{k} \exp\left(\frac{iT\hbar(k^2 - \epsilon^2)}{2m}\right) \psi(k) \right|^2 \right]. \end{aligned} \quad (49)$$

Notice that the ϵ dependence gives only a phase that disappears when we take the absolute value squared, namely,

$$\begin{aligned} \pi(T) &= \frac{\hbar}{2\pi m} \left[\left| \int_0^{\infty} dk \sqrt{k} \exp\left(\frac{iT\hbar k^2}{2m}\right) \psi(k) \right|^2 \right. \\ &\quad \left. + \left| \int_{-\infty}^0 dk \sqrt{k} \exp\left(\frac{iT\hbar k^2}{2m}\right) \psi(k) \right|^2 \right]. \end{aligned} \quad (50)$$

We thus have the result that for the states that do not include an amplitude for zero velocity, the time-of-arrival probability distribution computed (with ϵ sufficiently small) with the regulated operator $\hat{T}_{\epsilon}(x)$ is independent from ϵ .

The two terms in (50) correspond to the left and right moving components of the state. Therefore, we have immediately that the probability $\pi^+(T)$ [and $\pi^-(T)$] that the particle is detected in $X=0$ while moving in the positive (or negative) direction is

$$\pi^{\pm}(T) = \frac{\hbar}{2\pi m} \left| \int_0^{\pm\infty} dk \sqrt{k} \exp\left(\frac{iT\hbar k^2}{2m}\right) \psi(k) \right|^2. \quad (51)$$

Finally, the result generalizes immediately to the case in which the detector is not placed in the origin but rather in an arbitrary position X . The eigenstates of the operator $\hat{T}_{\epsilon}(X)$

$$\hat{T}_{\epsilon}(X) |T, \pm; X\rangle_{\epsilon} = T |T, \pm; X\rangle_{\epsilon} \quad (52)$$

are obtained using the unitary translation operator

$$|T, \pm; X\rangle_{\epsilon} = e^{-i/\hbar \hat{p}X} |T, \pm\rangle_{\epsilon}, \quad (53)$$

yielding

$${}_{\epsilon}g_{T;X}^{\pm}(k) = \langle k | T, \pm; X \rangle_{\epsilon} = e^{-ikX} {}_{\epsilon}g_T^{\pm}(k). \quad (54)$$

The projectors considered in Sec. II are given by

$$P(T; X) = |T, +; X\rangle_{\epsilon} \langle T, +; X| + |T, -; X\rangle_{\epsilon} \langle T, -; X|. \quad (55)$$

The probability density of being detected at time T by a detector in X that detects particles traveling with positive (negative) velocity is

$$\pi^\pm(T;X) = \frac{\hbar}{2\pi m} \left| \int_0^{\pm\infty} dk \sqrt{k} \exp\left(\frac{iT\hbar k^2}{2m} - ikX\right) \psi(k) \right|^2. \quad (56)$$

Equation (56) represents our final result for the probability distribution of the time of arrival at X of a free quantum particle. This probability distribution should be compared with experiments. (As is typical in standard quantum mechanics, we have considered ideal measurements with perfect detector efficiencies. In a realistic situation, one should of course take into consideration a model of the detector as well. For example, Muga *et al.* [12] have modeled the absorption process by complex potentials, which is perhaps more realistic than the ideal measurements considered here.)

C. Time of arrival of a Gaussian wave packet

As an example of an application of the above result, we compute the probability distribution for the time of arrival of a Gaussian wave packet. Consider a Gaussian wave packet localized about a point, say, to the left of the origin at time $t=0$ and moving, say, to the right. In the standard Schrödinger-picture position representation, let this wave packet be given by the normalized solution of the Schrödinger equation

$$\psi(x,t) = \left(\frac{\delta^2}{2\pi}\right)^{1/4} \frac{e^{-k_0^2 \delta^2}}{\sqrt{\delta^2 + \frac{i\hbar t}{2m}}} \exp\left(\frac{[2\delta^2 k_0 + i(x-x_0)]^2}{4\delta^2 + 2i\hbar t/m}\right). \quad (57)$$

Expectation values are

$$\langle p(t) \rangle = \hbar k_0, \quad \langle \Delta p(t) \rangle = \frac{\hbar}{2\delta}, \quad (58)$$

$$\langle x(t) \rangle = x_0 + \hbar k_0 t/m, \quad \langle \Delta x(t) \rangle = \delta \sqrt{1 + \frac{t^2 \hbar^2}{4\delta^4 m^2}}. \quad (59)$$

If we choose $|x_0| \gg \delta$, $|k_0| \delta \gg 1$, $x_0 < 0$, and $k_0 > 0$, this state represents a particle well localized to the left of the origin and with a well-defined positive momentum at time $t=0$. In the Heisenberg-picture momentum representation (23), this state is given by

$$\psi(k) = \left(\frac{2\delta^2}{\pi}\right)^{1/4} \exp[-(k-k_0)^2 \delta^2 - ikx_0]. \quad (60)$$

The envelope of this wave function is a Gaussian of width $1/\delta$ centered at k_0 .

