Resolutions of the identity in terms of SU(2) coherent states and their use for quantum-state engineering

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Resolutions of the identity in terms of line integrals that involve SU(2) coherent states and their complementary states are presented. They are used for the expansion of various states in terms of SU(2) coherent states on a line. The properties of the complementary states are also studied. [S1050-2947(96)08211-X]

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I. INTRODUCTION

SU(2) coherent states have been studied by various authors [1,2] and found various applications in quantum optics. They are realizable in various contexts, for example, in two-photon systems described by the Hamiltonian [3]

$$H=\omega_1 a^\dagger a+\omega_2 b^\dagger b+\lambda a b^\dagger+\lambda^* a^\dagger b, \tag{1.1}$$

where $a^\dagger$, a and $b^\dagger$, b are photon creation and annihilation operators.

SU(2) coherent states form an overcomplete set of states. In fact it can be proved using the language of analytic representations (see Appendix A), that if $\{z_N\}$ is a convergent sequence to some point z_0 in the extended complex plane (which is stereographically equivalent to a sphere), then the corresponding coherent states form an overcomplete set. This is undoubtedly a very powerful theorem. However, from a practical point of view in order to use the SU(2) coherent states as a basis in the Hilbert space, we need to have a resolution of the identity that can be used for the expansion of an arbitrary state in terms of SU(2) coherent states. Simply to know that a set of states is overcomplete without having a resolution of the identity is not practically useful; but, on the other hand, it encourages us to try to find resolutions of the identity in terms of these states. And sometimes even a weaker concept than a resolution of the identity, like the concept of frames in the context of wavelets [4], might also be useful. In the SU(2) case, the known resolution of the identity involves all the SU(2) coherent states in the extended complex plane. According to what we have said above, there are much smaller subsets that are also overcomplete, and we are encouraged to look for other resolutions of the identity in terms of fewer SU(2) coherent states.

In order to explain the philosophy of our approach we first point out that the resolutions of the identity for the various types of coherent states are in terms of surface integrals of the form

$$\int d\mu |s\rangle\langle s| = \mathbf{1}, \tag{1.2}$$

where $|s\rangle$ are coherent states, and the integration is over a certain manifold [e.g., the complex plane for Glauber coherent states, the extended complex plane for SU(2) coherent states, the unit disk for SU(1,1) coherent states, etc.]. Here we study resolutions of the identity in terms of the line integrals of the type

$$\int dl |s\rangle\langle s;\mathrm{com}| = \mathbf{1}, \tag{1.3}$$

where $|s\rangle$ are coherent states and $\langle s;\mathrm{com}|$ are ‘‘complementary’’ states that are not coherent states. The use of these states gives us great flexibility in constructing new resolutions of the identity; at the same time there is no loss in the strength of the resulting resolution of the identity. Indeed Eq. (1.3) can be used to expand an arbitrary ket state $|f\rangle$ in terms of the coherent states $|s\rangle$ as

$$|f\rangle = \int dl\ f(s)|s\rangle, \tag{1.4}$$

$$f(s)=\langle s;\mathrm{com}|f\rangle \tag{1.5}$$

and the corresponding bra state $\langle f|$ in terms of the coherent states, $\langle s|$ as

$$\langle f| = \int [dl\ f(s)]^* \langle s|. \tag{1.6}$$

Clearly we can also write the bra state $\langle f|$ in terms of the complementary states $\langle s;\mathrm{com}|$ as

$$\langle f| = \int dl g(s)\langle s;\mathrm{com}|, \tag{1.7}$$

$$g(s)=\langle f|s\rangle, \tag{1.8}$$

but we are usually interested in the expansions (1.4), (1.6) in terms of the coherent states $|s\rangle$ that are experimentally realizable, rather than in Eq. (1.7) in terms of the states $|s;\mathrm{com}\rangle$. The $|s;\mathrm{com}\rangle$ are auxiliary states that are used in the calculation of the coefficients (1.5).

The above ideas have been inspired by recent work [5] where quantum states have been expressed as quantum superpositions of Glauber coherent states on a certain line in phase space. The approximation of the exact expansion in terms of line integrals by a discrete sum leads to the possibility of producing experimentally, approximately any desired state as a superposition of Glauber coherent states. This is one approach within the more general framework of quantum-state engineering [6] and indicates one of the practical merits of expansions like Eq. (1.3). In Eq. (1.2) all the coherent states are involved and any good approximation of the integral as a sum (e.g., by using the coherent states on a von Neumann-type of lattice) will require a lot of terms. In Eq. (1.3) it is expected that such approximations will be more accurate with fewer terms. Related to [5] is also the contour representations studied in [7]. The latter formalism has recently been used in [8] to enlarge the usual Hilbert space in order to describe a harmonic oscillator at both positive and negative temperatures. All these references use the Glauber coherent states associated with the Heisenberg-Weyl group.

In this paper we construct a resolution of the identity using a line integral of the type Eq. (1.3), in terms of SU(2) coherent states. We also study the properties of the complementary states $|s;\mathrm{com}\rangle$ that enter in this integral. Using this resolution of the identity we can expand an arbitrary state in the Hilbert space in terms of SU(2) coherent states. As examples, we use this expansion to express the ''θ states'' that we studied in our previous work [9] within the general context of quantum systems with finite-Hilbert space [10–12], as superpositions of SU(2) coherent states. The results could be useful for the production of these states.

