PHYSICAL REVIEW A VOLUME 54, NUMBER 5 NOVEMBER 1996

Nonlinear coherent states

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(Received 23 October 1995; revised manuscript received 17 June 1996)

We consider a class of nonlinear coherent states, which are right-hand eigenstates of the product of the boson annihilation operator and a nonlinear function of the number operator. Such states may appear as stationary states of the center-of-mass motion of a trapped and bichromatically laser-driven ion far from the Lamb-Dicke regime. Besides coherence properties, they exhibit nonclassical features such as amplitude squeezing and self-splitting, which is accompanied by pronounced quantum interference effects. [S1050-2947(96)08811-7]

PACS number(s): 42.50.Dv, 03.65.Db, 32.80.Pj, 42.50.Vk

Coherent states, defined as the right-hand eigenstates $|\alpha\rangle$ of the non-Hermitian boson annihilation operator $\hat{a}$, $\hat{a}|\alpha\rangle=\alpha|\alpha\rangle$ [1], play an important role in quantum optics. The development of lasers made it possible to prepare light, fields which are very close to such states. Their behavior shows a close correspondence to that of a classical wave. The mean amplitude of the (electric or magnetic) field strength depends linearly on the eigenvalue α and the corresponding variance is insensitive to the amplitude, it is just given by the variance in the vacuum state.

In the context of the quantum nature of light, coherent states appear to be of less interest. Over the last two decades there have been several experimental demonstrations of nonclassical effects, such as photon antibunching [2], sub-Poissonian statistics [3,4], and squeezing [5,6]. Moreover, there exist interesting quantum effects, and related quantum states that are hard to prepare and to detect, namely superposition states exhibiting quantum interference effects [7]. Such states display the striking consequences of the superposition principle of quantum mechanics. Transient electronic states of this type have recently been prepared via pulsed excitation of atomic Rydberg wave packets [8]. Moreover, superpositions of coherent states can be prepared in the motion of a trapped ion [9,10]. With respect to the nonclassical effects, the coherent states turn out to define the limit between the classical and nonclassical behavior, so that they do not display any of these interesting features.

In the present contribution we will consider nonlinear coherent states (NCS's) of the harmonic oscillator and their realization in the motion of a trapped atom. Such states maintain typical features of the coherent states, such as the localization of their phase-space distributions around their (nonvanishing) mean complex amplitude. On the other hand, the NCS can display strongly nonclassical properties, such as amplitude squeezing and quantum interferences. The latter occur due to self-splitting of these states into pure substates which eventually gives rise to interferences of their own structures. Particular representatives of these nonlinear coherent states emerge as stationary states of the motion of an appropriately laser-driven trapped ion, which is in the resolved sideband limit and far from the Lamb-Dicke regime. These experimental requirements could be fulfilled using presently available trapped ion techniques [10–12]. For studying the properties of quantum states in detail, appropriate techniques are needed which allow their reconstruction from measured data [13–15]. To record the full quantum statistical information on the NCS including the quantum interference fringes, a highly efficient method for determining the quantum mechanical state of a trapped ion has been proposed [16], where the information on the motional state is transformed by appropriate bichromatic laser excitation of a weak electronic transition onto the electronic-state dynamics, which is subsequently probed by measuring the fluorescence of a strong electronic transition.

Let us consider the right-hand eigenstates $|\chi;f\rangle$ of the non-Hermitian operator $\hat{f}(\hat{n})\hat{a}$,

$$\hat{f}(\hat{n})\hat{a}|\chi;f\rangle=\chi|\chi;f\rangle, \tag{1}$$

where $\hat{a}$ is the annihilation operator of the harmonic oscillator, $\hat{f}(\hat{n})$ is an operator-valued function of the number operator $\hat{n}=\hat{a}^{\dagger}\hat{a}$, and χ is a complex eigenvalue. The notation $|\chi;f\rangle$ indicates the dependence of these states on the function $\hat{f}(\hat{n})$. The ordinary coherent states are recovered for the special choice of $\hat{f}(\hat{n})=\hat{1}$. In general, the features of those states are expected to alter significantly in dependence on their excitation, thus we call them nonlinear coherent states.

