# Simple algebraic method to solve a coupled-channel cavity QED model

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In this paper, we present a simple purely algebraic method to solve a coupled-channel cavity QED model with three nondegenerate quantized field modes proposed by Wang, Puri, and Eberly [Phys. Rev. A **46**, 7192 (1992)]. By transforming it into one describing a spin-$\frac{1}{2}$ particle in a magnetic field and utilizing the relations between field variables and an orbital angular momentum, we can easily obtain energy eigenvalues without the need to know concrete expressions for eigenstates in terms of the usual Fock states, and analytical expressions of evolution and atomic inversion operators. The eigenstates of energy and orbital angular momentum are also explicitly expressed in terms of Fock states.
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## I. INTRODUCTION

The exact solvability of fully quantum-mechanical models plays a critically important role in the field of light-atom interactions for the study of purely quantum features such as the collapse and revival of Rabi oscillations because it permits access to regimes which are incompatible with perturbation theory, which embrace most long-time low-loss near-resonance phenomena and include the domain of few-photon strong fields, and in which atomic response can be large even if the field is extremely weak by conventional measures [1]. Over the last two decades, there has been intensive study [2,3] of the solvable Jaynes-Cummings (JC) model and its various extensions such as intensity-dependent coupling constants [4], two-photon or multiphoton transitions [5], and two-cavity modes for three-level atoms [2,6,7,10–12], many of which have been proved to be exactly solvable. Here we mention only a few studies of Raman-coupled cavity QED models mostly relevant to this paper. Gerry and Eberly [6] and Cardimona *et al.* [7] treated a nondegenerate model in a two-mode cavity in which a three-level atom and only the pump Stokes transition are considered. This model has been solved exactly for the zero-detuning case [8,9] and for an arbitrary detuning [11,12]. We have found that this three-level probem can be exactly transformed to a two-level one for arbitary detuning [11], and that bare- and dressed-state solutions display quite different dynamics [12]. The corresponding three-mode model called the coupled-channel cavity QED model [1], which includes both the pump-Stokes and pump-anti-Stokes transitions, has also been proved partially solvable [1,13]. It displays nontrivial multiwave mixing, an interesting chain structure in a fully quantized treatment [1], and two-mode squeezing and a nontrivial phase correlation when treating a pump with constant amplitude classically.

The Hamiltonian of the fully quantized coupled-channel cavity QED model reads [1]

$$H=\omega_p N+E_{+-}J_z+H_{\rm int}, \tag{1a}$$

$$H_{\rm int}=g(a_S^\dagger a_P+a_P^\dagger a_A)\sigma_{+-}+g(a_P^\dagger a_S+a_A^\dagger a_P)\sigma_{-+}, \tag{1b}$$

where the subscripts $P$, $S$, and $A$ represent pump, Stokes, and anti-Stokes modes, respectively; $a$ and $a^\dagger$ are the creation and annihilation operators for the corresponding modes, subscripts $+$ and $-$ denote atomic levels $|+\rangle$ and $|-\rangle$; $\sigma$'s are the usual atomic transition operators satisfying $\sigma_{jk}\sigma_{mn}=\sigma_{jn}\delta_{mk}$ and $\sigma_{--}+\sigma_{++}=1$; the two constants of motion are the total photon number $N=n_P+n_S+n_A$ and $J_z=n_A-n_S+\frac{1}{2}(\sigma_{++}-\sigma_{--})$; and the two channels have an equal coupling $g$. Obviously, the second channel causes considerable difficulty in solving the eigenvalue problem, far from trivial compared to all known single-channel JC cases. Wang, Puri, and Eberly succeeded in exactly solving it partially [1]. They expanded the eigenvectors of the Hamiltonian in terms of $|n_P,n_S,n_A,\pm\rangle\equiv|n_P,n_S,n_A|\otimes|\pm\rangle$, where $|n_P,n_S,n_A\rangle$ denotes the usual Fock states (or eigenstates of photon numbers) to obtain a recursion relation for the expansion coefficients. Then the recursion relation was converted into a differential equation by introducing a generating function. By demanding its solution be suitable for the generating function defined, they finally obtained the eigenvalues and corresponding eigenvectors for the special case of $J_z=-\frac{1}{2}$. They also gave the eigenvalues but not the eigenvectors for the general case [1]. The same method could in principle deal with the general case of any allowable $J_z$, so long as the generated differential equation could be transformed into a standard one with known solution(s) which remain unclear right now. In their paper [1], they established an elegant connection between field variables and an orbital angular momentum $\mathbf{L}$ with $L_z=a_A^\dagger a_A-a_S^\dagger a_S$, and identified $S_z=\frac{1}{2}(\sigma_{++}-\sigma_{--})$ as the $z$ component of another (spin) angular momentum. Hence $J_z=L_z+S_z$ denotes the $z$ component of the sum of two angular momenta. They did not utilize this connection in solving the model because they felt [1] that to take advantage of it, a relation between the eigenstates of $L_z$ and the usual Fock states $|n_P,n_S,n_A\rangle$ was necessary, and they did not know this [1].

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In this paper, we present a simple algebraic method to solve this model for all possible values of $J_z$, by establishing its connection with the model for a spin-$\frac{1}{2}$ particle in a magnetic field, and taking advantage of the relations between field variables and the orbital angular momentum. The outline of this paper is as follows. In Sec. II, we calculate the eigenvalues of the Hamiltonian or energy eigenvalues. It is shown that these can be obtained quickly with no need to know the explicit relation between the eigenstates of $L_z$ and Fock states, and, in fact, without the need to know the concrete relations between the energy eigenstates and Fock states. This method is in fact a systematic and effective, perhaps the simplest, one to quickly obtain eigenvalues of all solvable JC models. Our results for eigenvalues are the same as the ones obtained by Wang, Puri, and Eberly for the special case of $J_z=-\frac{1}{2}$, but differ from theirs in the general case. In Sec. III, we first calculate the analytical expressions of the evolution operator and the atomic inversion operator, and explain why the Fock initial-field state in this model would in general give the collapse and revivals of Rabi oscillations. Then we explicitly express the eigenstates of the energy and orbital angular momentum in terms of Fock states for any allowable values of $J_z$. The connection between the angular quantum number and the total photon number is derived in Appendix A. In Appendix B, we present expressions of the eigenvectors of the Hamiltonian and the orbital momentum in terms of Fock states for a few photons (the total photon number $N=1$, 2, and 3).

