Swift-Hohenberg equation for optical parametric oscillators

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Pattern formation near threshold in single longitudinal mode and large aspect ratio optical parametric oscillators (OPOs) operating near resonance are shown to be described by a complex Swift-Hohenberg equation. Such an equation is capable of capturing the main wave-vector selection rules for both degenerate and nondegenerate OPOs. [S1050-2947(96)07609-3]

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Complex pattern formation in nonlinear optics has been the subject of extensive investigations in recent years. In particular, an increasing interest has been addressed toward large aspect ratio systems, where pattern formation is independent of transverse boundaries and is thus described by universal order parameter equations that provide a connection between pattern formation in optics and in other physical fields, particularly in hydrodynamics. Coupling of diffraction with an optical nonlinearity in two transverse spatial dimensions has revealed the appearance of transverse patterns in a wide range of passive optical devices [1–3]. Active optical systems are no exception and transverse pattern formation has been investigated both in lasers [4–7] and in optical parametric oscillators [8–10] in a cavity configuration with flat end mirrors of infinite transverse extension and uniform pumping. In the laser case, it was shown that the full Maxwell-Bloch equations admit exact solutions in the form of tilted traveling waves (TW's), and that the nature of the selected pattern depends on the sign of the detuning between the cavity and the atomic frequencies [4]. In particular, a single tilted wave is able to dominate and to suppress all others [4]; rhombic patterns arising from a four-wave interaction may be also stable states of the laser equation [7]. Pattern formation in optical parametric oscillators (OPO's) has been also investigated, and previous studies have been mainly restricted to the degenerate case, where signal and idler fields are indistinguishable [8–10]. As for laser systems, it was shown that the OPO dynamics above threshold for signal and idler generation strongly depends on the sign of detuning between signal and cavity frequencies [8]. However, contrary to laser systems, in degenerate OPO's threshold lowering in the negative detuning side corresponds to an off-axis emission of the signal field, which manifests itself as a standing-wave (roll) pattern. The tendency to yield roll patterns instead of a single tilted wave is due to the process of parametric down conversion, which leads to the simultaneous emission of two symmetric TW's with conversion of the transverse photon momentum [8]. However, as pointed out by one of the present authors, when the degeneracy constraint is removed the interference between these TW's, which is the basic reason for roll pattern formation, disappears and the full OPO equations have an exact continuum family of TW solutions for both signal and idler fields, which are preferred to roll states found in the degenerate case [11]. As for laser systems, we have recently shown that off-axis emission in a nondegenerate OPO manifests itself as a single TW [11] or as a rhombic pattern [12] for both signal and idler fields. This analogy suggests to us that the spatiotemporal dynamics in lasers and OPO's may be described by an order parameter equation of the same kind and that the degenerate OPO configuration may be captured by the same model equation, provided that signal and idler fields are assumed to be indistinguishable. Recent theoretical studies on the laser equations have shown that a global description of the laser dynamics for small detunings is provided by a complex Swift-Hohenberg equation (SHE) [5,6]. This equation seems to be very appealing because it explicitly contains the wave-vector selection properties of the full laser equations for both signs of detuning. The possibility of reducing the dynamics of the full OPO equations to that of simpler universal equations has been recently discussed [10,13], but these analyses were restricted to the degenerate case or to singly or doubly resonant OPO configurations. In particular, it was shown that the doubly resonant OPO configuration in the degenerate case can be conveniently described by a real SHE [10]. However, the degeneracy constraint profoundly affects pattern forming properties [11], and therefore we envisage that such equations do not represent a general model for the study of transverse effects in OPO's.

In this paper we show that pattern formation in large aspect ratio, single longitudinal mode OPO's operating near resonance can be described by a complex SHE analogous to that recently proposed for lasers systems. Such an equation is here derived as a solvability condition in a multiple scale expansion near threshold for signal and idler generation assuming small detunings for both fields from cavity resonances. In particular, we show that for a degenerate OPO this equation reduces to a real SHE, which has been proposed as a model of stationary convection in hydrodynamics [14].