An explicit expression for the state $|\chi;f\rangle$ in the number representation,

$$|\chi;f\rangle=\sum_{n}|n\rangle\langle n|\chi;f\rangle, \tag{2}$$

is readily given by

$$\langle n|\chi;f\rangle=\mathcal{N}\frac{g(n)}{\sqrt{n!}}\chi^{n}, \tag{3}$$

where

$$g(n)=\begin{cases}1 & \text{if}\quad n=0,\\ \prod_{k=0}^{n-1}[f(k)]^{-1} & \text{if}\quad n>0,\end{cases} \tag{4}$$

and the normalization constant reads as

$$\mathcal{N}=\left[\sum_{n}\frac{|\chi|^{2n}}{n!}|g(n)|^{2}\right]^{-1/2}. \tag{5}$$

The function $f(k)$ in Eq. (4) is simply derived from the operator function $\hat{f}(\hat{n})$ by replacing the number operator $\hat{n}$ by the integer k. It is seen from Eq. (3) that the phase-sensitivity of the NCS is determined by the phase φ_χ of the complex eigenvalue χ, in close analogy to the situation for ordinary coherent states. Thus it is expected that the NCS's are located in the phase space around a preferential phase value, as should be the case for any state exhibiting some coherence. To get more insight in the features of such states, let us first deal with a method of their generation in the motion of a trapped ion [17].

Consider a single ion trapped in a harmonic potential [18] of frequency ν and interacting with two laser fields, tuned, respectively, to an electronic transition of frequency ω_{21} and to the (first) lower vibrational sideband with respect to that transition. In the optical rotating-wave approximation the Hamiltonian of this system may be given as

$$\hat{H}=\hat{H}_0+\hat{H}_{\text{int}}(t), \tag{6}$$

where

$$\hat{H}_0=\hbar\nu\hat{a}^\dagger\hat{a}+\hbar\omega_{21}\hat{A}_{22} \tag{7}$$

describes the free motion of the internal and external degrees of freedom of the ion, and

$$\hat{H}_{\text{int}}(t)=\lambda[E_0e^{-i(k_0\hat{x}-\omega_{21}t)}+E_1e^{-i[k_1\hat{x}-(\omega_{21}-\nu)t]}]\hat{A}_{12}+\text{H.c.} \tag{8}$$

the interaction of the ion with the laser fields. The operators $\hat{a}$ and $\hat{A}_{ij}$ $(i,j=1,2)$, respectively, are the annihilation operator of a quantum of the ionic vibrational motion and the electronic (two-level) flip operator for the $|j\rangle\to|i\rangle$ transition, λ is the electronic coupling matrix element, and k_0,k_1 are the wave vectors of the driving laser fields. The operator of the center-of-mass position $\hat{x}$ may be written as

$$\hat{x}=\frac{\eta}{k_L}(\hat{a}+\hat{a}^\dagger), \tag{9}$$

η being the Lamb-Dicke parameter and $k_L\approx k_0\approx k_1$.

In the resolved sideband limit the vibrational frequency ν is much larger than the characteristic frequencies of the interaction problem. In the interaction picture, one may use a vibrational rotating wave approximation and neglect the contributions of all those terms quickly rotating with the frequency ν. This allows us to treat the interaction of the ion with the two lasers separately, using a nonlinear Jaynes-Cummings Hamiltonian [19–21] for each coupling. Thus, the interaction Hamiltonian (8) simplifies to

$$\hat{H}'_{\text{int}}=\hbar\Omega_1e^{-\eta^2/2}\hat{A}_{21}\left(\hat{F}+\frac{\Omega_0}{\Omega_1}\right)+\text{H.c.}, \tag{10}$$

where the operator $\hat{F}$ is given by

$$\hat{F}=\sum_{k=0}^{\infty}\frac{(i\eta)^{2k+1}}{k!(k+1)!}(\hat{a}^\dagger)^k\hat{a}^{k+1}+\frac{\Omega_0}{\Omega_1}\sum_{k=1}^{\infty}\frac{(i\eta)^{2k}}{(k!)^2}(\hat{a}^\dagger)^k\hat{a}^k, \tag{11}$$

Ω_0 and Ω_1 are the Rabi frequency of the lasers tuned to the electronic transition and the first sideband, respectively.