II. RESOLUTIONS OF THE IDENTITY IN TERMS OF SU(2) COHERENT STATES AND THEIR COMPLEMENTARY STATES

We consider the angular momentum operators

$$[J_z,J_\pm]=\pm J_\pm,\quad [J_+,J_-]=2J_z,$$
$$J^2=J_z^2+\tfrac{1}{2}(J_+J_-+J_-J_+) \tag{2.1}$$

and the usual $|J;j,n\rangle$ vectors ($j=1,2,3,\ldots,-j\leqslant n\leqslant j$),

$$J^2|J;j,n\rangle=j(j+1)|J;j,n\rangle, \tag{2.2}$$

$$J_z|J;j,n\rangle=n|J;j,n\rangle, \tag{2.3}$$

$$J_+|J;j,n\rangle=[j(j+1)-n(n+1)]^{1/2}|J;j,n+1\rangle, \tag{2.4}$$

$$J_-|J;j,n\rangle=[j(j+1)-n(n-1)]^{1/2}|J;j,n-1\rangle. \tag{2.5}$$

As in our previous work [9], we introduce an extra J to the usual notation because through a finite-Fourier transform, we can introduce dual angle states that we denote as $|\theta;j,n\rangle$. θ states and operators have been studied in [9] and will be used here later. The states $|J;j,n\rangle$ span a $(2j+1)$-dimensional Hilbert space H. We also consider the SU(2) operators:

$$T(\theta,\phi,\lambda)=\exp[-\tfrac{1}{2}\theta e^{-i\phi}J_+ +\tfrac{1}{2}\theta e^{i\phi}J_-]\exp(i\lambda J_z),$$
$$0\leqslant\theta\leqslant\pi,\quad 0\leqslant\phi<2\pi. \tag{2.6}$$

SU(2) coherent states are defined as

$$|J;z\rangle=(1+|z|^2)^{-j}\sum_{n=-j}^{j}\delta(j,n)z^{j+n}|J;j,n\rangle, \tag{2.7}$$

$$\delta(j,n)=\left[\frac{(2j)!}{(j+n)!(j-n)!}\right]^{1/2}, \tag{2.8}$$

where z belongs to the extended complex plane $[C\cup\{\infty\}]$ that we call the J plane, and which is topologically equivalent to the J sphere. An alternative equivalent definition is

$$|J;\theta\phi\lambda\rangle=T(\theta,\phi,\lambda)|J;j,-j\rangle=\exp(-i\lambda z)|J;z\rangle,$$
$$z=-\tan(\tfrac{1}{2}\theta)e^{-i\phi}, \tag{2.9}$$

z is the stereographic projection of the point (θ,ϕ) of the sphere onto the extended complex plane. The following resolution of the identity in terms of these states is well known:

$$\frac{2j+1}{\pi}\int d\mu(z)|J;z\rangle\langle J;z|=\mathbf{1},$$
$$d\mu(z)=(1+|z|^2)^{-2}d^2z. \tag{2.10}$$

We next introduce the ''complementary'' states to the SU(2) coherent states that we define for any $z\neq 0$, as

$$\langle J;z;\mathrm{com}|=[\mathcal{N}(|z|)]^{-1}\sum_{n=-j}^{j}[\delta(j,n)z^{j+n+1}]^{-1}\langle J;j,n|,$$

$$|J;z;\mathrm{com}\rangle=[\mathcal{N}(|z|)]^{-1}\sum_{n=-j}^{j}[\delta(j,n)(z^*)^{j+n+1}]^{-1}|J;j,n\rangle,$$

$$\mathcal{N}(|z|)=\left[\sum_{n=-j}^{j}\frac{(j+n)!(j-n)!}{(2j)!}\frac{1}{|z|^{2(j+n+1)}}\right]^{1/2}, \tag{2.11}$$

where ''com'' in the notation indicates complementary states. They are auxiliary states that will be useful in the calculation of the coefficients, in the expansion of an arbitrary state in terms of SU(2) coherent states.

Combining Eqs. (2.7) and (2.11) we show the resolution of the identity

$$\oint_c\frac{dz}{2\pi i}(1+|z|^2)^j\mathcal{N}(|z|)|J;z\rangle\langle J;z;\mathrm{com}|=\mathbf{1}, \tag{2.12}$$

where C is a contour around the origin in the anticlockwise direction. Using Eq. (2.12) we can expand an arbitrary pure state

$$|f\rangle=\sum_{n=-j}^{j}f_n|J;j,n\rangle;\ \sum_{n=-j}^{j}|f_n|^2=1 \tag{2.13}$$

in terms of SU(2) coherent states on a contour c around the origin as

$$|f\rangle=\oint_c\frac{dz}{2\pi i}f(z)|J;z\rangle, \tag{2.14}$$

where

$$f(z) = (1+|z|^2)^j \mathcal{N}(|z|)\langle J;z;\mathrm{com}|f\rangle = (1+|z|^2)^j \sum_{n=-j}^{j} \frac{f_n}{\delta(j,n) z^{j+n+1}}. \quad (2.15)$$