The starting point of our analysis is provided by a mean-field model for three optical fields (signal, idler, and pump waves), which simultaneously resonate in an optical cavity

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with flat end mirrors containing a nonlinear $\chi^{(2)}$ medium and driven by a homogeneous, plane-wave pump input field [8,9]. Assuming perfect phase matching, the single longitudinal mode approximation and the paraxial approximation, the dynamic equations for the three fields in the cavity are [9]

$$\partial_t B = \gamma_0[-(1+i\Delta_0)B + ia_0\nabla^2 B] - \gamma_0 A_1 A_2, \quad (1a)$$

$$\partial_t A_1 = \gamma_1[-(1+i\Delta_1)A_1 + ia_1\nabla^2 A_1 + \mu A_2^*] + \gamma_1 A_2^* B, \quad (1b)$$

$$\partial_t A_2 = \gamma_2[-(1+i\Delta_2)A_2 + ia_2\nabla^2 A_2 + \mu A_1^*] + \gamma_2 A_1^* B, \quad (1c)$$

where A_1 and A_2 are the normalized slowly varying envelopes for signal and idler fields, respectively, $B=A_0-\mu$ is the difference of the normalized pump field in the cavity A_0 from its stationary value $\mu=E(1-i\Delta_0)/(1+\Delta_0^2)$ below threshold for idler and signal generation, and E is the normalized amplitude of the plane-wave pump input field. Δ_0, Δ_1, and Δ_2 are three detuning parameters for pump, signal, and idler fields, respectively, defined by $\Delta_0=(\omega_0-\omega_L)/\gamma_0$, $\Delta_1=(\omega_1-\nu_1\omega_L)/\gamma_1$, $\Delta_2=(\omega_2-\nu_2\omega_L)/\gamma_2$, where γ_0, γ_1, and γ_2 are the cavity decay rates of the three fields, ω_0, ω_1, and ω_2 are the three longitudinal cavity frequencies closest to the pump frequency ω_L, the signal frequency $\nu_1\omega_L$, and the idler frequency $\nu_2\omega_L$, respectively, with $\nu_1+\nu_2=1$. Finally, in Eqs. (1) the diffraction parameters a_0, a_1, a_2 for the three fields are given by $a_0=c/2k_z\gamma_0$, $a_1=c/2\nu_1 k_z\gamma_1$, $a_2=c/2\nu_2 k_z\gamma_2$, where c is the velocity of light and k_z longitudinal wave vector of the field at frequency ω_L. Note that, with a suitable choice of the phase of the input pump field E, the parametric gain μ in Eqs. (1) may be assumed to be real and positive. Strictly speaking, Eqs. (1) hold if signal and idler fields are distinguishable, i.e., in case of non-degenerate OPO's. But, we might use Eqs. (1) again if signal and idler fields were degenerate with the condition $A_1=A_2$. In this case, since $\gamma_1=\gamma_2$, $a_1=a_2$, $\nu_1=\nu_2$, $\Delta_1=\Delta_2$, Eqs. (1b) and (1c) are the same and there is total match between signal and idler fields. The stationary solution $A_1=A_2=B=0$ Eqs. (1) becomes unstable beyond threshold for signal and idler generation as the parametric gain μ is increased. Stability of the trivial zero solution can be reached by the standard linear stability method; substitution of the perturbation $\binom{A_1}{A_2^*}\propto\exp(\sigma t+i\mathbf{k}\cdot\mathbf{r})$ into Eqs. (1b) and (1c) gives the following algebraic equation for the eigenvalues σ in the linearized system

$$\sigma^2+[\gamma_1+\gamma_2+i(\gamma_1\tilde{\Delta}_1-\gamma_2\tilde{\Delta}_2)]\sigma + \gamma_1\gamma_2[1+\tilde{\Delta}_1\tilde{\Delta}_2+i(\tilde{\Delta}_1-\tilde{\Delta}_2)]-\gamma_1\gamma_2\mu^2=0, \quad (2)$$