Using the theory developed, we can compute the projection of this state on the eigenstates of the time-of-arrival operator. We assume here that ϵ can be taken arbitrarily close to 0. (See [24] for the relevant integrals.) We obtain

$$\begin{aligned} \epsilon \langle T, +; X | \psi \rangle &= \sqrt{\frac{\hbar}{m}} \left(\frac{\delta^2}{2^5 \pi^3}\right)^{1/4} e^{-\delta^2 k_0^2} \Gamma(5/4) \left(\delta^2 + \frac{i\hbar T}{2m}\right)^{-5/4} \\ &\times \left\{ \left[[2\delta^2 k_0 + i(X-x_0)] \right] \right. \\ &\times \Phi\left(\frac{5}{4}, \frac{3}{2}, \frac{[2\delta^2 k_0 + i(X-x_0)]^2}{4\delta^2 + 2i\hbar T/m}\right) \\ &+ \sqrt{\pi \left(\delta^2 + \frac{i\hbar T}{2m}\right)} \exp\left(\frac{[2\delta^2 k_0 + i(X-x_0)]^2}{4\delta^2 + 2i\hbar T/m}\right) \\ &\left. \times \mathcal{L}\left(\frac{1}{4}, -\frac{1}{2}, -\frac{[2\delta^2 k_0 + i(X-x_0)]^2}{4\delta^2 + 2i\hbar T/m}\right) \right\}, \quad (61) \end{aligned}$$

$$\begin{aligned} \epsilon \langle T, -; X | \psi \rangle &= \sqrt{\frac{\hbar}{m}} \left(\frac{\delta^2}{2^5 \pi^3}\right)^{1/4} e^{-\delta^2 k_0^2} \Gamma(5/4) (-i) \left(\delta^2 + \frac{i\hbar T}{2m}\right)^{-5/4} \\ &\times \left\{ \left(-[2\delta^2 k_0 + i(X-x_0)] \right) \right. \\ &\times \Phi\left(\frac{5}{4}, \frac{3}{2}, \frac{[2\delta^2 k_0 + i(X-x_0)]^2}{4\delta^2 + 2i\hbar T/m}\right) \\ &+ \sqrt{\pi \left(\delta^2 + \frac{i\hbar T}{2m}\right)} \exp\left(\frac{[2\delta^2 k_0 + i(X-x_0)]^2}{4\delta^2 + 2i\hbar T/m}\right) \\ &\left. \times \mathcal{L}\left(\frac{1}{4}, -\frac{1}{2}, -\frac{[2\delta^2 k_0 + i(X-x_0)]^2}{4\delta^2 + 2i\hbar T/m}\right) \right\}, \quad (62) \end{aligned}$$

where we have ignored the components in $|k| < \epsilon$ since ϵ can be taken arbitrarily small. [$\Phi(n, m; z) = {}_1F_1(n; m; z)$ is the Kummer confluent hypergeometric function, $\mathcal{L}(n, a; z) = \mathcal{L}_n^a(z)$ is the n th generalized Laguerre polynomial, and $\Gamma(n)$ is the Euler gamma function.] The probability distribution is then given by (48). The expression above for the probability distribution of the time of arrival of a Gaussian wave packet is a bit heavy; in order to unravel its content, we have expanded it in powers of small quantities in the Appendix and we have plotted the total probability density $\pi(T;X)$ at various detector positions X in Fig. 1 [choosing a Gaussian state (57) with $x_0 = -5$, $k_0 = 20$, $\delta = 0.5$, and $\hbar = m = 1$]. To begin with, the term corresponding to negative velocities is exponentially small; indeed, it derives from the scalar product of a Gaussian wave packet concentrated around a positive k with a function having support on $k < 0$.

The total detection probability density at time T for the detector in position X is a function (more precisely, it is a density in T) on the T - X plane. This function is concentrated around the classical trajectory of the particle $X = x_0 + p_0 T/m$, with a (quantum) spread in T that increases with the spread of the wave packet, namely, with the distance of the detector from the initial state.

In the Appendix we compare our result with the probability density obtained indirectly using the Schrödinger probability density. We find good agreement within second order

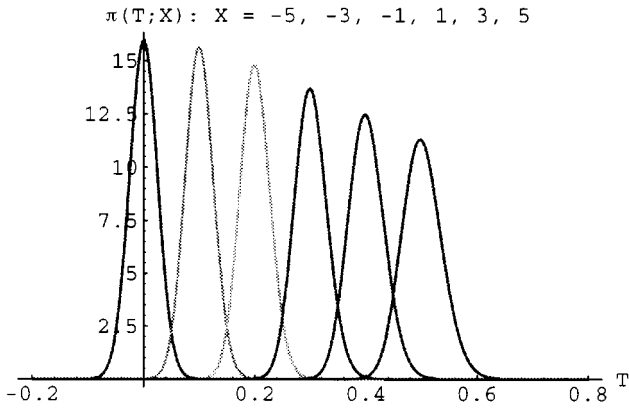


FIG. 1. Time-of-arrival probability densities $\pi(T;X)$, plotted at the detector positions $X = -5, -3, -1, 1, 3, 5$.

in the de Broglie wavelength of the particle and we discuss the order of the discrepancies. Thus our result is reasonable to leading order. Whether or not it is physically correct to all orders is a question that can perhaps be decided experimentally. A discrepancy with an experimental result may indicate an incorrect ordering of the time-of-arrival operator or a more general difficulty with our approach.