## II. ENERGY EIGENVALUES

To determine the energy eigenvalues of the coupled-channel cavity QED model, we begin with a connection between field variables and an orbital angular momentum **L** introduced by Wang, Puri, and Eberly [1],

$$L_+=L_x+iL_y=\sqrt{2}(a_S a_P^\dagger+a_A^\dagger a_P), \tag{2a}$$

$$L_-=L_+^\dagger=\sqrt{2}(a_P a_S^\dagger+a_P^\dagger a_A), \tag{2b}$$

$$L_z=a_A^\dagger a_A-a_S^\dagger a_S, \tag{2c}$$

$$L^2=(n_A-n_S)^2+(n_A+n_S)(2n_P+1)+2n_P +2(a_P^2 a_A^\dagger a_S^\dagger+a_P^{\dagger 2}a_S a_A), \tag{2d}$$

where $L_-$ and $L_+$ are the lowering and raising operators of the momentum. The interaction Hamiltonian can be put into the form [1]

$$H_{\rm int}=\frac{g}{\sqrt{2}}(L_-\sigma_{+-}+L_+\sigma_{-+}). \tag{3}$$

From the theory of angular momentum and the expression of $L_z$, we know that **L** represents an orbital angular momentum, and $L^2$ and $L_z$ have a complete set of common eigenvectors $|l,m\rangle$ satisfying $L^2|l,m\rangle=l(l+1)|l,m\rangle$, and $L_z|l,m\rangle=m|l,m\rangle$ with $l=0,1,2,3,\ldots$ and $m=o,\pm1,\pm2,\pm3,\ldots,\pm l$. In Appendix A, we show how to obtain the restriction of the total photon number $N$ on the quantum numbers $l$ and $m$ without the need to know the concrete expressions of $|l,m\rangle$ in terms of Fock states $|n_P,n_S,n_A\rangle$. The results are as follows. For a given $N$, the allowable values of the angular quantum number are $l=N,N-2,N-4,\ldots,(N-2\,\mathrm{Int}(N/2))$, where $\mathrm{Int}(N/2)=N/2$ for even $N$ and $\mathrm{Int}(N/2)=(N-1)/2$ for odd $N$ or, equivalently, $l=0,2,4,\ldots,N$ for even $N$ and $l=1,3,5,\ldots,N$ for odd $N$. Also, $m=0,\pm1,\pm2,\ldots,\pm l$ for any allowable $l$. Because $J_z=L_z+\frac{1}{2}\sigma_z$, where $\sigma_z=\sigma_{++}-\sigma_{--}$, and the operators $L^2,L_z$, and $J_z$ are mutually commutative, we see that they have a complete set of common eigenvectors $|l,m\rangle\otimes|\pm\rangle$ where $|\pm\rangle$ are the atomic states, and the eigenvalues of $J_z$ are $\pm\frac{1}{2},\pm\frac{3}{2},\ldots,\pm(l+\frac{1}{2})$ for any given $l$.

We now consider the Hamiltonian and its eigenvalues. Noting that $(L_-\sigma_{+-}+L_+\sigma_{-+})^2=(L^2-J_z^2+\frac{1}{4})$ which obviously commutes with $(L_-\sigma_{+-}+L_+\sigma_{-+})$, we can rewrite the interaction Hamiltonian as

$$H_{\rm int}=\frac{g}{\sqrt{2}}\sqrt{L^2-J_z^2+\tfrac{1}{4}}\,\sigma_y=\tfrac{1}{2}\Omega\sigma_y \tag{4}$$

by introducing operators

$$\sigma_y=\frac{L_-\sigma_{+-}+L_+\sigma_{-+}}{\sqrt{L^2-J_z^2+\dfrac{1}{4}}}, \tag{5}$$

$$\Omega=g\sqrt{2(L^2-J_z^2+\tfrac{1}{4})}. \tag{6}$$

It is easy to check that $\sigma_x=i\sigma_z\sigma_y=i(L_-\sigma_{-+}-L_+\sigma_{+-})/\sqrt{L^2-J_z^2+\frac{1}{4}},\sigma_y$, and $\sigma_z=\sigma_{++}-\sigma_{--}$ satisfy the well-known Pauli operators' commutation and anticommutation relations $\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij}$ and $[\sigma_k,\sigma_l]=2i\epsilon_{klm}\sigma_m$; in particular; $\sigma_y^2=1$, and $\sigma_y$ has two eigenvalues $\pm1$. Obviously, this form of interaction Hamiltonian $H_{\rm int}$ describing the coupled-channel cavity QED model is identical to the model for a spin-$\frac{1}{2}$ particle in a magnetic field $B$ along the $y$ direction with its magnitude proportional to the quantity $\Omega$, except that $B$ and $\Omega$ are now operators and not $c$ numbers. The quantity $\Omega$ is nothing but the Rabi operator (its eigenvalues give all the frequencies in the Rabi oscillations of the atomic inversion) for the cavity QED model and the gyration frequency (operator) for the latter model. This similarity between the two models permits us to obtain quickly the energy eigenvalues and solutions of the evolution operator and other operators such as the atomic inversion as well as the expression of Rabi operator. It also permits us to determine energy eigenvalues quickly when including the Stark term and off-resonance effect, as will be explained below. In addition, this similarity immediately shows that the system's energy levels manifest a doublet structure, with the doublets' energy differences equal to the eigenvalues of the Rabi operator. The operators' solutions will be discussed in Sec. III. Here we decide the energy eigenvalues.