where $\tilde{\Delta}_{1,2}\equiv\Delta_{1,2}+a_{1,2}k^2$ and $k=|\mathbf{k}|$. For a given transverse wave number k of the perturbation, the real part of one of the two eigenvalues, say $\mathrm{Re}(\sigma_1)$, crosses zero from negative as μ is increased. Thus $\mathrm{Re}(\sigma_1)=0$ gives the neutral stability curve $\mu=\mu_0(k)$ and $\omega=\mathrm{Im}(\sigma_1)$ defines the frequency of the Hopf bifurcation; they are given by [9,11]

$$\mu_0(k)=\sqrt{1+\tilde{\Delta}^2}, \quad (3a)$$

$$\omega(k)=\gamma_1\gamma_2[\tilde{\Delta}_2-\tilde{\Delta}_1]/(\gamma_1+\gamma_2), \quad (3b)$$

where $\tilde{\Delta}=[\gamma_1\Delta_1+\gamma_2\Delta_2+k^2(\gamma_1 a_1+\gamma_2 a_2)]/(\gamma_1+\gamma_2)$. Threshold for oscillation is obtained by minimizing $\mu_0(k)$ with respect to k and, as previously shown in Refs. [9], [11], the nature of the instability at threshold strongly depends on the sign of the averaged detuning parameter $\Delta=(\gamma_1\Delta_1+\gamma_2\Delta_2)/(\gamma_1+\gamma_2)$. In particular, if $\Delta>0$ the threshold condition corresponds to $\mu_{\mathrm{th}}=(1+\Delta^2)^{1/2}$ and the consequent bifurcation leads to a uniform phase-wave state for both signal and idler fields. Conversely, for $\Delta<0$ a lowering of threshold to $\mu_{\mathrm{th}}=1$ is predicted and the instability for signal and idler generation gives rise to a couple of symmetric traveling waves with transverse wave vector $k_c=\sqrt{-\Delta(\gamma_1+\gamma_2)/(a_1\gamma_1+a_2\gamma_2)}$. In the degenerate case, these waves correspond to the same optical field and interference that give roll patterns in the transverse plane. Motivated by recent studies on pattern formation in lasers [5,6], we argue that such asymmetric behavior should be manifested as a result of multiple-scale analysis with the assumption of small Δ. For this purpose, we perform a weakly nonlinear analysis of the full OPO equations (1), based on a multiple-scale expansion around threshold for signal and idler generation: this is done by assuming small cavity detunings $\Delta_1=\epsilon\Omega_1$, $\Delta_2=\epsilon\Omega_2$, where ϵ is the kernel parameter for power expansion. Therefore, we look for OPO equations with solutions in the form

$$\mathbf{V}=\epsilon^1\mathbf{V}^{(1)}+\epsilon^2\mathbf{V}^{(2)}+\epsilon^3\mathbf{V}^{(3)}+\cdots, \quad (4)$$

where $\mathbf{V}=(B,A_1,A_2^*)^T$ contains the field variables, which slowly depend on time and space. The right scalings in the perturbation analysis may be deduced by considering the dependence of the growth rate $\mathrm{Re}(\sigma_1)$ on the wave number k and parametric gain μ near to the critical point ($k_c=0$, $\mu^2=1$). From Eq. (2) with $\Delta_1=\Delta_2=0$, we find that at leading order in k and (μ^2-1) the growth rate $\mathrm{Re}(\sigma_1)$ is given by $\mathrm{Re}(\sigma_1)=(\mu^2-1)\gamma_1\gamma_2/(\gamma_1+\gamma_2)-\beta k^4+\cdots$, where $\beta=\gamma_1\gamma_2[(\gamma_1 a_1+\gamma_2 a_2)^2 + a_1 a_2(\gamma_1+\gamma_2)^2]/(\gamma_1+\gamma_2)^3$. Thus a band of wave-vector k of width $(\mu^2-1)^{1/4}$ centered around $k_c=0$ is experiencing growth. To take into account these modes, we introduce the slow spatial variables $X=(\mu^2-1)^{1/4}x$, $Y=(\mu^2-1)^{1/4}y$. In order to have the terms in Δ_1 and Δ_2 of the same order of magnitude as the diffraction terms in Eqs. (1b) and (1c), we assume $\mu^2-1=\epsilon^2$: so doing, the slow spatial variables become $X=\epsilon^{1/2}x$, $Y=\epsilon^{1/2}y$. Finally, two slow time variables $T_1=\epsilon t$ and $T_2=\epsilon^2 t$ are needed.