The time evolution of the system is characterized by the master equation

$$\frac{d\hat{\rho}}{dt}=-\frac{i}{\hbar}[\hat{H}'_{\text{int}},\hat{\rho}]+\frac{\Gamma}{2}(2\hat{A}_{12}\hat{\tilde{\rho}}\hat{A}_{21}-\hat{A}_{22}\hat{\rho}-\hat{\rho}\hat{A}_{22}), \tag{12}$$

where the last term describes spontaneous emission with energy relaxation rate Γ, and

$$\hat{\tilde{\rho}}=\frac{1}{2}\int_{-1}^{1}ds W(s)e^{i\eta(\hat{a}+\hat{a}^\dagger)s}\hat{\rho}e^{-i\eta(\hat{a}+\hat{a}^\dagger)s} \tag{13}$$

accounts for changes of the vibrational energy due to spontaneous emission. $W(s)$ is the angular distribution of spontaneous emission and $\hat{\rho}$ the vibronic density operator.

The stationary solution $\hat{\rho}_s$ of Eq. (12) can be found by setting $d\hat{\rho}/dt=0$ on its left-hand side. This yields

$$\hat{\rho}_s=|1\rangle|\xi\rangle\langle\xi|\langle 1|, \tag{14}$$

where $|1\rangle$ is the electronic ground state and $|\xi\rangle$ is a vibrational right-hand eigenstate of the operator $\hat{F}$ given in Eq. (11),

$$\hat{F}|\xi\rangle=\xi|\xi\rangle, \quad \xi=-\frac{\Omega_0}{\Omega_1}. \tag{15}$$

Note that the ion stops to interact with the laser fields when it reaches the steady-state and remains in a ''dark-state'' [22]. This ensures a high stability of the state, since any perturbation switches on the interaction with the lasers, which restores the state.

Using Eqs. (1)–(5) it is straightforward to show that $|\xi\rangle$ belongs to the class of NCS considered above,

$$|\xi\rangle\equiv|\chi;f\rangle, \tag{16}$$

with

$$\chi=\frac{i\Omega_0}{\eta\Omega_1}, \tag{17}$$

$$f(k)=L_k^1(\eta^2)[(k+1)L_k^0(\eta^2)]^{-1}, \tag{18}$$

where L_m^n is an associated Laguerre polynomial. It is noteworthy that the state $|\xi\rangle$ represents the exact stationary solution of the master equation (12), valid for arbitrary values of the Lamb-Dicke parameter.

The properties of the particular NCS $|\xi\rangle$ will depend on the nonlinear function $f(k)$, characterized by the Lamb-Dicke parameter η. Moreover, the state depends on the complex eigenvalue χ, which is controlled by the amplitudes and the phase difference of the two lasers. In order to get some insight in the behaviour of the state $|\xi\rangle$, we show in Fig. 1 the coefficient $g(n)/\sqrt{n!}$, appearing in Eq. (3), for various