As discussed above, the total Hamitonian can be written as follows:

$$H=\omega_p N+E_{+-}J_z+\frac{g}{\sqrt{2}}\sqrt{L^2-J_z^2+\tfrac{1}{4}}\,\sigma_y. \tag{7}$$

It is easily found that the Hermitian operators $N$, $L^2$, $J_z$, and $\sigma_y$ are mutually commutative, and hence have a complete set of common eigenvectors, which is also the complete set of the energy eigenvectors, with its typical member expressed as $|N,l,J_z,m_y\rangle$, where $m_y=\pm 1$ are the eigenvalues of $\sigma_y$, and we use the symbols $N$ and $J_z$ to express the operators and their eigenvalues for simplicity. We immediately obtain the energy eigenvalues

$$E_{N,l,J_z,m_y}=\omega_p N+E_{+-}J_z+\frac{gm_y}{\sqrt{2}}\sqrt{l(l+1)-J_z^2+\tfrac{1}{4}}, \quad (8)$$

where

$$N=0,1,2,\ldots, \quad (9a)$$

$$l=N,N-2,N-4,\ldots,\left(N-2\,\mathrm{Int}\left(\frac{N}{2}\right)\right), \quad (9b)$$

$$J_z=\tfrac{1}{2},\pm\tfrac{3}{2},\ldots,\pm(l+\tfrac{1}{2}), \quad (9c)$$

$$m_y=\pm 1. \quad (9d)$$

We discuss the system's energy structure by means of the expression of energy eigenvalues.

It is seen that the energy structure displays a singlet plus doublet characteristic quite similar to the case of the JC model. For any given set of $(N,l,J_z)$ except for $J_z=\pm(l+\frac{1}{2})$, $m_y=\pm 1$ gives a doublet with its energy difference equal to $\Omega_{l,J_z}=g\sqrt{2}\sqrt{l(l+1)-J_z^2+\frac{1}{4}}\equiv g\sqrt{2}\sqrt{(l+\frac{1}{2})^2-J_z^2}$, which is the eigenvalues of the Rabi operator. These characteristics resemble those of the doublets of the JC model. For $J_z=(l+\frac{1}{2})$ or $-(l+\frac{1}{2})$, we obtain a singlet state, since $\Omega_{l,J_z}=0$. It is easily seen that $|l,m=l\rangle\otimes|\sigma_z=1\rangle$ is the eigenvector of two operators: a Hamiltonian and $J_z$ for $J_z=(l+\frac{1}{2})$, and so is $|l,m=-l\rangle\otimes|\sigma_z=-1\rangle$ for $J_z=-(l+\frac{1}{2})$. It is interesting to see that these singlets represent states of no coupling between the atom and the fields, since the vectors can be expressed by the direct product of the atomic states $|\sigma_z=\pm 1\rangle$ and the field states $|l,m=\pm 1\rangle$, quite similar to the ground state of the JC model, which is also a singlet and represents a state of no coupling between the atom and the field. There is some ambiguity in the definitions of operators $\sigma_y$ and $\sigma_x$ as $J_z=\pm(l+\frac{1}{2})$, since their denominator $\sqrt{L^2-J_z^2+\frac{1}{4}}$ becomes zero. However, it is easy to show that their numerators $(L_-\sigma_{+-}+L_+\sigma_{-+})$ and $(L_-\sigma_{-+}-L_+\sigma_{+-})$ also become zero, and this ambuguity does not affect the expression of the energy eigenvalues.

For a given $N$, the angular quantum number $l$ will still have several choices. We have $2l$ doublets and two singlets for any allowable $l$, and thus $4l+2$ different energy states for a given set of $(N,l)$. Thus, it is easy to show that the total number of energy states for a given $N$ is $\Sigma_{\text{allowable }l}(4l+2)=(N+1)(N+2)$, which is the exact number of all the possible $|n_P,n_S,n_A,\sigma_z=\pm 1\rangle$ for a fixed $N$. This means that all energy levels are nondegenerate.

We now compare our results with the ones given by Wang, Puri, and Eberly [1]. They [1] derived an expression for energy eigenvalues for the special case of $J_z=-\frac{1}{2}$ ($J_z$ is denoted $C$ by them), and gave the eigenvalues for the general case of any allowable $J_z$ without presenting derivations. We cite their result [their Eq. (B1.9)] here;

$$\lambda=\pm\sqrt{2n(n+|J_z|)}, \quad (10)$$

where, depending on the parity of the integer $(N-|J_z|+\frac{1}{2})$, the allowed values of $n$ are either $n=0,1,\ldots,\frac{1}{2}(N-|J_z|+\frac{1}{2})$ or $n=\frac{1}{2},\frac{3}{2},\ldots,\frac{1}{2}(N-|J_z|+\frac{1}{2})$. In this equation, $\lambda$ is the eigenvalue of the interaction Hamiltonian divided by g. Our corresponding result is

$$\lambda=\pm\left(\frac{l(l+1)-J_z^2+\frac{1}{4}}{2}\right)^{1/2}, \quad (11)$$

where $l=N,N-2,N-4,\ldots,(N-2\,\mathrm{Int}(N/2))$ and $J_z=\frac{1}{2},\pm\frac{3}{2},\ldots,\pm(l+\frac{1}{2})$. Obviously, our result is same as theirs for $J_z=-\frac{1}{2}$, but differs from theirs generally since the two expressions for eigenvalues have different functional dependences on the quantum number $J_z$. Let us give a numerical example to illustrate that the two expressions indeed lead to different results. We follow their convention by specifying $N$ and $J_z$ first. Taking $N=10$ and $J_z=5+\frac{1}{2}$, one finds that their expression gives $\lambda=\pm 2(n=\frac{1}{2})$, $\pm\frac{39}{2}(n=\frac{3}{2})$, and $\pm\frac{85}{2}(n=\frac{5}{2})$, while ours gives $\lambda=\pm 6(l=6)$, $\pm\sqrt{21}(l=8)$, and $\pm\sqrt{40}(l=10)$, since $l=0$, 2, 4, 6, 8, and 10 for $N=10$, and $J_z=5+\frac{1}{2}$, implying $l\geqslant 5$.