IV. DISCUSSION

A. Time representation

Anytime we have a self-adjoint operator in quantum mechanics, we may define a representation that diagonalizes this operator. Namely, we may choose the eigenbasis of the operator as our working basis on the theory's Hilbert space. Nothing prevents us from doing so with the operators $\hat{T}_\epsilon(X)$ as well. Let us therefore introduce a "time-of-arrival representation," or, for short, a "time representation." We fix an ϵ and define

$$\psi_\epsilon^\pm(T, X) := {}_\epsilon\langle T, \pm; X | \psi \rangle = \int dk {}_\epsilon\overline{g_{T, X}^\pm(k)} \psi(k). \quad (63)$$

Clearly, we can do quantum mechanics in the $\psi_\epsilon^\pm(T, X)$ representation as well as we do quantum mechanics in the position, momentum, or energy representations. Since the eigenstates ${}_\epsilon g_{T, X}^\pm(k)$ have support on positive k , we must interpret $\psi_\epsilon^+(T, X)$ as the amplitude for the particle to be detected by a detector placed at X in an infinitesimal neighborhood of T coming from the left and $\psi_\epsilon^-(T, X)$ as the amplitude to be detected at X in a neighborhood of T coming from the right.

What is the relation between the $\psi_\epsilon^\pm(T, X)$ amplitude and the conventional Schrödinger wave function $\psi(x, t)$? Notice that the first is defined by $\psi_\epsilon^\pm(T, X) = {}_\epsilon\langle T, \pm; X | \psi \rangle$, where $|T, \pm; X\rangle_\epsilon$ is an eigenstate of $T_\epsilon(X)$ with eigenvalue T , while the second can be viewed as defined by $\psi(x, t) = \langle x; t | \psi \rangle$, where $|x; t\rangle$ is the eigenstate of the operator $\hat{x}(t)$ with eigenvalue x . At first sight, the two seem to be related to the same quantity (up to the ϵ and the distinction between the two directions of the velocity): They both refer to probabilities of being detected at some space point and at some time. However, this naive observation is very misleading. The quantity

$|\psi(x, t)|^2 dx$ is the probability *in space* that the particle happens to be between the positions x and $x + dx$ at time t , as opposed to *being elsewhere* at time t , whereas the quantity $|\psi^\pm(T, X)|^2 dT$ is the probability *in time* that the particle happens to arrive between times T and $T + dT$ at the position X , as opposed to *reaching X at some other time*. The two bases $|T, \pm; X\rangle_\epsilon$ and $|x; t\rangle$ are two well-defined (generalized one-parameter families of) bases in the Hilbert space, but they are distinct.

In particular, the two bases $|T, \pm; X\rangle_\epsilon$ and $|x; t\rangle$ have distinct dimensions because $|\psi(x, t)|^2 dx$ and $|\psi^\pm(T, X)|^2 dT$ must both be dimensionless probabilities. Thus the transformation factor between $|T, \pm; X\rangle_\epsilon$ and $|x; t\rangle$ has the dimension of the square root of a velocity. Indeed, let us write the two (generalized) states explicitly in the Heisenberg momentum representation. Restricting ourselves to $k > 0$ and taking ϵ to zero for simplicity, we have from (42)

$$\langle k | T, +; X \rangle_\epsilon = \sqrt{\frac{\hbar}{2\pi}} \sqrt{\frac{k}{m}} \exp\left(\frac{i\hbar T}{2m} k^2 - ikX\right), \quad (64)$$

while, as it is well known,

$$\langle k | x, t \rangle = \sqrt{\frac{1}{2\pi}} \exp\left(\frac{i\hbar t}{2m} k^2 - ikx\right). \quad (65)$$

Therefore, taking $x = X$ and $t = T$ we have

$$\langle k | T, +; X \rangle_\epsilon = \sqrt{\frac{\hbar k}{m}} \langle k | x, t \rangle. \quad (66)$$

A physical understanding of the curious $\sqrt{\hbar k/m}$ factor that characterizes the eigenstates of the time-of-arrival operator can be obtained as follows. Consider a well-localized wave packet traveling with velocity $v = \hbar k_0/m$. We have approximately $|\psi^\pm(T, X)|^2 \approx v |\psi(x, t)|^2$. Now consider the $x-t$ plane. The wave function $\psi(x, t)$ is significantly different from zero on a band around the classical trajectory. The classical trajectory is a straight line with a slope given by the velocity v . The ratio between a vertical and an horizontal section of the band is therefore precisely v . Thus, in order to have both total probabilities normalized to 1 when integrating along a $t = \text{const}$ or a $x = \text{const}$ line, the probability density in space and the probability density in time must be related by a factor v . This hypothesis (66) has also been made by Kijowski [7], who used it to obtain a probability distribution similar to (50).

In Fig. 2 we have plotted the usual Schrödinger probability density in x , $\pi(x; t) = |\psi(x, t)|^2$, at various times, for the same state used for Fig. 1. The time-of-arrival probability densities are $v = \hbar k_0/m$ times as large as the Schrödinger probability densities for values of $X = x$ and $T = t$ near the classical trajectory.

B. Time-energy uncertainty relation

Position-momentum uncertainty relations are of wide use in quantum mechanics and can be cleanly derived from the formalism on very general grounds. The commutation rela-

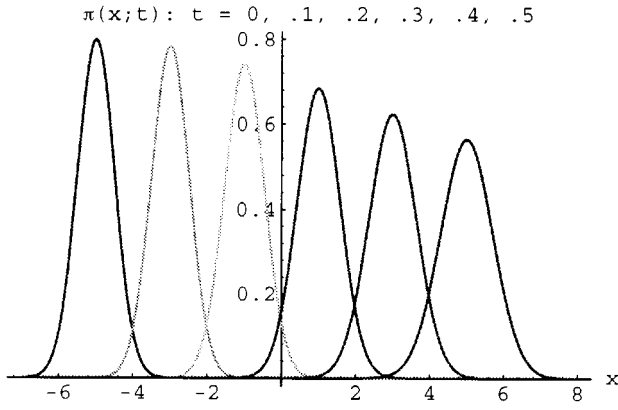


FIG. 2. Schrödinger probability densities $\pi(x;t)$ for position, plotted at times $t=0,0.1,0.2,0.3,0.4,0.5$.