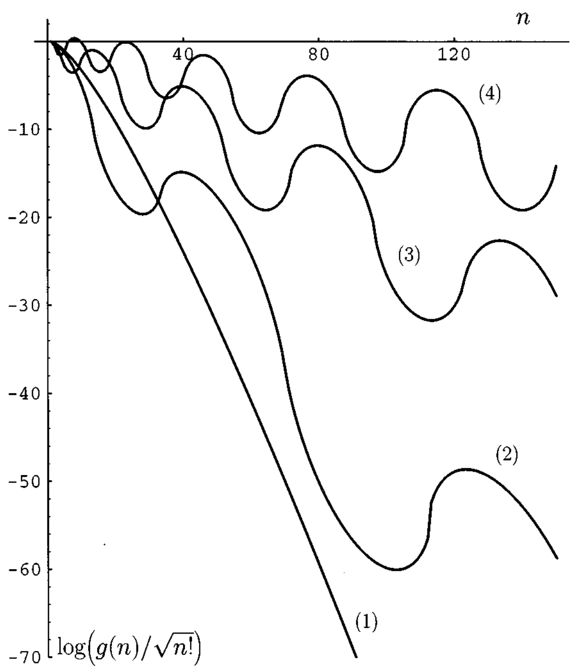

FIG. 1. The function $g(n)/(n!)^{1/2}$ according to Eq. (4) together with Eq. (18) is shown for various values of η. Curve (1), $\eta=0.01$; curve (2), $\eta=0.33$; curve (3), $\eta=0.6$; curve (4), $\eta=0.8$.

values of η. In the Lamb-Dicke regime, where $\eta \ll 1$, this coefficient reduces to $1/\sqrt{n!}$ corresponding to an ordinary coherent state. When η increases, $g(n)/\sqrt{n!}$ is no longer a monotone function of n and displays several local maxima. For a given parameter η, the properties of the state $|\xi\rangle$ will therefore depend on the value of χ in a rather complex, nonlinear manner.

For an appropriately chosen value of χ the number distribution $P_n=|\langle n|\xi\rangle|^2$ of the state can be localized rather close to one local maximum of the expansion coefficient, which gives rise to a narrow distribution since the probabilities for the occupation of number states somewhat apart from the maximum are strongly suppressed, cf. Fig. 1. Consequently, a NCS of this type may display amplitude squeezing. Changing the excitation somewhat, it may split into two well separated peaks. This yields a coherent superposition of two quantum states, appearing due to self-splitting of the NCS, giving rise to quantum interference effects. For sufficiently large values of the Lamb-Dicke parameter η, where the expansion coefficients are strongly structured, even a multiple splitting of the NCS can occur. This gives rise to rich structures of the phase-space distributions.

In Fig. 2 we show the Wigner function for a typical NCS that exhibits strong amplitude squeezing: $\langle(\Delta\hat{n})^2\rangle = 0.07\langle\hat{n}\rangle$. Moreover, this state has a large coherent amplitude. To achieve such a single-peaked state the Lamb-Dicke parameter should not be too large. Otherwise several maxima of the expansion coefficients are close together and the NCS exhibits more complex structures.

When the excitation of the state given in Fig. 2 is somewhat decreased, the NCS exhibits a two-peak structure as

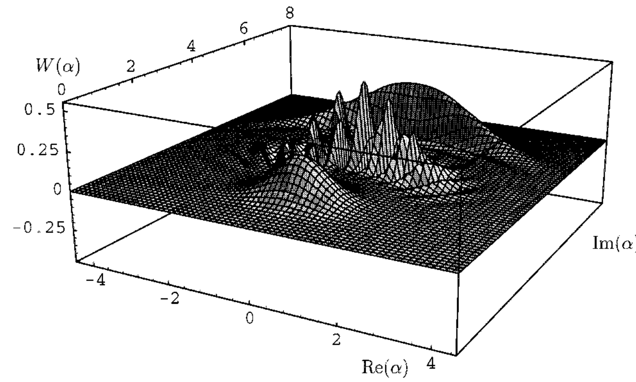

FIG. 3. Wigner function of the NCS $|\xi\rangle$ for $\eta=0.33$, $|\chi|=2.3$, and $\varphi_\chi=\pi/2$.