In order to derive an amplitude equation as a solvability condition in the amplitude scale expansion, it is worthwhile to take into account the complex conjugate equation of (1c) and to write the OPO equations in the compact notation

$$\partial_t\mathbf{V}=L\mathbf{V}+\mathbf{N}, \quad (5)$$

where L is the linear operator and $\mathbf{N}$ the nonlinear operator of the system, i.e.,

$$L=\begin{pmatrix} -\gamma_0(1+i\Delta_0)+ia_0\gamma_0\nabla^2 & -0 & 0 \\ 0 & -\gamma_1(1+i\Delta_1)+ia_1\gamma_1\nabla^2 & \gamma_1\mu \\ 0 & \gamma_2\mu & -\gamma_2(1-i\Delta_2)-ia_2\gamma_2\nabla^2 \end{pmatrix}, \quad \mathbf{N}=\begin{pmatrix} -\gamma_0 A_1 A_2 \\ \gamma_1 A_2^* B \\ \gamma_2 A_1 B^* \end{pmatrix}.$$

Substituting expansion (4) into Eq. (5), using $\mu^2=1+\epsilon^2$ and the derivative rules partials $\partial_t=\epsilon\partial_{T_1}+\epsilon^2\partial_{T_2}$, $\nabla^2=\epsilon\nabla_x^2$, where ∇_x^2 is the transverse Laplacian with respect to the slow spatial coordinates, a hierarchy of equations for successive corrections of $\mathbf{V}$ is obtained. $O(\epsilon^1)$: $L_0\mathbf{V}^{(1)}=0$; $O(\epsilon^2)$: $L_0\mathbf{V}^{(2)}=\partial_{T_1}\mathbf{V}^{(1)}-L_1\mathbf{V}^{(1)}-\mathbf{N}_2\equiv\mathbf{S}_2$; $O(\epsilon^3)$: $L_0\mathbf{V}^{(3)}=\partial_{T_2}\mathbf{V}^{(1)}+\partial_{T_1}\mathbf{V}^{(2)}-L_2\mathbf{V}^{(1)}-L_1\mathbf{V}^{(2)}-\mathbf{N}_3\equiv\mathbf{S}_3$, where

$$L_0=\begin{pmatrix} -\gamma_0(1+i\Delta_0) & 0 & 0 \\ 0 & -\gamma_1 & \gamma_1 \\ 0 & \gamma_2 & -\gamma_2 \end{pmatrix},$$

$$L_1=\begin{pmatrix} ia_0\gamma_0\nabla_x^2 & 0 & 0 \\ 0 & -\gamma_1 i\Omega_1+ia_1\gamma_1\nabla_x^2 & 0 \\ 0 & 0 & \gamma_2 i\Omega_2-ia_2\gamma_2\nabla_x^2 \end{pmatrix},$$

$$L_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma_1/2 \\ 0 & \gamma_2/2 & 0 \end{pmatrix}$$

and $\mathbf{N}_2$, $\mathbf{N}_3$ are the nonlinear terms of the system at second and third order, respectively. The solution of equation at leading order is given by $\mathbf{V}^{(1)}=(0,1,1)^T\psi$, where ψ is a complex function that depends on the slow space-time variables. For a degenerate OPO, the further condition $A_1^{(k)}=A_2^{(k)}$ at any order $O(\epsilon^k)$ requires that ψ be a real variable. In order to solve ϵ^2 and ϵ^3 equations, solvability conditions must be satisfied, which determine the behavior of the function ψ at the time scales T_1 and T_2, respectively. Such conditions may be written as $\mathbf{S}_k\cdot\mathbf{u}=0$ $(k=2,3)$, where $\mathbf{u}=(0,1/\gamma_1,1/\gamma_2)^T$ is the eigenvector of the adjoint of L_0.