As examples of this expansion we consider the states $|J;j;n\rangle$ and the SU(2) coherent states $|J;w\rangle$ for which we easily prove

$$|J;j,n\rangle \to f(z) = \frac{(1+|z|^2)^j}{\delta(j,n) z^{j+n+1}}, \quad (2.16)$$

$$|J;w\rangle \to f(z) = \frac{1}{2}\left(\frac{1+|z|^2}{1+|w|^2}\right)^j S\left(\frac{w}{z}\right), \quad (2.17)$$

where

$$S(z) = \frac{z^{2j+1}-1}{z-1} \quad \text{if } z \neq 1,$$

$$S(1) = 2j+1. \quad (2.18)$$

Note that if

$$\omega = \exp\left(i\,\frac{2\pi}{2j+1}\right) \quad (2.19)$$

then

$$S(\omega^m) = 0; \quad m = 1,\ldots,(2j). \quad (2.20)$$

In order to elucidate the relationship between the resolution of the identity (2.10) in terms of a surface integral, and the resolution of the identity (2.12) in terms of a line integral, we give an alternative proof of Eq. (2.12) starting from Eq. (2.10). In order to do this, we first use the relation

$$\oint_c \frac{dw}{2\pi i}\,\frac{(1+z^*w)^{2j}}{w^{j+n+1}} = (z^*)^{j+n}[\delta(j,n)]^2, \quad (2.21)$$

where c is an anticlockwise contour around the origin, to prove,

$$\langle J;z| = (1+|z|^2)^{-j} \oint_c \frac{dw}{2\pi i}\,\mathcal{N}(|w|)(1+wz^*)^{2j}\langle J;w;\mathrm{com}|. \quad (2.22)$$

We also use the resolution of the identity (2.10) to prove that

$$|J;w\rangle = \frac{2j+1}{\pi}\int d\mu(z)\,\frac{(1+z^*w)^{2j}}{(1+|z|^2)^j(1+|w|^2)^j}\,|J,z\rangle. \quad (2.23)$$

Inserting Eq. (2.22) into Eq. (2.10) and using Eq. (2.23) we have an alternative proof of Eq. (2.12).

III. RESOLUTION OF THE IDENTITY IN THE DUAL θ PLANE

In Refs. [9] we have studied the Fourier transform

$$U_F = (2j+1)^{-1/2}\sum_{m,n} \omega^{mn}|J;j,m\rangle\langle J;j,n|, \quad (3.1)$$

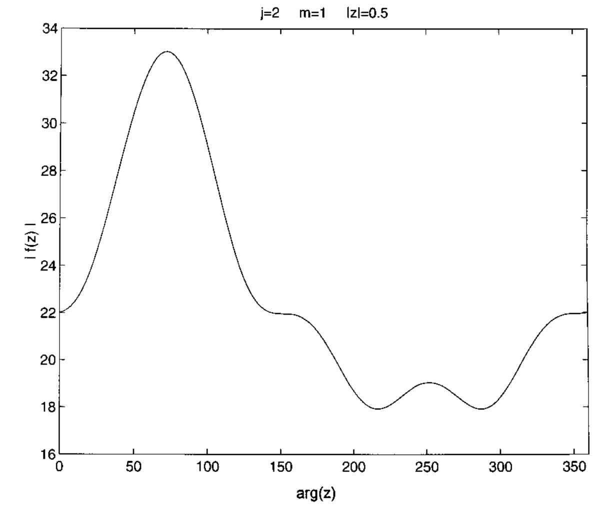

FIG. 1. Coefficients of the expansions of $|\theta;j,m\rangle$ (with $j=2$, $m=1$) in the terms of J-coherent states on the circle $|z|=0.5$. The $|f(z)|$ is plotted as a function of arg(z).

$$U_F U_F^+ = U_F^+ U_F = \mathbf{1}, \quad (3.2)$$

$$U_F^4 = \mathbf{1}, \quad (3.3)$$

where ω has been defined in Eq. (2.19). Using these Fourier transforms we have introduced the θ basis of Euler angle states $|\theta;j,m\rangle$

$$|\theta;j,m\rangle = U_F|J;j,m\rangle = (2j+1)^{-1/2}\sum_{n=-j}^{j} \omega^{mn}|J;j,n\rangle. \quad (3.4)$$

We have also introduced the Euler angle operators θ_+, θ_-, and θ_z which can obey the SU(2) algebra

$$\theta_z = U_F J_z U_F^+, \quad (3.5)$$

$$\theta_+ = U_F J_+ U_F^+, \quad (3.6)$$

$$\theta_- = U_F J_- U_F^+, \quad (3.7)$$

$$[\theta_z, \theta_\pm] = \pm\theta_\pm, \quad [\theta_+, \theta_-] = 2\theta_z. \quad (3.8)$$

The θ operators act on the θ states in an analogous way to the J operators acting on the J states. Therefore we have a θ sphere (and through stereographic projection a θ-extended complex plane) which are dual to the J sphere (and to the J-extended complex plane). Using Eq. (3.4) in conjunction with Eq. (2.15) we express the $|\theta;j,m\rangle$ states as superpositions of the J-coherent states [Eq. (2.14)] with coefficients

$$f(z;m) = (1+|z|^2)^j(2j+1)^{-1/2}\sum_{n=-j}^{j} \frac{\omega^{mn}}{\delta(j,n) z^{j+n+1}}. \quad (3.9)$$

Numerical results for the $|f(z)|$ and $\arg[f(z)]$ are shown in Figs. 1 and 2 correspondingly, for the expansion of the states $|\theta;j,m\rangle$ with $j=2, m=1$ in terms of the J-coherent states on the circle $|z|=0.5$. These results can be useful for the production of these states.

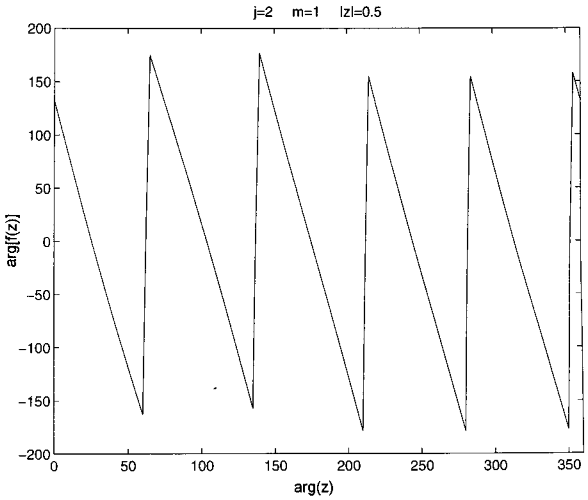

FIG. 2. Coefficients of the expansion of $|\theta;j,m\rangle$ (with $j=2$, $m=1$) in terms of J-coherent states on the circle $|z|=0.5$. The $\arg(f(z))$ is plotted as a function of $\arg(z)$.