Our method of obtaining eigenvalues still works when the Stark term $g(n_A-n_S)\sigma_{--}$ is included and the frequency resonant condition is not exactly satisfied, which amounts to an extra term $\delta\sigma_{--}$ [1]. To see this, We rewrite the two terms as $[g(n_A-n_S)+\delta]\sigma_{--}=\frac{1}{2}[g(J_z+\frac{1}{2})+\delta](1-\sigma_z)$. The Hamiltonian then becomes

$$H=\left(\frac{g}{4}+\frac{\delta}{2}\right)+\omega_p N+\left(E_{+-}+\frac{g}{2}\right)J_z+\frac{g}{\sqrt{2}}\sqrt{L^2-J_z^2+\tfrac{1}{4}}\sigma_y -\tfrac{1}{2}[g(J_z+\tfrac{1}{2})+\delta]\sigma_z. \quad (12)$$

Noting that operators $L^2$ and $J_z$ commute with $\sigma_y$ and $\sigma_z$, we can put the Hamiltonian into the form

$$H=\left(\frac{g}{4}+\frac{\delta}{2}\right)+\omega_p N+\left(E_{+-}+\frac{g}{2}\right)J_z+\frac{1}{2}\omega\sigma_n, \quad (13)$$

where

$$\omega=\sqrt{[g(J_z+\tfrac{1}{2})+\delta]^2+2g^2(L^2-J_z^2+\tfrac{1}{4})}, \quad (14)$$

$$\sigma_n=\mathbf{n}\cdot\sigma=-\frac{\left[g\left(J_z+\frac{1}{2}\right)+\delta\right]}{\omega}\sigma_z+\frac{\Omega}{\omega}\sigma_y, \quad (15)$$

where $\omega$ and $\Omega$ denote, respectively, the Rabi operator with and without considering the Stark and $\delta$ terms, and $\Omega$ is given in Eq. (6); $\mathbf{n}$ represents a unit vector (operator). It is easy to see that $\sigma_n^2=1$, that $\sigma_n$ has eigenvalues $m_n=\pm 1$, and that it commutes with $L^2$, $J_z$, and $N$. This form of the Hamiltonian still resembles the model for a spin-$\frac{1}{2}$ particle in a magnetic field along the $\mathbf{n}$ direction. Therefore, the typical energy eigenvector is $|N,l,J_z,m_n\rangle$, and the corresponding energy eigenvalue is

$$E_{N,l,J_z,m_n}=\left(\frac{g}{4}+\frac{\delta}{2}\right)+\omega_p N+\left(E_{+-}+\frac{g}{2}\right)J_z + \frac{m_n}{2}\left\{\left[g\left(J_z+\frac{m_n}{2}\right)+\delta\right]^2 + 2g^2[l(l+1)-J_z^2+\tfrac{1}{4}]\right\}^{1/2}, \tag{16}$$

which gives all the energy eigenvalues when $N$, $l$, and $J_z$ run through all the values given by Eq. (9) and $m_n=\pm 1$.

## III. OPERATOR SOLUTIONS AND ENERGY EIGENVECTORS

In this section, we calculate the analytical expressions of the evolution operator and the atomic inversion operator, and explicitly express the eigenstates of the energy and orbital angular momentum in terms of Fock states for any allowable values of $J_z$. We begin with the first task.

### A. Dynamics

In this part, we show that it is a simple matter to obtain analytical expressions for the evolution operator and atomic inversion by recognizing the similarity between this model and the one for a spin-$\frac{1}{2}$ particle in a magnetic field; then we explain why there is no longer any dramatic difference, in this model, between the role of a Fock state and a coherent state in the collapse and revival of Rabi oscillations.

The dynamics in Heisenberg's scheme is represented by the transformations $A\equiv A(0)\rightarrow A(t)=U(t)AU^{\dagger}(t)$, where $U(t)$ is the system's evolution operator, and has, in our case, the form

$$U(t)=\exp(iHt)=\exp(iH_0t)U_{\rm int}(t), \tag{17}$$

where

$$U_{\rm int}(t)=\exp(iH_{\rm int}t). \tag{18}$$

Utilizing $H_{\rm int}=\frac{1}{2}\Omega\sigma_y$ and the formula $\exp(if\sigma_y)=\cos(f)+i\sigma_y\sin(f)$, We immediately obtain the analytical expression of the evolution operator as follows:

$$U_{\rm int}(t)=\cos\left(\frac{\Omega t}{2}\right)+i\sigma_y\sin\left(\frac{\Omega t}{2}\right), \tag{19}$$

where $\sigma_y=(L_-\sigma_{+-}+L_+\sigma_{-+})/\sqrt{L^2-J_z^2+\frac{1}{4}}$, and $\Omega=g\sqrt{2(L^2-J_z^2+\frac{1}{4})}$ is the Rabi operator. They commute with each other and both are constants of motion. The atomic inversion operator $\sigma_z(t)=\sigma_{++}(t)-\sigma_{--}(t)$ is also easy to obtain by using the expression of evolution operator and noting that it commutes with Rabi operator and $H_0$, and satisfies $[\sigma_y,\sigma_z]=2i\sigma_x$ and $\sigma_y\sigma_z=-\sigma_z\sigma_y$. The result is

$$\sigma_z(t)=\sigma_z\cos(\Omega t)-\sigma_x\sin(\Omega t), \tag{20}$$

where $\sigma_x=i(L_-\sigma_{-+}-L_+\sigma_{+-})/\sqrt{L^2-J_z^2+\frac{1}{4}}$ $\sigma_x(t)$ is simply obtained by $\sigma_x(t)=i\sigma_z(t)\sigma_y$ as