The solvability condition at order $O(\epsilon^2)$ requires

$$(1/\gamma_1+1/\gamma_2)\partial_{T_1}\psi=i[(\Omega_2-\Omega_1)-(a_2-a_1)\nabla_x^2]\psi \quad (6)$$

and the solutions at this order can be chosen as follows: $B^{(2)}=-|\psi|^2/(1+i\Delta_0)$, $A_1^{(2)}=-(1/2\gamma_1)\partial_{T_1}\psi-(i/2)(\Omega_1-a_1\nabla_x^2)\psi$, $A_2^{*(2)}=-(1/2\gamma_2)\partial_{T_1}\psi+(i/2)(\Omega_2-a_2\nabla_x^2)\psi$. At order $O(\epsilon^3)$, the solvability condition yields

$$\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\partial_{T_2}\psi=\psi+\frac{1}{2}\left[\left(\frac{1}{\gamma_1}\partial_{T_1}+\hat{P}_1\right)^2+\left(-\frac{1}{\gamma_2}\partial_{T_1}+\hat{P}_2\right)^2\right]\psi-\frac{2|\psi|^2\psi}{1+\Delta_0^2}, \quad (7)$$

where $\hat{P}_{1,2}\equiv i(\Omega_{1,2}-a_{1,2}\nabla_x^2)$. Finally, if we use the identities $[(1/\gamma_1)\partial_{T_1}+\hat{P}_1]\psi=[-(1/\gamma_2)\partial_{T_1}+\hat{P}_2]\psi=(\gamma_1\hat{P}_1+\gamma_2\hat{P}_2)/(\gamma_1+\gamma_2)\psi$ we get the behavior of ψ at the time scale T_2

$$\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\partial_{T_2}\psi=\psi-\left[\frac{\gamma_1\Omega_1+\gamma_2\Omega_2-(a_1\gamma_1+a_2\gamma_2)\nabla_x^2}{\gamma_1+\gamma_2}\right]^2\times\psi-2|\psi|^2\psi/(1+\Delta_0^2). \quad (8)$$

The final equation for ψ is obtained by collecting all the terms and looks like

$$\partial\psi/\partial t=\epsilon\,(\partial\psi/\partial T_1)+\epsilon^2(\partial\psi/\partial T_2). \quad (9)$$

We can now substitute Eqs. (6) and (8) into Eq. (9), reintroduce the original variables $x=X/\sqrt{\epsilon}$ and $y=Y/\sqrt{\epsilon}$, $\Delta_1=\epsilon\Omega_1$ and $\Delta_2=\epsilon\Omega_2$, $\mu^2-1=\epsilon^2$, and redefine $\epsilon\psi$ as ψ: in this way, we get the following complex SHE:

$$\tau\partial_t\psi=(\mu^2-1)\psi+i[(\Delta_2-\Delta_1)-(a_2-a_1)\nabla^2]\psi-(\Delta-a\nabla^2)^2\psi-2|\psi|^2\psi/(1+\Delta_0^2), \quad (10)$$

where $\tau\equiv(\gamma_1+\gamma_2)/\gamma_1\gamma_2$, $a\equiv(a_1\gamma_1+a_2\gamma_2)/(\gamma_1+\gamma_2)$, $\Delta\equiv(\Delta_1\gamma_1+\Delta_2\gamma_2)/(\gamma_1+\gamma_2)$. This equation is very similar to the laser SHE recently derived for class A and C lasers [5]. As will be shown below, it is capable of describing the main features of the OPO dynamics for both degenerate and nondegenerate configurations. Moreover, it contains the asymmetric wave-vector selection properties around the zero cavity detuning $\Delta=0$.

In order to discuss the main properties of Eq. (10), we distinguish the degenerate configuration from the nondegenerate one.

(1) Degenerate OPO. If this is the case, the field variable ψ must be real, and Eq. (10) reduces to a real SHE with a real order parameter:

$$\tau\partial_t\psi=(\mu^2-1)\psi-(\Delta-a\nabla^2)^2\psi-2\psi^3/(1+\Delta_0^2). \quad (11)$$

Equation (11) in its present form has been extensively studied as a model of stationary convection in hydrodynamics [14] and has been previously derived by Staliunas in the OPO context by an adiabatic elimination approach [10]. The spatial structure of the unstable mode that arises when the null solution loses its stability (as the driving parameter is increased) is a plane-wave state for $\Delta>0$ and a periodic state (roll pattern) for $\Delta<0$. In the latter case, the zero solution bifurcates at $\mu_{\text{th}}=1$. In the former case instead, the instability arises at $\mu_{\text{th}}=\sqrt{1+\Delta^2}$: this is just the threshold value predicted by full OPO equations [8]. Beyond threshold, the wave-vector selection properties can be derived from a variational approach and the roll pattern may undergo phase (Eckhaus and zigzag) instabilities [14].