tions $[\hat{x},\hat{p}]=i\hbar\hat{1}$ imply $\Delta x\Delta p\geq\hbar/2$. Time-energy uncertainty relations are of wide use as well (e.g., between the width of a spectral line and the lifetime), but their general derivation is notoriously tricky. If one tries to reproduce the position-momentum derivation for the time-energy case by assuming the existence of an operator \hat{T} such that

$$[\hat{T},\hat{H}]=-i\hbar\hat{1} \quad (67)$$

from which

$$\Delta T\Delta E\geq\hbar/2 \quad (68)$$

would follow, then one would clash against a well-known nonexistence theorem for \hat{T} . The theorem states that the commutation relations $[\hat{A},\hat{B}]=i\hbar\hat{1}$ between two self-adjoint operators \hat{A} and \hat{B} implies that the spectrum of both operators is the real line. However, the spectrum of the Hamiltonian is bounded from below in all reasonable systems. Therefore a time operator \hat{T} satisfying (67) does not exist on the usual Hilbert space. See, e.g. [4], for a rigorous proof.

This theorem might have been the reason for which a time-of-arrival operator has virtually never been considered in quantum mechanics. In fact, it is often stated that time cannot be an operator in quantum mechanics, with the above theorem as a proof. Here we show how one can rigorously derive time-energy uncertainty relations for a quantum particle and how the existence of \hat{T}_ϵ circumvents the theorem.

The commutation relations between \hat{T}_ϵ and the Hamiltonian are easy to compute. In the momentum representation we have

$$[\hat{T}_\epsilon,\hat{H}]=-i\hbar(\hat{1}-h_\epsilon(k)), \quad (69)$$

where

$$h_\epsilon(k)=1-kf_\epsilon(k). \quad (70)$$

The function $h_\epsilon(k)$ vanishes for $|k|>\epsilon$, and in the small interval where it has support, it is bounded [by 1, if we choose $f_\epsilon(k)$ as in (36), which we do for simplicity]. For the particle in the state $\psi(k)$ the uncertainty relations are then

$$(\Delta T_\epsilon)^2(\Delta E)^2\geq\frac{\hbar^2}{4}[1-\langle\psi|h_\epsilon(k)|\psi\rangle]^2. \quad (71)$$

Now

$$\langle\psi|h_\epsilon(k)|\psi\rangle=\int_{-\epsilon}^{\epsilon}dk|\psi|^2h_\epsilon(k)\leq\int_{-\epsilon}^{\epsilon}dk|\psi|^2. \quad (72)$$

For all states $\psi(k)$ as in Sec. III B, whose support lies entirely outside an arbitrarily small region $|k|<\delta$ around the origin, and for all $\epsilon<\delta$, we have $\int_{-\epsilon}^{\epsilon}dk|\psi|^2=0$. Thus, for all such states with support away from the origin, we have

$$\Delta T_\epsilon\Delta E\geq\frac{\hbar}{2} \quad (73)$$

for all sufficiently small ϵ . Note that we have defined the standard mathematical uncertainties as defined for the position and momentum operators and thus (73) is subject to the standard interpretation of the product of the uncertainties of two noncommuting operators.

With varying degrees of rigor, it is manifest that the various ‘‘time as energy derivative’’ approaches cited in the Introduction yield the desired uncertainty relation (68).

C. Definition of state in quantum mechanics

Finally, let us return to the problem we briefly discussed at the end of Sec. II, which is the interpretation of the time of arrival T when $T<0$, namely, when the detection time is earlier than the time $t=0$ at which the initial state is given. We have suggested that in this case the correct interpretation of T is the following. T is the detection time for a state that arbitrarily in the past was in a state that would have evolved to the $t=0$ initial state if undisturbed. A cleaner way of dealing with the general situation is to make use of a fully time-independent notion of ‘‘state’’ and a fully time-independent version of phase space and quantum state space. This can be done as follows. Consider first classical mechanics. Let us denote a single solution of the equation of motion as a ‘‘physical history’’ of the system. (A physical history should not be confused with the histories considered in sum-over-histories theories: A physical history here is a history satisfying the equations of motion.) Let Γ_s be the space of these physical histories. A point in Γ_s represents an entire evolution of the system. Γ_s can be coordinatized by the $2n$ integration constants z^i . We ask for the time of arrival at x of a system following one of the motions in Γ_s . This time of arrival is given by (13). The key to the matter is that there is no need to choose a time in order to specify a physical history.

In quantum mechanics we can define the Hilbert space \mathcal{H}_S of the solutions of the Schrödinger equation. A vector in \mathcal{H}_S represents an entire (quantum) motion of the system, without reference to any particular time. The conventional Heisenberg operators are defined on \mathcal{H}_S . The operator (14) is properly defined on \mathcal{H}_S . We may choose to represent the vectors in \mathcal{H}_S by means of the value that the Schrödinger state would take (if undisturbed) at $t=0$; therefore it makes sense to define \hat{T} as a function of the operators \hat{x}_0 and \hat{p}_0 that

are defined on the states at $t=0$.

In this regard, it is interesting to notice that the original definition of the ‘‘Heisenberg picture Hilbert space’’ given by Dirac in the first edition of his book [22] is the definition of \mathcal{H}_S given above. It is only later that the Heisenberg picture Hilbert space came to be mostly identified with the state space at a fixed time (both interpretations of \mathcal{H}_S can be found in the literature). A crucial advantage of using the definition of \mathcal{H}_S given here is that this definition can be extended to systems without Newtonian time at all [19,20,25]. We will exploit this point of view in [21].