SU(2) θ-coherent states can be defined as [9]

$$|\theta;z\rangle=U_F|J;z\rangle=(1+|z|^2)^{-j}\sum_{n=-j}^{j}\delta(j,n)z^{j+n}|\theta;j,n\rangle. \tag{3.10}$$

With respect to the θ operators they have the same properties as the (ordinary) J-coherent states with respect to the J operators. It is the properties of the θ-coherent states with respect to the J operators or the properties of the J-coherent states with respect to the θ operators, that are novel. Using Eq. (3.4) we show that

$$|\theta;w\rangle=(1+|w|^2)^{-j}(2j+1)^{-1/2}\sum_{n,m}\delta(j,n)w^{j+n}\omega^{nm}|J;j,m\rangle \tag{3.11}$$

that we use in conjunction with Eq. (2.15) to show that the θ-coherent states can be expressed as superpositions of the J-coherent states [Eq. (2.14)] with coefficients

$$f(z;w)=\left[\frac{1+|z|^2}{1+|w|^2}\right]^j(2j+1)^{-1/2}\sum_{n,m}\left[\frac{(j-m)!(j+m)!}{(j-n)!(j+n)!}\right]^{1/2}\times\omega^{mn}\frac{w^{j+n}}{z^{j+m+1}}. \tag{3.12}$$

Numerical results for the $|f(z)|$ and $\arg[f(z)]$ are shown in Figs. 3 and 4, for the expansion of the θ-coherent states $|\theta;w=2\rangle$ (with $j=2$) in terms of the J-coherent states on the circle $|z|=0.5$. These results can be useful for the production of the θ-coherent states.

The θ-complementary states are introduced as

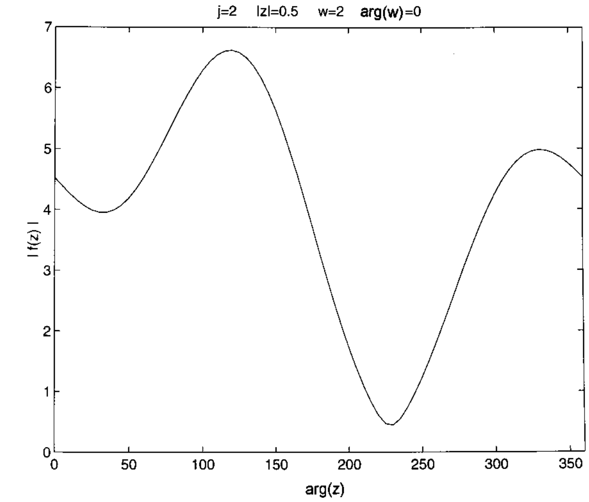

FIG. 3. Coefficients of the expansion of the θ-coherent states $|\theta;w\rangle$ [with $j=2$, $w=2$, $\arg(w)=0$] in terms of J-coherent states on the circle $|z|=0.5$. The $|f(z)|$ is plotted as a function of $\arg(z)$.

$$|\theta,z,\mathrm{com}\rangle=U_F|J;z;\mathrm{com}\rangle=[\mathcal{N}(|z|)]^{-1}\sum_{n=-j}^{j}[\delta(j,n)(z^*)^{j+n+1}]^{-1}\times|\theta;j,n\rangle. \tag{3.13}$$

It is clear that the following resolution of the identity is valid:

$$\oint_c\frac{dz}{2\pi i}(1+|z|^2)^j\mathcal{N}(|z|)|\theta;z\rangle\langle\theta;z;\mathrm{com}|=\mathbf{1}, \tag{3.14}$$

which is analogous to Eq. (2.12).

The J_z-θ_z phase space is the discretized torus $Z(2j+1)\times Z(2j+1)$ [where $Z(2j+1)$ denotes the integers modulo

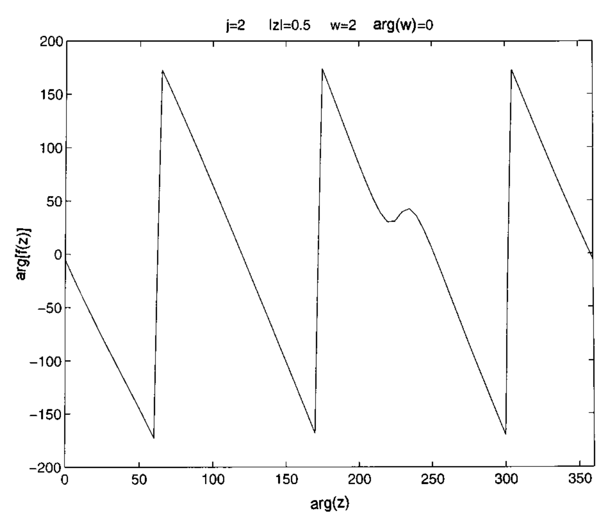

FIG. 4. Coefficients of the expansion of the θ-coherent states $|\theta;w\rangle$ [with $j=2$, $w=2$, $\arg(w)=0$] in terms of J-coherent states on the circle $|z|=0.5$. The $\arg[f(z)]$ is plotted as a function of $\arg(z)$.