$$\sigma_x(t)=\sigma_x\cos(\Omega t)+\sigma_z\sin(\Omega t). \tag{21}$$

The population operators are determined by $\sigma_{++}(t)=(1+\sigma_z(t))/2$ and $\sigma_{--}(t)=(1-\sigma_z(t))/2$. The calculation of the transition operator $\sigma_{+-}$ is a little more difficulty since it does not commute with $H_0$ and $\Omega$. However, it is easy to show that $\sigma_{+-}F(H_0)=F(H_0-E_{+-}/2)\sigma_{+-}$ and $G(\Omega)\sigma_{+-}=\sigma_{+-}G(\Omega_+)$ where $F$ and $G$ are arbitrary functions, and $\Omega_+=\Omega_{J_z\rightarrow J_z+1/2}$. Using these commutation relations and the expressions of $U_{\rm int}$ and $\sigma_y$, we arrive at

$$\sigma_{+-}(t)\exp\left(i\frac{E_{+-}t}{2}\right)=\sigma_{+-}\cos\left(\frac{\Omega_+t}{2}\right)\cos\left(\frac{\Omega t}{2}\right) + i\frac{L_+\sigma_{--}}{\sqrt{L^2-J_z^2+\frac{1}{4}}}\sin\left(\frac{\Omega_+t}{2}\right)\cos\left(\frac{\Omega t}{2}\right) - i\frac{L_+\sigma_{++}}{\sqrt{L^2-J_z^2+\frac{1}{4}}}\cos\left(\frac{\Omega_+t}{2}\right)\sin\left(\frac{\Omega t}{2}\right) + \frac{L_+^2\sigma_{-+}}{L^2-J_z^2+\frac{1}{4}}\sin\left(\frac{\Omega_+t}{2}\right)\sin\left(\frac{\Omega t}{2}\right). \tag{22}$$

The operator $L_-$ can be obtained by the relation $L_-(t)=\sqrt{L^2-J_z^2+\frac{1}{4}}(\sigma_y\sigma_{-+}(t)-i\sigma_x(t)\sigma_{+-}(t))$.

The photon number operators satisfy the relations $2n_A=(J_z-\sigma_z(t)/2+N-n_P)$ and $2n_S=(-J_z+\sigma_z(t)/2+N-n_P)$, and $n_P(t)$ relates its initial operator $n_P$ by the relation

$$n_P(t)=\left[\cos\left(\frac{\Omega t}{2}\right)+i\sigma_y\sin\left(\frac{\Omega t}{2}\right)\right] \times n_P\left[\cos\left(\frac{\Omega t}{2}\right)-i\sigma_y\sin\left(\frac{\Omega t}{2}\right)\right].$$

This formula is not very useful if we do not have the explicit relation between Fock states and energy eigenstates, or if we do not know the results of the photon number operator $n_P$ acting on the energy eigenstates. However, things are totally different about the atomic quantities $\sigma_{++}$, $\sigma_{--}$ and $\sigma_{+-}$ as well as the field variables $L_-$ and $L_+$, because their actions on energy eigenstates are known even if we do not know the relations between the Fock states and energy eigenstates. So long as the (initial) state or density operator in (Schrödinger) the Heisenberg picture is given in the energy representation, the dynamical properties of these operators, including their expectation values, can be discussed and determined solely within the energy representation by using the above formulas, without the need to know about Fock states and the relations between the Fock states and energy eigenstates; this does not hold for photon numbers if we do not know such relations. Let us illustrate this point by calculating the atomic inversion. The most general form of a density operator in energy representation is

$$\rho=\sum_{l,J_z,m_y;l',J_z',m_y'}C_{l,J_z,m_y;l',J_z',m_y'}|N,l,J_z,m_y\rangle \times\langle N,l',J_z',m_y'|.$$

Using the formulas in Eq. (24) and $\sigma_x = i\sigma_z\sigma_y$, we obtain

$$\langle N,l',J_z',m_y'|\sigma_z|N,l,J_z,m_y\rangle = \delta_{J_z,J_z'}\delta_{l,l'}\delta_{m_y,-m_y'},$$

$$\langle N,l',J_z',m_y'|\sigma_x|N,l,J_z,m_y\rangle = im_y\delta_{J_z,J_z'}\delta_{l,l'}\delta_{m_y,-m_y'}.$$

Then the atomic inversion $W(t)=\mathrm{Tr}(\sigma_z(t)\rho)$ is easily shown to have the form

$$W(t)=\sum_{l,J_z,m_y} C_{l,J_z,m_y;l,J_z,-m_y}[\cos(\Omega_{lJ_z}t)-im_y\sin(\Omega_{lJ_z}t)],$$

where $\Omega_{lJ_z}=g\sqrt{2[(l+\frac{1}{2})^2-J_z^2]}$. There is no unknown quantities in the expression of the atomic inversion, since the coefficients $C_{l,J_z,m_y;l',J_z',m_y'}$ are supposed to be known (we use the Heisenberg picture, so that the density operator does not vary with time, which corresponds to the initial density operator in Schrödinger picture. In order to obtain the density operator at any time in Schrödinger picture, one has to assume the form of the initial density operator). In the expression of the average pump photon number $\mathrm{Tr}(n_P(t)\rho)$, there is a quantity $\langle N,l',J_z',m_y'|n_P|N,l,J_z,m_y\rangle$ which is unknown if one does not know the relations between the Fock states and energy eigenstates.