V. CONCLUSION: $X \leftrightarrow T$ EQUIVALENCE IN QUANTUM THEORY

Let us summarize our results. We have considered the problem of computing the time of arrival T of a quantum particle at a position X . Relying on the general validity of the superposition principle, we have argued that the probability distribution for T can be obtained by means of an operator \hat{T} . This operator is, in general, not self-adjoint. However, it admits an orthogonal basis of eigenstates. The eigenstates with real eigenvalues correspond to (generalized) states for which the detection time is sharp. The eigenstates with complex eigenvalues correspond to states that are never detected.

The time-of-arrival operator is partially characterized by its classical limit, which fixes its dependence on the position and momenta operators, up to ordering. We have considered the simple case of a nonrelativistic free particle, using a tentative natural ordering. A regulation procedure allows us to find a self-adjoint time-of-arrival operator (there is no nondetection in this case) and we have studied the probabilities the operator yields. These are given explicitly in Eq. (56) for an arbitrary initial state and in Eqs. (61), (62), and (A8) for a Gaussian wave packet. Remarkably, in the theoretical literature there is no agreement about these probabilities [1,13,15]. We suggest that an experimental investigation of the problem could be of interest. It could indicate which one is the correct approach to calculating these probabilities and it could shed some light on the conceptual issues raised by this seemingly very simple aspect of quantum mechanics.

We showed that the regulated time-of-arrival operator can be used to derive time-energy uncertainty relations, circumventing a well-known nonexistence theorem, and that it yields a well-defined ‘‘time representation’’ for the system.

For more general systems, the classical limit is not likely to be sufficient for constructing the operator. In a forthcoming paper [21] we investigate a general technique for constructing the time-of-arrival operator in general cases. We will study in detail the particle in an exponential potential, where the \hat{P} operator is nontrivial. We will also investigate parametrized systems and theories without a Newtonian time, arguing that the ideas presented here may be relevant for the interpretations of quantum gravity.

We close with a general comment on the significance of the result obtained. In classical mechanics there is a hidden equivalence between the independent time variable t and the dependent dynamical variables (the position x in the present case). This equivalence is made manifest by expressing the theory in parametrized form, namely, by representing the

evolution in terms of the functions $t(\tau)$ and $x(\tau)$ of an arbitrary parameter τ . In more elegant and general mathematical terms, there are formulations of mechanics, e.g., the presymplectic formulation, in which the distinction between dependent and independent variables is inessential; see, for instance, Arnold’s classic text [26]. A parametrized representation is commonplace in special relativity (where T is called x^0) since it allows manifest Lorentz covariance.

It is commonly stated that this equivalence between independent (time) and dependent variables is lost in quantum mechanics. The arguments in support of this claim are common in the quantum gravity literature and take various forms. For instance, it is said that the wave function must be normalized by integrating in x and cannot be normalized by integrating in t . Or it is said that probabilities are always probabilities of different outcomes happening at the same time, never at the same position. It is our impression that these claims are misleading. The mistake is to assume that the $x \leftrightarrow t$ equivalence has to be realized as an equivalence in the arguments of the Schrödinger wave function $\psi(x,t)$.

The conventional formulation of quantum mechanics in terms of the Schrödinger wave function $\psi(x,t)$ has *already* broken the $x \leftrightarrow t$ equivalence. Indeed, it is a formulation tailored to answer the following (experimental) question: What is the probability of the particle being *here now*, as opposed to that of being *elsewhere now*? The corresponding representation diagonalizes the Heisenberg operators $\hat{x}(t)$. It is the experimental question considered and the related choice of basis that break the $x \leftrightarrow t$ equivalence. Quantum mechanics allows us to consider the following question as well: What is the probability of the particle getting *here now* as opposed to getting *here some other time*? In order to answer this question, one is led naturally to the $\psi^\pm(T,X)$ representation in which the roles of position and time are to a large extent interchanged. In particular, the wave function is normalized in time and probabilities of events at the same position are considered.

To avoid misunderstandings, let us make clear that we certainly do not claim that space and time have the same nature, nor that their role in the quantum mechanics of a particle is exactly the same. What we suggest is that the common arguments that x and t can be treated on equal footing in classical mechanics, but not in quantum mechanics, might lose force under closer scrutiny. Contrary to the above arguments, our analysis reveals an underlying hidden equivalence between ‘‘dependent’’ and ‘‘independent’’ variables in the quantum theory of a free particle.

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APPENDIX: IS THE COMPUTED PROBABILITY DENSITY REASONABLE?

Let us now investigate whether the result we have obtained for the Newtonian free particle is physically reasonable. The simplest check is to compute the expectation value of the time of arrival of a wave packet with initial position x_0 and momentum p_0 . The result should agree with the expected classical time of arrival $T = t(X; x_0, p_0)$. Since the operator was constructed via a factor ordering and regulation of the classical solution, the expectation values obviously satisfy Ehrenfest's theorem. A more accurate check that goes beyond the semiclassical approximation is to compare the probability distribution we have obtained with the one we can *estimate* by indirect but intuitive methods. In Sec. 1 of the Appendix we first compute an approximation to the probability amplitude (61) and (62), and the resulting probability density, for a Gaussian state. This also gives us some intuition for the behavior of the distribution. Then in Sec. 2 of the Appendix we compute the Schrödinger current through the detector position and compare this with the approximation we obtained in Sec. 1.