$2j+1$]. Displacements in this phase space are performed with the operators (see Appendix B)

$$E=\exp\left[-i\,\frac{2\pi}{2j+1}\,\theta_z\right], \tag{3.15}$$

$$F=\exp\left[i\,\frac{2\pi}{2j+1}\,J_z\right], \tag{3.16}$$

$$E^{2j+1}=F^{2j+1}=\mathbf{1}, \tag{3.17}$$

$$E^{\alpha}F^{\beta}=F^{\beta}E^{\alpha}\omega^{-\alpha\beta}, \tag{3.18}$$

where α,β are integers (modulo $2j+1$). The general displacement operator in the J_z-θ_z phase space, can be written as

$$D(\alpha,\beta)=F^{\alpha}E^{\beta}\omega(-2^{-1}\alpha\beta), \tag{3.19}$$

$$D(\alpha_1,\beta_1)D(\alpha_2,\beta_2)=D(\alpha_1+\alpha_2,\beta_1+\beta_2)\times\omega[2^{-1}\alpha_1\beta_2-2^{-1}\alpha_2\beta_1], \tag{3.20}$$

$$D(0,0)=D(2j+1,0)=D(0,2j+1)=D(2j+1,2j+1)=\mathbf{1}, \tag{3.21}$$

where $\omega(x)$ is a shorthand notation for ω^x. We next consider the parity operator

$$P_0=U_F^2 \tag{3.22}$$

and prove that acting on SU(2) coherent states, we get

$$P_0|J;z\rangle=\left(\frac{z}{|z|}\right)^{2j}\left|J;\,\frac{1}{z}\right\rangle. \tag{3.23}$$

The displaced parity operators are defined as

$$P(\alpha,\beta)=D(\alpha,\beta)P_0[D(\alpha,\beta)]^{\dagger}=D(2\alpha,2\beta)P_0=P_0[D(2\alpha,2\beta)]^{\dagger} \tag{3.24}$$

and can be used in expressing the Wigner function as [13]

$$W(\alpha,\beta)=\mathrm{Tr}[\rho P(\alpha,\beta)]. \tag{3.25}$$

Combining Eqs. (2.12), (4.20), (4.21), and (4.22) we express the Wigner function as

$$W(\alpha,\beta)=\oint_c\frac{dz}{2\pi i}\,(1+|z|^2)^j\mathcal{N}(|z|)\left(\frac{z}{|z|}\right)^{2j}\times\langle J;z;\mathrm{com}|\rho D(2\alpha,2\beta)\left|J;\,\frac{1}{z}\right\rangle. \tag{3.26}$$

IV. COMPLEMENTARY STATES

The expansion Eq. (2.14) is in terms of the usual SU(2) coherent states whose properties are well known. However, the complementary states of Eq. (2.11) also enter in the calculation, in the coefficients $f(z)$ of Eq. (2.15). In this sense there is merit in studying some of the properties of these states. We start by making clear that the complementary states are not SU(2) coherent states. Indeed they can be expressed as

$$|J;z;\mathrm{com}\rangle=[z^*\mathcal{N}(|z|)]^{-1}\sum_{k=0}^{2j}\frac{(2j-k)!}{(2j)!}\left(\frac{J_+}{z^*}\right)^k|J;j,-j\rangle=[(z^*)^{2j+1}\mathcal{N}(|z|)]^{-1}\sum_{k=0}^{2j}k!(z^*J_-)^k|J;j,j\rangle. \tag{4.1}$$

We see that the operators acting on $|J;j,-j\rangle$ or $|J;j,j\rangle$ are not of the type (2.6). In order to get a better understanding of the complementary states, we compare and contrast the directions of the vectors

$$a_i=\langle J;z;\mathrm{com}|J_i|J;z,\mathrm{com}\rangle, \tag{4.2}$$

$$b_i=\langle J;z|J_i|J;z\rangle, \tag{4.3}$$

$$c_i=\langle\theta;z|J_i|\theta;z\rangle, \tag{4.4}$$

where $i=x,y,z$. We also use the notation a_+,b_+,c_+ for

$$a_+=\langle J;z;\mathrm{com}|J_+|J;z;\mathrm{com}\rangle=a_x+ia_y, \tag{4.5}$$

$$b_+=\langle J;z|J_+|J;z\rangle=b_x+ib_y, \tag{4.6}$$

$$c_+=\langle\theta;z|J_+|\theta;z\rangle=c_x+ic_y. \tag{4.7}$$

It is easy to show that

$$a_z=[\mathcal{N}(|z|)]^{-2}\sum_{n=-j}^{j}\frac{(j+n)!(j-n)!}{(2j)!}\,\frac{n}{|z|^{2(j+n+1)}}, \tag{4.8}$$

$$a_+=[\mathcal{N}(|z|)]^{-2}\,\frac{1}{z}\sum_{n=-j}^{j-1}\frac{(j-n)!(j+n+1)!}{(2j)!}\,\frac{1}{|z|^{2(j+n+1)}} \tag{4.9}$$

and it is well known that

$$b_z=j\,\frac{|z|^2-1}{|z|^2+1}, \tag{4.10}$$

$$b_+=2j\,\frac{z^*}{|z|^2+1}. \tag{4.11}$$

It is also easy to show [using Eq. (3.4)] that

$$c_z=(1+|z|^2)^{-2j}(2j+1)^{-1}\times\sum_{n,m,l}\left[\frac{2(j)!}{(j+n)!(j-n)!(j+l)!(j-l)!}\right]^{1/2}\times m\,\omega^{m(n-l)}z^{j+n}(z^*)^{j+l}, \tag{4.12}$$