Before ending this subsection, we discuss the Rabi frequency $\Omega=g\sqrt{2(L^2-J_z^2+\frac{1}{4})}$. The atomic inversion [the expectation value of the operator $\sigma_z(t)$] is easily seen from the expression of the operator $\sigma_z(t)$ to oscillate, with all its frequencies given by the eigenvalues of the Rabi operator, and determined by those of $L^2$ and $J_z$. The atomic inversion is exactly a periodic function if the system's state is one of the eigenvectors of the Rabi operator ($|l,m\rangle\otimes|\pm\rangle$ or its suitable combinations such as $c_1|l,m\rangle\otimes|+\rangle+c_2|l,m+1\rangle\otimes|-\rangle$). It is noted that the vectors $|n_P,n_S,n_A\rangle\otimes|\pm\rangle$, with $|n_P,n_S,n_A\rangle$ denoting Fock states, are generally not the eigenvectors of the Rabi operator. Their expressions in terms of Rabi operator eigenvectors generally contain many terms corresponding to different oscillation frequencies. The interference of these oscillations with different frequencies will generally lead to collapse and revivals of the Rabi oscillations. The same is true for a coherent state. This is the reason why the Fock initial-field state in this model generally gives the collapse and revivals of Rabi oscillations. In contrast, the direct products of atomic states $|\pm\rangle$ and Fock states are Rabi operator eigenvectors in almost all other models. For instance, the JC model's Rabi operator is $\sqrt{(\Delta/2)^2+g^2[n+(1+\sigma_z)/2]}$ which has an eigenvector $|n\rangle\otimes|\pm\rangle$ or suitable combinations such as $c_1|n\rangle\otimes|+\rangle+c_2|n+1\rangle\otimes|-\rangle$. Therefore, the corresponding atomic inversion is a periodic function if the field part of the system state is given by a Fock state $|n\rangle$, while it manifests collapse and revivals if the field is specified by a coherent state. From the above discussions, we also know that the role of vectors $|l,m\rangle$ in this coupled-channel model replaces the role of Fock states in the JC model. To be more specific, the quantum number $m$ describing $L_z$ in this model is similar to the photon number $n$ in the JC model, which becomes obvious by comparing the two models, since $L_-$ ($L_+$) lowers (raises) $m$ by a unit, while operator $a$ ($a^\dagger$) lowers (raises) photon number $n$ by a unit. This similarity helps to construct the energy eigenvectors in term of vectors $|l,m\rangle$.

### B. Energy eigenstates

According to the above discussions, and using the formulas

$$L_-|l,m\rangle=\sqrt{l(l+1)-m(m-1)}|l,m-1\rangle, \tag{23a}$$

$$L_+|l,m\rangle=\sqrt{l(l+1)-m(m+1)}|l,m+1\rangle, \tag{23b}$$

one easily sees that the energy eigenvectors can be expressed in terms of the eigenvectors of the orbital angular momentum as

$$|N,l,J_z=m+\tfrac{1}{2},m_y=1\rangle = \frac{1}{\sqrt{2}}(|l,m\rangle\otimes|+\rangle+|l,m+1\rangle\otimes|-\rangle), \tag{24a}$$

$$|N,l,J_z=m-\tfrac{1}{2},m_y=-1\rangle = \frac{1}{\sqrt{2}}(|l,m-1\rangle\otimes|+\rangle-|l,m\rangle\otimes|-\rangle), \tag{24b}$$

where the two atomic levels $|\pm\rangle$ satisfy $\sigma_z|\pm\rangle=\pm|\pm\rangle$, $l=N,N-2,\ldots,(N-2\,\mathrm{Int}(N/2))$, and $m=0,\pm1,\pm2,\ldots,\pm l$. The normalization constant $1/\sqrt{2}$ is replaced by 1 as $J_z=\pm(l+\frac{1}{2})$ since $|l,m=\pm(l+1)\rangle=0$. These results can also be obtained by the relations $\sigma_z=\exp(-i(\pi/4)\sigma_x)\sigma_y\exp(i(\pi/4)\sigma_x)$, $\sigma_z|l,m\rangle\otimes|\pm\rangle=\pm|l,m\rangle\otimes|\pm\rangle$, $\exp(i(\pi/4)\sigma_x)=(1+i\sigma_x)/\sqrt{2}$, and $|N,l,J_z=m\pm\frac{1}{2},m_y=\pm1\rangle=\exp(i\pi/4\sigma_x)|l,m\rangle\otimes|\pm\rangle$.

We now establish the relations between the Fock states and the eigenstates of the orbital angular momentum. We construct the eigenvectors $|l,m=-l\rangle\equiv|l,-l\rangle$, and then obtain the rest by the formula [which is obtained by recursively using Eq. (23.b)]

$$|l,m\rangle=\frac{1}{\sqrt{(2l)!}}\sqrt{\frac{(l-m)!}{(l+m)!}}L_+^{l+m}|l,-l\rangle \tag{25}$$

where $m=0,\pm1,\pm2,\ldots,\pm l$. The $(2l+1)$ vectors $|l,m\rangle$ for a fixed $l$ are normalized and mutually orthogonal, which comes only from the theory of angular momentum without using the connection between the field variables and the argular momentum. Suppose

$$|l,-l\rangle=\sum_{k=0}^{n}(-1)^k\alpha_k|2k,l+n-k,n-k\rangle, \tag{26}$$

where $l=N,N-2,\ldots,(N-2\,\mathrm{Int}(N/2))$, $n=(N-l)/2$ is a nonnegative integer, and the coefficients $\alpha_k, k=0,1,\ldots,n$ are determined by $L_-|l,-l\rangle=0$, which gives the recursive relation $\alpha_k\sqrt{(2k+1)(n-k)}=\alpha_{k+1}\sqrt{(2k+2)(l+n-k)}$, $k=0,1,2,\ldots,n-1$. Using this recursive relation, we can obtain that

$$|l,-l\rangle = c_l \sum_{k=0}^{n} (-1)^k \frac{1}{2^k k!} \left( \frac{(2k)!(l+n-k)!}{(n-k)!} \right)^{1/2} \times |2k, l+n-k, n-k\rangle, \quad (27)$$

where $c_l$ is a normalization constant given by

$$c_l = \left[ \sum_{r=0}^{n} \frac{1}{4^r r!^2} \frac{(2r)!(l+n-r)!}{(n-r)!} \right]^{-1/2}. \quad (28)$$