1. Gaussian state

Consider the Gaussian wave packet of Sec. III C. In the momentum representation, this state is given by

$$\psi(k) = \left(\frac{2\delta^2}{\pi}\right)^{1/4} \exp[-(k-k_0)^2\delta^2 - ikx_0]. \quad (\text{A1})$$

The envelope of this wave function is a Gaussian of width $1/\delta$ at positive momentum k_0 . In order to simplify the calculations, we slightly modify this state by assuming it to be zero for $k < \epsilon$. Clearly, since this modification is far out on the tail of the Gaussian, the error we make is very small [more precisely, one can show that it vanishes as $\exp(-1/k_0\delta)$]. We thus replace (A1) by

$$\psi(k) = \theta(k-\epsilon) \left(\frac{2\delta^2}{\pi}\right)^{1/4} \exp[-(k-k_0)^2\delta^2 - ikx_0], \quad (\text{A2})$$

where $\theta(x)$ is the characteristic function of the positive line. Substituting this expression into (63), we find that the amplitude for the particle to be detected between T and $T+dT$ at the position x is

$$\begin{aligned} \psi^+(T, X) &= \sqrt{\frac{\hbar}{2\pi m}} \left(\frac{2\delta^2}{\pi}\right)^{1/4} \int_{\epsilon}^{\infty} dk \sqrt{k} \exp\left(- (k-k_0)^2\delta^2 \right. \\ &\quad \left. - i\frac{\hbar T}{2m}k^2 + ik(X-x_0)\right) \\ &= \sqrt{\frac{\hbar}{2\pi m}} \left(\frac{2\delta^2}{\pi}\right)^{1/4} \left[\int_{-\infty}^{\infty} dk \sqrt{k} \exp\left(- (k-k_0)^2\delta^2 \right. \right. \\ &\quad \left. \left. - i\frac{\hbar T}{2m}k^2 + ik(X-x_0)\right) - R_1 \right], \end{aligned} \quad (\text{A3})$$

where R_1 is again exponentially small $\sim \exp(-1/k_0\delta)$ and we have disregarded an ϵ -dependent phase, which is not going to affect the probability distribution.

Now we can expand \sqrt{k} about the peak of the Gaussian as

$$\sqrt{k} = \sqrt{k_0} \left[1 + \frac{k-k_0}{2k_0} + O\left(\frac{k-k_0}{k_0}\right)^2 \right]. \quad (\text{A4})$$

Using this expansion, we can compute the Gaussian integral in (A3)

$$\begin{aligned} &\int_{-\infty}^{\infty} dk \sqrt{k} \exp\left(- (k-k_0)^2\delta^2 - i\frac{\hbar T}{2m}k^2 + ik(X-x_0)\right) \\ &= e^{ik_0(X-x_0) - i(\hbar T/2m)k_0^2} \left[\sqrt{\frac{k_0^2\pi}{\delta^2 + i\frac{\hbar T}{2m}}} \exp \right. \\ &\quad \times \left[-\frac{\left(X-x_0 - \frac{\hbar T k_0}{m}\right)^2}{4\left(\delta^2 + i\frac{\hbar T}{2m}\right)} \right] \\ &\quad \left. \times \left(1 + i\frac{X-x_0 - \frac{\hbar T k_0}{m}}{4k_0\left(\delta^2 + i\frac{\hbar T}{2m}\right)} + R_2 \right) \right], \end{aligned} \quad (\text{A5})$$

where

$$|R_2| \leq \frac{1}{8k_0^{3/2}} \int_{-\infty}^{\infty} dk k^2 e^{-k^2\delta^2} = \sqrt{k_0\pi} \frac{1}{16k_0^2\delta^2} \frac{1}{\delta} \quad (\text{A6})$$

is of second order in $1/\delta k_0$. Thus, to first order in the (small) quantity $1/\delta k_0$, we have

$$\begin{aligned} \psi^+(T; x) &= \sqrt{\frac{\hbar k_0}{m}} \left(\frac{\delta^2}{\pi}\right)^{1/4} e^{ik_0(X-x_0) - i(\hbar T/2m)k_0^2} \\ &\quad \exp\left(-\frac{\left(X-x_0 - \frac{\hbar T k_0}{m}\right)^2}{4\left(\delta^2 + i\frac{\hbar T}{2m}\right)}\right) \\ &\quad \times \frac{1}{4\left(\delta^2 + i\frac{\hbar T}{2m}\right)^{1/2}} \\ &\quad \times \left(1 + i\frac{X-x_0 - \frac{\hbar T k_0}{m}}{4k_0\left(\delta^2 + i\frac{\hbar T}{2m}\right)} \right). \end{aligned} \quad (\text{A7})$$

To this approximation, the probability density $\psi^+(T; x) = |\psi^+(T; x)|^2$ is then

$$\begin{aligned} \pi^+(T;x) &= \frac{\hbar}{m\sqrt{2\pi}} \\ &\times \frac{\left(k_0\delta^2 + \frac{(X-x_0)T\hbar}{4m\delta^2} + \frac{k_0(X-x_0-k_0\hbar T/m)^2}{16k_0^2\delta^2}\right)}{\left(\delta^2 + \frac{T^2\hbar^2}{4m^2\delta^2}\right)^{3/2}} \\ &\times \exp\left(-\frac{(X-x_0-k_0T\hbar/m)^2}{2\left(\delta^2 + \frac{T^2\hbar^2}{4m^2\delta^2}\right)}\right). \end{aligned} \quad (\text{A8})$$

Due to the Gaussian factor, the probability distribution is centered on the classical time of arrival $T = m[(X-x_0)/\hbar k_0]$, with width δ , and vanishes exponentially outside such a region.