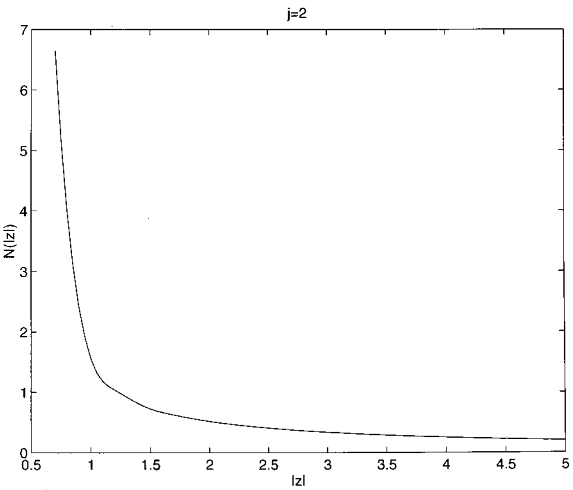

FIG. 5. The normalization constant $\mathcal{N}(|z|)$ for the complementary states (with $j=2$) as a function of $|z|$.

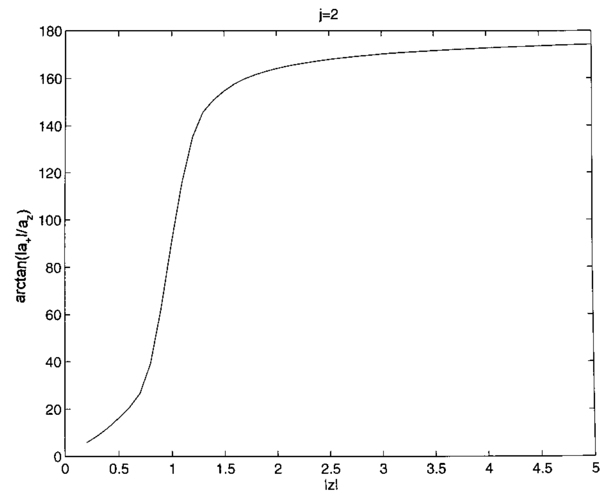

FIG. 7. The angle $\arctan(|a_+|/a_z)$ for complementary states with $j=2$ as a function of $|z|$.

$$c_+=(1+|z|^2)^{-2j}(2j+1)^{-1} \times \sum_{n,m,l}\left[\frac{(2j)!}{(j+n)!(j-n)!(j+l)!(j-l)!}\right]^{1/2} \times \omega^{mn-l(m+1)}z^{j+n}(z^*)^{j+l}[(j-m)(j+m+1)]^{1/2}. \tag{4.13}$$

We first present results for the normalization constant. In Fig. 5 we plot $\mathcal{N}(|z|)$ as a function of $|z|$ for $j=2$. As expected, for $|z|<1$ the $\mathcal{N}(|z|)$ takes large values.

We next consider the spherical angles $\arg(a_+)$, $\arctan(|a_+|/a_z)$ for the vector a_i, and the corresponding angles for the vectors b_i, c_i. It is clear that

$$\arg(a_+)=\arg(b_+)=-\arg(z) \tag{4.14}$$

and therefore both vectors a and b are on the same half plane through the z axis, defined by the angle $-\arg(z)$. $\arg(c_+)$ as a function of $\arg(z)$ is shown in Fig. 6 from which it is seen clearly that the vector c_+ is on a different half plane through the z axis, than the vectors a_+ and b_+.

Equations (4.10) and (4.11) show that

$$\frac{|b_+|}{b_z}=\frac{2|z|}{|z|^2-1} \tag{4.15}$$

from which it is clear that the $\arctan(|b_+|/b_z)$ is independent of j and $\arg(z)$. The corresponding angle for the vector a_i is shown in Figs. 7 and 8 where it is seen that it depends on both $|z|$ and j [although it is independent of $\arg(z)$]. From Fig. 7 and Eq. (4.15) it is seen that when the vector a_i points in the north hemisphere, the vector b_i points in the south hemisphere and vice versa. Only for $|z|=1$, both vectors a_i, b_i point in the same direction on the x-y plane. Results for $\arctan(|c_+|/c_z)$ are shown in Figs. 9–11 where it is seen that it depends on $|z|$, $\arg(z)$, and j.

We next evaluate the overlaps:

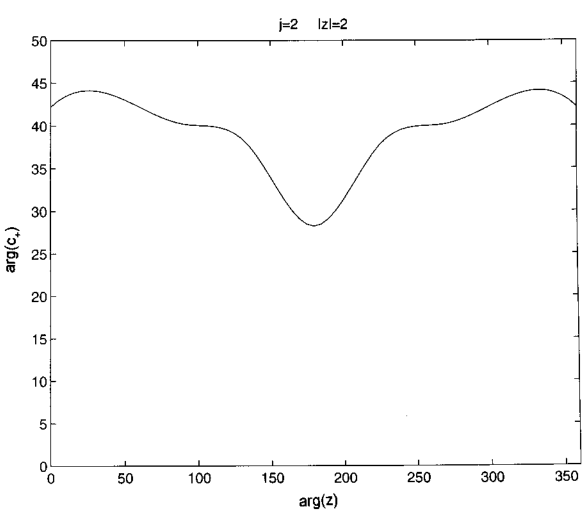

FIG. 6. The angle $\arg(c_+)$ for θ-coherent states $|\theta;z\rangle$ (with $j=2$, $|z|=2$) as a function of $\arg(z)$.