It is seen that all the vectors $|l,m\rangle$ ($l = N, N-2, \ldots, (N-2\,\mathrm{Int}(N/2))$; $m = 0, \pm 1, \pm 2, \ldots, \pm l$) generated by Eqs. (25) and (27) form a orthogonal and normalized set whose number is $(N+1)(N+2)/2$, the number of the Fock states $|n_P, n_S, n_A\rangle$ with a fixed total photon number $N$. They are the explicit relations between the Fock states and the eigenvectors of the orbital angular momentum. Substituting them into Eq. (24), we also obtain the explicit relations between the Fock states and the energy eigenstates. These forms of explicit relations are particularly simple to use when the field part of the system's (initial) state in Heisenberg's (Schrödinger's) scheme is given by a Fock state with $n_S = 0$ and/or $n_P = 0$. To illustrate this, we assume the atom is in its ground state and the fields in a Fock state $|2M,0,0\rangle$, and express this state in terms of energy eigenvectors. It is simple to obtain

$$L_-^l |2M,0,0\rangle = 2^{1/2} \left( \frac{(2M)!\,l!}{(2M-l)!} \right)^{1/2} |2M-l, l, 0\rangle, \quad (29)$$

$$\langle l,0|2M,0,0\rangle = \frac{1}{\sqrt{(2l)!}} \langle l,-l|L_+^l|2M,0,0\rangle = (-1)^{M-(1/2)} \frac{2^{l-M}}{\sqrt{(2l)!}} \frac{l!\sqrt{(2M)!}}{\left(M-\frac{l}{2}\right)!} c_l. \quad (30)$$

Using these expressions and Eq. (24), we arrive at

$$|2M,0,0\rangle \otimes |-\rangle = \sum_{k=0}^{M} (-1)^{M-k} \frac{2^{2k-M}}{\sqrt{(4k)!}} \frac{(2k)!\sqrt{(2M)!}}{(M-k)!} \times c_{2k} (|m_y = 1\rangle - |m_y = -1\rangle), \quad (31)$$

where $|m_y = \pm 1\rangle \equiv |N = 2M, l = 2k, J_z = -\frac{1}{2}, m_y = \pm 1\rangle$ represent energy eigenstates, and $c_{2k}$ is given by Eq. (28). This simple example also serves the purpose of comparison, as will be seen below.

Next we go one step further to calculate the concrete forms of vectors $|l,m\rangle$ without expressing them in terms of the lowering operator acting on $|l,-l\rangle$. Using the property of a creation operator $a^{\dagger n}|0\rangle = \sqrt{n!}|n\rangle$, we express $|l,-l\rangle$ in a compact and convenient form

$$|l,-l\rangle = \frac{c_l}{n!} a_S^{\dagger l} (a_A^\dagger a_S^\dagger - \tfrac{1}{2} a_P^{\dagger 2})^n |0,0,0\rangle, \quad (32)$$

where $l = N, N-2, \ldots, (N-2\,\mathrm{Int}(N/2))$, $n = (N-l)/2$, and $|0,0,0\rangle$ is the field state without any photon. Denoting $L_+ = \sqrt{2}A, A = (a_S a_P^\dagger + a_A^\dagger a_P)$, and noting that $A$ commutes with $(a_A^\dagger a_S^\dagger - \frac{1}{2} a_P^{\dagger 2})$, we find that we need to calculate quantity $A^{l+m} a_S^{\dagger l} |0,0,0\rangle$, which is transformed to calculate the coefficient of $l$th power of $-\alpha$ in the expression of $A^{l+m} \exp(-\alpha a_S^\dagger)|0,0,0\rangle$. It is easy to show $A^{l+m} \exp(-\alpha a_S^\dagger) = \exp(-\alpha a_S^\dagger)(A - \alpha a_P^\dagger)^{l+m}$ by noting $\exp(\alpha a_S^\dagger) A \exp(-\alpha a_S^\dagger) = (A - \alpha a_P^\dagger)$. The calculation of $(A - \alpha a_P^\dagger)^{l+m}|0,0,0\rangle$ is very similar to calculating the eigenfunctions of the harmonic oscillator in quantum mechanics. It is noted that $A|0,0,0\rangle = 0$, $[A, a_P^{\dagger n}] = n a_A^\dagger a_P^{\dagger n-1}$ which means that $A$ resembles $a_A^\dagger d/dx$ and $a_P^\dagger$, similar to $x$ in calculating $(A - \alpha a_P^\dagger)^{l+m}|0,0,0\rangle$. While doing so, $a_A^\dagger$ can be treated as a $c$ number, since it commutes with $A$ and $a_P^\dagger$ Therefore, we obtain $(A - \alpha a_P^\dagger)^{l+m}|0,0,0\rangle = W|0,0,0\rangle$, where $W = (\beta(d/dx) - \alpha x)^{l+m}_{x = a_P^\dagger, \beta = a_A^\dagger}$. Obviously, we have

$$\left( \beta \frac{d}{dx} - \alpha x \right)^{l+m} = \beta^{l+m} \left[ \exp\left( \frac{\alpha}{2\beta} x^2 \right) \frac{d}{dx} \exp\left( -\frac{\alpha}{2\beta} x^2 \right) \right]^{l+m} = \beta^{l+m} \exp\left( \frac{\alpha}{2\beta} x^2 \right) \frac{d^{l+m}}{dx^{l+m}} \exp\left( -\frac{\alpha}{2\beta} x^2 \right) = (-1)^{l+m} \left( \left( \frac{\alpha\beta}{2} \right)^{1/2} \right)^{l+m} H_{l+m}\left( \frac{\alpha}{2\beta} x \right)^{1/2}, \quad (33)$$

where use has been made of the expression [14] of the Hermitian polynomial of order, $H_n(x) = (-1)^n \exp(x^2)[d^n \exp(-x^2)]/dx^n$ which has the series form [14]