Notice also that due to the Gaussian factor the third term in the numerator of the amplitude of the Gaussian is of order $(1/\delta k_0)^2$ in all the region where the probability density is not exponentially small, so we may rewrite the probability density as

$$\begin{aligned} \pi(T;x) &= \frac{\hbar}{m\sqrt{2\pi}} \frac{\left(k_0\delta^2 + \frac{(X-x_0)T\hbar}{4m\delta^2}\right)}{\left(\delta^2 + \frac{T^2\hbar^2}{4m^2\delta^2}\right)^{3/2}} \\ &\times \exp\left(-\frac{(X-x_0-k_0T\hbar/m)^2}{2\left(\delta^2 + \frac{T^2\hbar^2}{4m^2\delta^2}\right)}\right) \end{aligned} \quad (\text{A9})$$

as an approximation correct to order $1/\delta k_0$.

2. Current

In ordinary quantum mechanics, one can define a current whose time and space components are

$$j^0(x,t) = \rho(x,t) = \overline{\psi(x,t)}\psi(x,t), \quad (\text{A10})$$

$$j^i(x,t) = \frac{\hbar}{2mi}(\overline{\psi}\partial_i\psi - \psi\partial_i\overline{\psi})(x,t), \quad (\text{A11})$$

where $\psi(x,t)$ is the quantum state in the conventional Schrödinger-picture position representation. Since the state satisfies Schrödinger's equation, this current is conserved

$$\partial_0 j^0 + \partial_i j^i = 0. \quad (\text{A12})$$

Consider, for a particle in one dimension, the space-time region (a half strip) defined by $x \leq X, t_1 \leq t \leq t_2$ and integrate the divergence free current over this volume. Dropping the boundary terms at $x = -\infty$, we find that the outgoing (i.e., rightward) flux of this current through the timelike boundary at $x = X$ in the time interval (t_1, t_2) is

$$\begin{aligned} \int_{t_1}^{t_2} dt j_x &\equiv \frac{\hbar}{2mi} \int_{t_1}^{t_2} dt (\overline{\psi}\partial_x\psi - \psi\partial_x\overline{\psi})|_{x=X} \\ &= \int_{-\infty}^X dx \rho \Big|_{t_1} - \int_{-\infty}^X dx \rho \Big|_{t_2}. \end{aligned} \quad (\text{A13})$$

The last line in the above equation is the probability that the particle is found in the left-hand side region ($X \leq x$) at time t_1 minus the probability that the particle is found in the left-hand side region at a later time t_2 . It is tempting to identify the flux between the times T and $T + dT$ (through the time-like surface at $x = X$) of the current density

$$j_X(T)dT = \frac{\hbar}{m} \text{Im}[\overline{\psi(x,T)}\partial_x\psi(x,T)]_{x=X}dT \quad (\text{A14})$$

as the probability density that a detector placed at $x = X$ will detect the particle. Note that the Schrödinger current density has the correct dimensions of a density in time, namely, $[T]^{-1}$. The problem, of course, is that the current represents the *net* flux of probability across $x = X$ and thus corresponds to the difference between the probability of crossing to the right and the probability of crossing to the left. Namely, we may expect that

$$j_X(T) \sim \pi^+(T;X) - \pi^-(T;X), \quad (\text{A15})$$

within some approximation. In fact, the current $j_X(T)$ is *not* positive definite and we believe that this is related to the difficulties in the approaches of [17,15]. Equivalently, we may try to interpret the current of a *pure right moving state* as the probability density that the particle crosses $x = X$. We are reassured in doing this by the fact that in this case the current is positive definite and its integral over all times gives one. This conclusion is based on the assumption that the particle cannot ‘‘zigzag’’ across the $X = x$ line, an assumption that might be valid only if we look at sufficiently large times.

So, when ψ is a pure right moving state, the rightward flux density is positive² and integrates to 1 over all time. [To see this, take the limits $t_1 \rightarrow -\infty$ and $t_2 \rightarrow \infty$ in (A13).] It is therefore at least consistent to interpret this as an estimate of the probability density. Does this estimate yield the correct semiclassical limit? The expectation values of the usual position and momentum operators satisfy Ehrenfest's theorem since the probability densities are associated with the decomposition of a state onto the spectrum of some self-adjoint operator. The same is true of the probability densities we have obtained from the time-of-arrival operator. The above current density is *not* obtained via a spectral projection, however, and is *not* associated with a ‘‘time operator.’’ How do the expectation values of the time of arrival behave with respect to this estimated probability density and do they correspond to the classical limit in some way? We next proceed to analyze this issue.