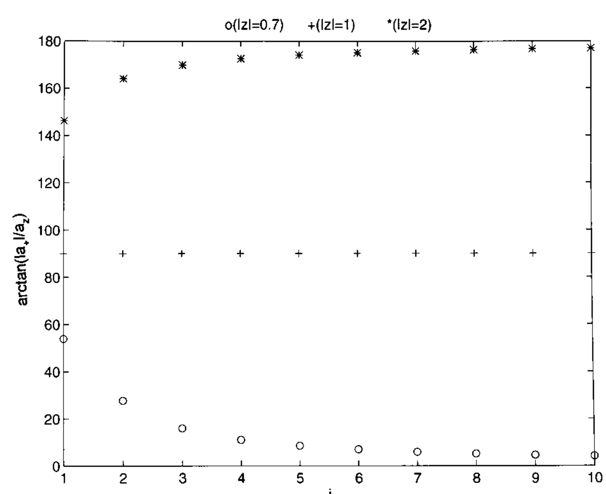

FIG. 8. The angle $\arctan(|a_+|/a_z)$ for complementary states [with $|z|=0.7$ (○); $|z|=1$ (+); $|z|=2$ (∗)] as a function of j.

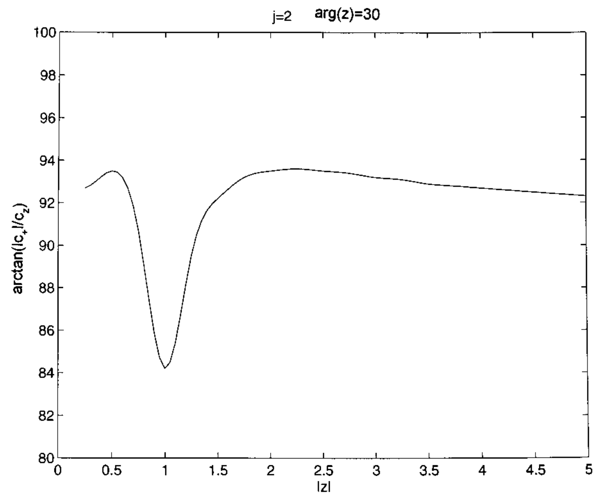

FIG. 9. The angle $\arctan(|c_+|/c_z)$ for θ-coherent states $|\theta;z\rangle$ [with $j=2$, $\arg(z)=30$] as a function of $|z|$.

$$\langle J,z_1;\mathrm{com}|J;z_2;\mathrm{com}\rangle=[\mathcal{N}(|z_1|)\mathcal{N}(|z_2|)]^{-1}\times\sum_{n=-j}^{j}\frac{(j+n)!(j-n)!}{(2j)!}\times\frac{1}{(z_1z_2{}^*)^{j+n+1}}, \tag{4.16}$$

and also

$$\langle J;z_1;\mathrm{com}|J;z_2\rangle=[\mathcal{N}(|z_1|)]^{-1}(1+|z_2|^2)^{-j}\,\frac{1}{z_1}\,S\left(\frac{z_2}{z_1}\right), \tag{4.17}$$

where $S(z)$ has been defined in Eq. (2.18). It is clear that the above overlap is equal to zero, when

$$z_1=z_2\omega^m,\quad m=1,\ldots,(2j). \tag{4.18}$$

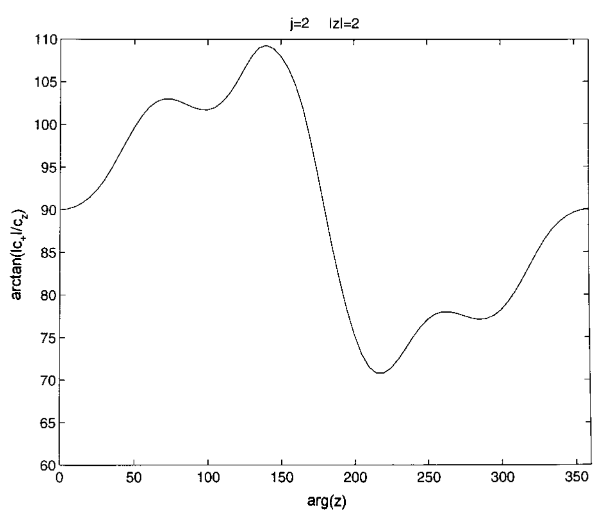

FIG. 10. The angle $\arctan(|c_+|/c_z)$ for θ-coherent states $|\theta;z\rangle$ (with $j=2$, $|z|=2$) as a function of $\arg(z)$.

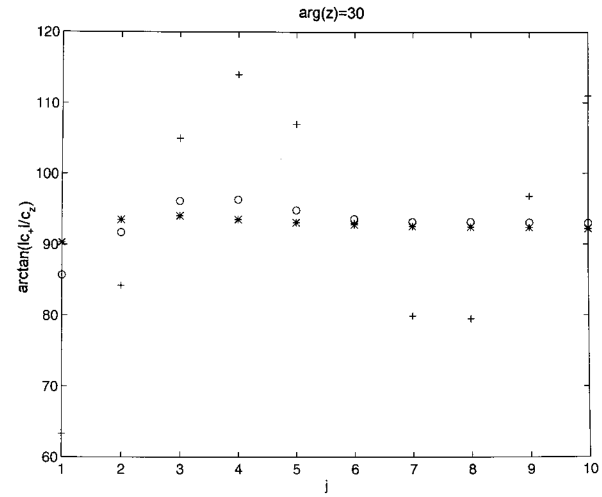

FIG. 11. The angle $\arctan(|c_+|/c_z)$ for θ-coherent states $|\theta;z\rangle$ [with $\arg(z)=30$ and $|z|=0.7$ (○); $|z|=1$ (+); $|z|=2$ (*)] as a function of j.