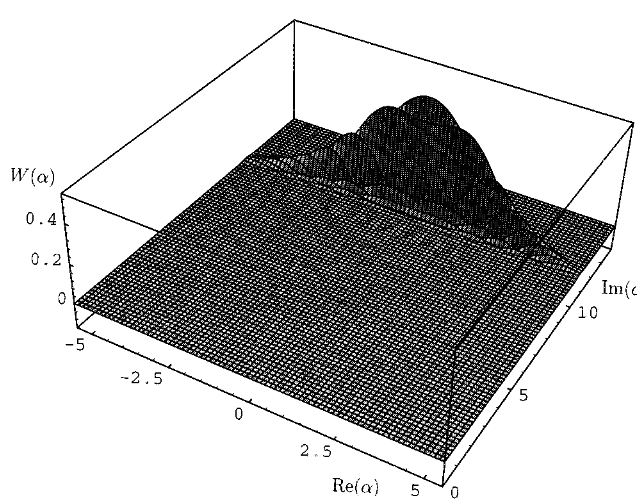

FIG. 2. Wigner function for the NCS $|\xi\rangle$ for $\eta=0.33$, $|\chi|=2.6$, and $\varphi_\chi=\pi/2$.

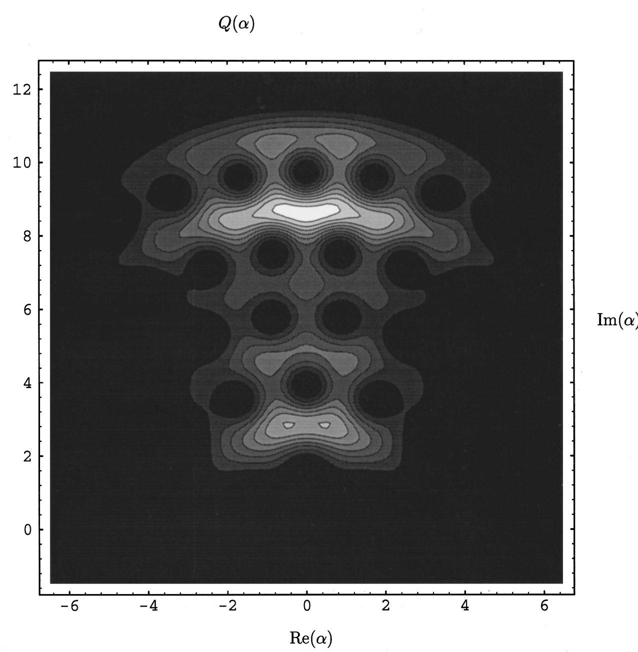

FIG. 4. Contour plot of the Q function of the NCS $|\xi\rangle$ for $\eta=0.82$, $|\chi|=1.1$, and $\varphi_\chi=\pi/2$. Light regions indicate large values of the function.

shown in Fig. 3. As expected, both peaks are centered in the phase space at the same phase angle. They are well separated from each other since the expansion coefficients are very small between the two local maxima. Therefore, the number distributions are essentially nonoverlapping so that the NCS effectively represents a quantum superposition of two partially coherent states, occurring due to self-splitting of the state.

Let us finally consider a typical example for an NCS exhibiting more structures, as it can be obtained for larger values of the Lamb-Dicke parameter η. In Fig. 4 we show the Q function of such a quantum state, which is centered at a given phase and contains contributions at several amplitudes. The quantum interference effects inherent in this state are so strong that even the Q function is strongly structured; it displays several localized regions where it becomes extremely small. This is due to the fact that the separate peaks of the number distribution of that state are rather close together, as suggested by Fig. 1.

In summary we have considered nonlinear coherent states showing strongly nonclassical features. They maintain typical properties of coherent states such as the localization of their phase-space distributions around a nonvanishing mean coherent amplitude. Moreover, these coherence effects may be accompanied by nonclassical effects such as strong amplitude squeezing and self-splitting into two or more substates, which eventually gives rise to pronounced quantum interferences. Such states could be generated as stationary states of the center-of-mass motion of a laser-driven trapped ion, in the resolved sideband regime and far from the Lamb-Dicke limit. When the motional state is prepared in an NCS, the ion is decoupled from the driving laser fields. Consequently, any perturbation of the corresponding motional state leads to the switching on of the interaction yielding a self-stabilization of the NCS. In view of their interesting properties, states of that type might be of more general interest, e.g., in the context of optical and microwave fields or for molecular vibrations. Eventually, they turned out to be of general interest from the point of view of quantum groups.

This research was supported by the Deutsche Forschungsgemeinschaft.

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