The expectation value of the time of arrival of the particle at the position X is naturally defined only when the state has support only on the $k \geq 0$ region and is then given by

²See (A14) and recall that this is a free particle.

$$\begin{aligned}
\langle T \rangle_X &= \int_{-\infty}^{\infty} dT T j_X(T) = \frac{\hbar}{2\pi m} \text{Re} \left\{ \int_{-\infty}^{\infty} dT \int_0^{\infty} dk \int_0^{\infty} dk' T \exp\left(\frac{i\hbar T k^2}{2m} - ikX\right) \overline{\psi(k)} \exp\left(\frac{-i\hbar T k'^2}{2m} + ik'X\right) \psi(k') \right\} \\
&= \frac{\hbar}{2\pi m} \text{Re} \left\{ \int_{-\infty}^{\infty} dT \int_0^{\infty} dk \int_0^{\infty} dk' \frac{m}{\hbar} \frac{1}{k} \left[\left(\frac{1}{i} \frac{\partial}{\partial k} + X \right) \exp\left(\frac{i\hbar T k^2}{2m} - ikX\right) \right] \overline{\psi(k)} \exp\left(\frac{-i\hbar T k'^2}{2m} + ik'X\right) \psi(k') \right\} \\
&= \frac{m}{\hbar} \text{Re} \left\{ \int_0^{\infty} dk \overline{\psi(k)} \left(\frac{X}{k} - \frac{i}{k} \frac{\partial}{\partial k} \right) \psi(k) \right\} \equiv mX \left\langle \frac{1}{\hat{p}_0} \right\rangle - \frac{m}{2} \left\langle \frac{1}{\hat{p}_0} \hat{x}_0 + \hat{x}_0 \frac{1}{\hat{p}_0} \right\rangle, \tag{A16}
\end{aligned}$$

where we have dropped a surface term at $k=0$ since we assume that $\psi(k)_{k \rightarrow 0} \rightarrow k^{2+}$. The final expression in the above equation is the symmetric factor ordering of the classical expression $T = m(X - x_0)/p_0$, thus we do recover the correct semiclassical limit.

For the localized right moving wave packet previously considered, the current is easily computed, giving

$$\begin{aligned}
j_X(T) &= \frac{\hbar}{2\pi m} \text{Re} \left[\int_0^{\infty} dk \exp\left(\frac{iT\hbar k^2}{2m} - ikX\right) \overline{\psi(k)} \right. \\
&\quad \left. \times \int_0^{\infty} dk' k' \exp\left(\frac{-iT\hbar k'^2}{2m} + ik'X\right) \psi(k') \right]. \tag{A17}
\end{aligned}$$

The integral can be explicitly done, yielding

$$\begin{aligned}
j_X(T) &= \frac{\hbar}{m\sqrt{2\pi}} \frac{\left(k_0 \delta^2 + \frac{(X-x_0)T\hbar}{4m\delta^2} \right)}{\left(\delta^2 + \frac{T^2\hbar^2}{4m^2\delta^2} \right)^{3/2}} \\
&\quad \times \exp\left(-\frac{(X-x_0 - k_0 T\hbar/m)^2}{2\left(\delta^2 + \frac{T^2\hbar^2}{4m^2\delta^2} \right)} \right), \tag{A18}
\end{aligned}$$

which is precisely the approximate form for the probability we computed from the time-of-arrival operator [see (A9)]. Thus the probability distributions computed with the time-of-arrival operator and by means of the current agree for a right moving localized wave packet to order $1/k_0\delta$, which is one order beyond the classical limit.

For a general state, roughly localized in momentum state around a momentum k_0 , we can compare the current (A14) with the difference between the probability of being detected moving towards the right minus the probability of being detected moving left. Namely, we can estimate

$$d_X(T) = j_X(T) - [\pi_X^+(T) - \pi_X^-(T)]. \tag{A19}$$

Explicitly, using (A14) and (50), we have

$$\begin{aligned}
d_X(T) &= \frac{\hbar}{m} \text{Im} \left[\overline{\psi(X;T)} \partial_X \psi(X;T) \right]_{x=X} \\
&\quad - \frac{\hbar}{2\pi m} \left| \int_0^{+\infty} dk \sqrt{k} \exp\left(\frac{-iT\hbar k^2}{2m} + ikX\right) \psi(k) \right|^2 \\
&\quad + \frac{\hbar}{2\pi m} \left| \int_{-\infty}^0 dk \sqrt{k} \exp\left(\frac{-iT\hbar k^2}{2m} + ikX\right) \psi(k) \right|^2. \tag{A20}
\end{aligned}$$

We now set $X=0$ for simplicity, without loss of generality. A little algebra gives

$$\begin{aligned}
d_0(T) &= \frac{\hbar}{2\pi m} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' (\sqrt{k} - \sqrt{k'})^2 \\
&\quad \times \exp\left(\frac{-iT\hbar(k^2 - k'^2)}{2m}\right) \psi(k) \overline{\psi(k')}. \tag{A21}
\end{aligned}$$

Now suppose that our measurement has a precision δT . Then the difference of the probability densities, averaged around T is

$$\Delta P(T, \delta T) = \frac{1}{2\delta T} \int_{T-\delta T}^{T+\delta T} dT' d_0(T'). \tag{A22}$$

The integral in dT' can be done easily, which yields

$$\begin{aligned}
\Delta P(T, \delta T) &= \frac{\hbar}{2\pi m} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' (\sqrt{k} - \sqrt{k'})^2 \\
&\quad \times \exp\left(\frac{-iT\hbar(k^2 - k'^2)}{2m}\right) \psi(k) \overline{\psi(k')} \\
&\quad \times \left\{ \frac{\sin\left[\frac{\delta T\hbar}{2m}(k^2 - k'^2)\right]}{\frac{\delta T\hbar}{2m}(k^2 - k'^2)} \right\}. \tag{A23}
\end{aligned}$$

For large δT , compared to the ‘‘de Broglie time’’ of the particle $2m/k^2\hbar$ (where k is the highest momentum in the support of the wave function), the factor in curly brackets in the integrand has non-negligible support only for $k \sim k'$ and the integral is then suppressed by the $(\sqrt{k} - \sqrt{k'})^2$ factor. This indicates that the two ways of computing the probability for the time-of-arrival approach each other when our resolution time is larger than the particle’s de Broglie time.

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