(2) Nondegenerate OPO. In this case, the order parameter ψ may be complex and the SHE (10) has both zero solution $\psi=0$ and TW's, given by

$$\psi=\sqrt{C}\,\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)], \quad (12)$$

where

$$C=(1+\Delta_0^2)/2\,[\mu^2-1-(\Delta+ak^2)^2], \quad (13a)$$

$$\omega=[\Delta_2-\Delta_1+(a_2-a_1)k^2]/\tau, \quad (13b)$$

and $k=|\mathbf{k}|$ is the transverse wave number of the TW. It is straightforward to show that the neutral stability curve of the zero solution for the SHE (10) coincides with that given by Eq. (3a). Therefore, for $\Delta>0$ the uniform solution $k=0$ has the lowest threshold and is excited first, whereas for $\Delta<0$ a traveling wave with transverse wave vector $k_c=(-\Delta/a)^{1/2}$ is the most unstable mode.

Because of the rotational symmetry in the transverse plane, in the latter case there is a continuous set of critical modes near threshold. As far as linear dynamics is concerned, all these modes are equally amplified, but usually only a few of them survive and saturate due to nonlinear interactions [3,15]. The competition among these modes near threshold can be captured by deriving with standard nonlinear analysis a set of nonlinear equations for the amplitudes of competing modes [15]. In particular in our context, it is interesting to investigate the interaction among four TW's with wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, and $\mathbf{k}_4$ where $|\mathbf{k}_i|=k_c$ $(i=1,2,3,4)$ and $\mathbf{k}_3=-\mathbf{k}_1$ and $\mathbf{k}_4=-\mathbf{k}_2$. Neglecting finite bandwidth effects, from the SHE (10) we find the following equations for TW amplitudes z_1, z_2, z_3, and z_4:

$$\frac{1}{2}\,\tau\partial_t z_n=\frac{1}{2}\,(\mu^2-1)z_n-\frac{z_n}{1+\Delta_0^2}\left(2\sum_{m=1}^{4}|z_m|^2-|z_n|^2\right)-\frac{2}{1+\Delta_0^2}\,z_{n+2}^{*}z_{n+1}z_{n+3}, \tag{14}$$

where n,m runs cyclically over 1,2,3,4,. . . . Equation (14) is similar to those derived in Ref. [7] for laser systems and it has two stable stationary solutions: single TW's $(z_1,z_2,z_3,z_4)=(h,0,0,0)$ and alternating rolls $(z_1,z_2,z_3,z_4)=(h,ih,h,ih)$ [7,16]. In particular, a standing-wave pattern $(z_1,z_2,z_3,z_4)=(h,0,h,0)$ is known to be unstable [16]. Thus the near field of signal and idler waves emitted by a nondegenerate OPO does not show stripe patterns as in the degenerate case. To better understand the OPO emission in the nondegenerate case, we point out that, at leading order in the expansion (4), the TW solution (12) of the OPO SHE corresponds to the following pump, signal, and idler fields: $B=-C/(1+\Delta_0^2)$, $A_1=\sqrt{C}\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$, $A_2=\sqrt{C}\exp(-i\mathbf{k}\cdot\mathbf{x}+i\omega t)$. Physically, this solution describes off-axis emission for both signal and idler fields in the form of tilted waves, and a plane-wave pump field. Signal and idler fields are emitted along two opposite directions of the transverse plane and at frequencies equally detuned (in opposite sides) from their reference frequencies ω_1 and ω_2 by the amount ω. In this way, energy and photon momentum are maintained in the parametric conversion process.