We next express the states $|J;z;\mathrm{com}\rangle$ as superpositions of the SU(2) coherent states $|J;z\rangle$. Due to the overcompleteness of the states $[|J;z\rangle]$, there are many such expansions and we give some of them.

We start with the expansion (2.14) and using Eq. (2.15) we find

$$|J;w;\mathrm{com}\rangle\rightarrow f(z)=[\mathcal{N}(|w|)]^{-1}(1+|z|^2)^j\times\sum_{j=-n}^{n}\frac{(j+n)!(j-n)!}{(2j)!}\,\frac{1}{(zw^*)^{j+n+1}}. \tag{4.19}$$

Another expansion can be found using the integral

$$\int_0^\infty ds\; s^{2j}\int_0^\infty dt\; e^{-(t+s)}\left(\frac{t}{s}\right)^{j+m}=(j+m)!(j-m)! \tag{4.20}$$

to prove that

$$|J;z;\mathrm{com}\rangle=[z^*(2j)!\mathcal{N}(|z|)]^{-1}\int_0^\infty ds\int_0^\infty dt\; e^{-(t+s)}\times\left[s^2+\frac{t^2}{|z|^2}\right]^j\left|J;\frac{t}{sz^*}\right\rangle. \tag{4.21}$$

Finally we use the integral

$$[\delta(j,n)]^{-2}=(j-n)B(j+n+1,j-n)=(j-n)\int_0^\infty\frac{x^{j+n}}{(1+x)^{2j}}\,dx \tag{4.22}$$

where B is a beta function, to prove

$$|J;z;\mathrm{com}\rangle = [z^*\mathcal{N}(|z|)]^{-1}\int_0^\infty \frac{dx}{(1+x^2)^{2j}}\,(2j - x\partial_x)\times\left\{\left[1+\frac{x^2}{|z|^2}\right]^j \left|J;\ \frac{x}{z^*}\right\rangle\right\}. \quad (4.23)$$

V. DISCUSSION

Resolutions of the identity are important for the practical use of coherent states as a basis in a Hilbert space. In this paper we have derived the resolution of the identity (2.12) in the context of SU(2) coherent states. This involves both the SU(2) coherent states $|J;z\rangle$ and the auxiliary states appearing in the calculation of the coefficients in Eq. (2.15). Some properties of the complementary states have been studied in Sec. IV. Using Eq. (2.12) we can expand any state in the Hilbert space in terms of SU(2) coherent states [Eqs. (2.14) and (2.15)]. Expansions for the states $|J;j,n\rangle$, $|\theta;j,n\rangle$, $|J;z\rangle$, and $|\theta;z\rangle$ have been given in Eqs. (2.16), (3.9), (2.17), and (3.12), correspondingly, and related numerical results have been presented. The Wigner function in the J_z-θ_z phase space has also been given in the form of a contour integral in Eq. (3.26). The results have been presented in a general SU(2) context but they could be readily applied in the two-photon realization of these states with the Hamiltonian (1.1).

More work is required on resolutions of the identity in terms of line integrals [like Eq. (1.3)]. One approach is to use contour integrals in an appropriate complex region and for coherent states (associated with a certain group) of the type

$$|z\rangle = \sum_N a_N z^N |N\rangle, \quad \sum_N |a_N|^2 = 1 \quad (5.1)$$

to construct the complementary states as

$$\langle z;\mathrm{com}| = [\mathcal{N}(|z|)]^{-1}\sum_N [a_N z^{N+1}]^{-1}\langle N|, \quad (5.2)$$

where $\mathcal{N}(|z|)$ is a normalization factor and $|N\rangle$ is an orthonormal basis. The difficulty with certain groups might be that the normalization factor diverges and then we have to think very carefully of how the complementary states are defined and in which space they belong. In the SU(2) case studied in this paper all the sums are finite and we had no difficulties of this nature.

APPENDIX A

In the language of analytic representations (e.g., [14]) we can represent the arbitrary state $|f\rangle$ of Eq. (2.13) with the analytic function in the extended complex plane:

$$f(z) = \sum_{n=-j}^{j} f_N \delta(j,n) z^{j+n} = (1+|z|^2)^j \langle z^*|f\rangle. \quad (A1)$$

Let $\{z_N\}$ be sequence in the complex plane that converges to some point z_0. We want to prove that the set of the corresponding SU(2) coherent states $\{|J;z_n\rangle\}$ is overcomplete. Indeed if it is not complete there will be some state $|g\rangle$ that will be orthogonal to all $\{|J;z_N\rangle\}$ and consequently $g(z_N)=0$ for all $\{z_N\}$. But this is not possible because the zeros of analytic functions are ''isolated'' and cannot converge to a point z_0. Therefore the set $\{|J;z_N\rangle\}$ is at least complete. In fact it is overcomplete because the same argument is also valid, even if we omit a finite number of terms from the sequence $\{z_N\}$.

APPENDIX B

In this appendix we briefly review the properties of the operators E and F of Eqs. (3.15) and (3.16). The operators E and F perform displacements along the J_z and θ_z axes, correspondingly,

$$E^\alpha|J;j,m\rangle = |J;j,m+\alpha\rangle, \quad (B1)$$

$$E^\alpha|\theta;j,m\rangle = \omega(-m\alpha)|\theta;j,m\rangle, \quad (B2)$$

$$F^\beta|J;j,m\rangle = \omega(m\beta)|J;j,m\rangle, \quad (B3)$$

$$F^\beta|\theta;j,m\rangle = |\theta;j,m+\beta\rangle. \quad (B4)$$

Combining (B1), (B3) or (B2), (B4) we prove the important relation (3.18). More details about these operators and their properties are given in Ref. [9].

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