Stability of the TW's (12) above threshold can be studied in the framework of the reduced SHE by using a standard linear stability method. As for the laser SHE [6], it turns out that the stability domain of TW's in the plane (k,μ) is delimited by the phase (Eckhaus and zigzag) instability boundaries, which can be analytically computed by deriving a phase-diffusion equation for TW's [4,6,15]. The derivation of the phase-diffusion equation for the OPO SHE is similar to that given in Ref. [6] for the laser SHE, and, therefore, we omit details of calculations. Assuming that the wave vector of TW is oriented along the x axis, the phase-diffusion equation reads

$$\tau(\partial_t\vartheta-\omega)=D_{\mathrm{Eck}}\vartheta_{xx}+D_{zig}\vartheta_{yy}, \tag{15}$$

where the diffusion coefficients are given by

$$D_{\mathrm{Eck}}=-(4a^2k^2/C)(1+\Delta_0^2)(\Delta+ak^2)^2+2a(\Delta+3ak^2), \tag{16a}$$

$$D_{\mathrm{zig}}=2a(\Delta+ak^2). \tag{16b}$$

The stability requirements for the TW with wave number $\mathbf{k}=(k,0)$ are thus $D_{\mathrm{Eck}}(k)>0$ and $D_{\mathrm{zig}}(k)>0$. The passage of D_{Eck} through zero states the Eckhaus stability boundary and the zero crossing of D_{zig} gives the zigzag stability boundary. Although a detailed comparison of the stability results obtained from the SHE (10) with those derived in Ref. [11] for the full OPO equations goes beyond the purpose of this paper, we note that the Eckhaus stability boundary as computed from Eq. (15) coincides with that derived in Ref. [11], only near threshold and for small detunings Δ_1,Δ_2. Moreover, the phase equation (15) for the reduced SHE is not able to capture the dependence of Eckhaus stability boundary on the pump detuning Δ_0, as predicted in Ref. [11]. This is due to the fact that the SHE (10) has been derived as a solvability condition in a perturbation expansion at third order and, as a consequence, its TW solutions coincide with the exact TW states of the full OPO equations only for small detunings Δ_1,Δ_2 and near threshold.

In conclusion, the present derivation of a complex SHE provides a connection between spatiotemporal behavior of large aspect ratio OPO's and other pattern forming systems in nonlinear optics and hydrodynamics. Such an equation also allows a unified treatment of the different wave-vector selection properties in degenerate and nondegenerate OPO's.

[1] W. J. Firth *et al.*, Phys. Rev. A **46**, R3609 (1992); M. Tlidi *et al.*, *ibid.* **48**, 4605 (1993).
[2] W. J. Firth, J. Mod. Opt. **37**, 151 (1990); G. D' Alessandro and J. W. Firth, Phys. Rev. Lett. **66**, 2597 (1991); E. Pampaloni *et al.*, *ibid.* **74**, 258 (1995).
[3] J. B. Geddes *et al.*, Phys. Rev. A **50**, 3471 (1994).
[4] P. K. Jakobsen *et al.*, Phys. Rev. A **49**, 4189 (1994); J. Lega *et al.*, *ibid.* **49**, 4201 (1994).
[5] J. Lega *et al.*, Phys. Rev. Lett. **73**, 2978 (1994).
[6] J. Lega *et al.*, Physica D **83**, 478 (1995).
[7] K. Staliunas and C. O. Weiss, Physica D **81**, 79 (1995); Q. Feng *et al.*, Phys. Rev. A **50**, R3601 (1994).
[8] G.-L. Oppo *et al.*, Phys. Rev. A **49**, 2028 (1994).
[9] G.-L. Oppo *et al.*, J. Mod. Opt. **41**, 1151 (1994).
[10] K. Staliunas, J. Mod. Opt. **42**, 1261 (1995).
[11] S. Longhi, Phys. Rev. A **53**, 4488 (1996).
[12] S. Longhi, J. Mod. Opt. **43**, 1569 (1996).
[13] S. Longhi, J. Mod. Opt. **43**, 1089 (1996).
[14] Y. Pomeau and P. Manneville, Phys. (Paris) Lett. **23**, 609 (1979).
[15] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[16] M. Silber *et al.*, Physica D **61**, 260 (1992).