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k} \quad (34)$$

where $[n/2] \equiv \mathrm{Int}(n/2)$ or $[n/2] = n/2$ for even $n$ and $[n/2] = (n-1)/2$ for odd $n$. From the above discussions, we obtain

$$A^{l+m} a_S^{\dagger l} |0,0,0\rangle = l!(l+m)! \sum_{k=k_0}^{[(l+m)/2]} \times \frac{a_S^{\dagger k-m} a_A^{\dagger k} a_P^{\dagger l+m-2k}}{2^k k!(l+m-2k)!(k-m)!} |0,0,0\rangle, \quad (35)$$

where $k_0 = m$ for $m \geq 0$ and $k_0 = 0$ for any negative $m$, or $k_0 = \max(m,0)$. It is now a simple task to express $|l,m\rangle$ in terms of Fock states by using Eqs. (25), (32), and (35). The final results are as follows:

$$|l,m\rangle=\sum_{k=0}^{(N-|m|)/2} B_{lmk}^{(e)}\left|2k,\frac{N-m}{2}-k,\frac{N+m}{2}-k\right\rangle \quad \text{for even } (l+m), \tag{36a}$$

$$|l,m\rangle=\sum_{k=0}^{(N-|m|-1)/2} B_{lmk}^{(o)}\left|2k+1,\ \frac{N-m-1}{2}-k,\ \frac{N+m-1}{2}-k\right\rangle \quad \text{for odd } (l+m), \tag{36b}$$

where

$$B_{lmk}^{(e)}=2^k\left(\frac{(l+m)!(l-m)!}{(2l)!}\right)^{1/2} l!\,c_l,$$

$$\sum_{r=r_{\min}}^{r_{\max}}(-1)^r\frac{\left[(2k)!\left(\frac{N-m}{2}-k\right)!\left(\frac{N+m}{2}-k\right)!\right]^{1/2}}{4^r r!\left(\frac{N-l}{2}-r\right)!\left(\frac{l-m}{2}-k+r\right)!\left(\frac{l+m}{2}-k+r\right)!(2k-2r)!}, \tag{37a}$$

$$B_{lmk}^{(o)}=2^k\sqrt{2}\left(\frac{(l+m)!(l-m)!}{(2l)!}\right)^{1/2} l!\,c_l,$$

$$\sum_{r=r_{\min}}^{r_{\max}}(-1)^r\frac{\left[(2k+1)!\left(\frac{N-m-1}{2}-k\right)!\left(\frac{N+m-1}{2}-k\right)!\right]^{1/2}}{4^r r!\left(\frac{N-l}{2}-r\right)!\left(\frac{l-m-1}{2}-k+r\right)!\left(\frac{l+m-1}{2}-k+r\right)!(2k-2r+1)!}, \tag{37b}$$

where $r_{\max}=\min(k,(N-l)/2)$, $r_{\min}=\max(0,k-[(l-|m|)/2])$, and $c_l$ is given by Eq. (28). Equation (36) gives all the eigenvectors of the orbital angular momentum explicitly expressed in terms of the Fock states as $l$ and $m$ run through the values $l=N,N-2,\ldots,(N-2\,\mathrm{Int}(N/2))$, where $\mathrm{Int}(N/2)=N/2$ for even $N$ and $\mathrm{Int}(N/2)=(N-1)/2$ for odd $N$, and $m=0,\pm1,\pm2,\ldots,\pm l$. Substituting Eq. (36) into Eq. (24), we also obtain the concrete forms of energy eigenstates in terms of Fock states for any allowable $J_z$.

In their paper, Wang, Puri, and Eberly did not establish the relations between the eigenvectors of the orbital angular momentum and the Fock states [1], but they [1] derived the explicit relations between the energy eigenstates and Fock states for the special case of $J_z=-\frac{1}{2}$. It can be seen that the first and second terms of their Eq. (3.2) are $|l,m=0\rangle\otimes|-\rangle/\sqrt{2}$ and $|l,m=-1\rangle\otimes|+\rangle/\sqrt{2}$. However, we cannot compare their results with ours for the special case of $J_z=-\frac{1}{2}$ at present, since factorials of a negative integer appear in the denominators in their expressions of expansion coefficients [their Eq. (3.6)] while we do not know their convention on the factorial of a negative integer. In Appendix B, we present expressions of the eigenvectors of the Hamiltonian and the orbital angular momentum in terms of Fock states for a few photons (the total photon number $N=1$, 2, and 3) by using Eqs. (24) and (36). It is easily checked directly by the expression of Hamiltonian that the vectors $|N,l,J_z,m_y\rangle$ in Appendix B are indeed the energy eigenvectors.

## IV. SUMMARY

In this paper, we presented a very simple algebraic method to investigate the eigenvalue problem and dynamics of the coupled-channel cavity QED model with three nondegenerate quantized field modes proposed and partially solved by Wang, Puri, and Eberly. We have shown that by establishing the similarity between this model and the one for a spin-$\frac{1}{2}$ particle in the presence of a magnetic field, we can quickly obtain the system's energy eigenvalues and Rabi operator, and its eigenvalues, evolution, and atomic inversion operators without having to know the explicit relation between the Fock states and the energy eigenstates. We have also explicitly expressed the energy eigenstates in terms of Fock states not only for the special case of $J_z=-\frac{1}{2}$ as in previous studies [1], but also those for any allowable values of $J_z$. With the energy eigenvalues and eigenstates known for the general situations, we can study the model's dynamical and statistical properties of the atomic and field variables such as the time evolution of atomic inversion, average photon numbers, self-made, and intermode correlations for any allowable $J_z$.

In their work, Wang, Puri, and Eberly established an elegant connection between the field variables and an orbital angular momentum, but were not able to obtain a relation between the Fock states and eigenvectors of the orbital angular momentum. We have gone one step further to establish such relations, and thus have finally given a complete connection between the field modes and the orbital angular momentum.

According to our results, we find that the model still shows a considerable similarity to the JC model, although, unlike the JC model, it manifests a chain structure pointed out first by Wang, Puri, and Eberly. For instance, the model displays many singlets [such as $J_z=\pm(l+1/2)$] representing states without effective coupling between the field modes

and the atom, very similar to the ground state in the JC model, and it has many more doublets, with the energy difference of each doublet given by one of the eigenvalues of the Rabi operator, while these eigenvalues also determine all the oscillating frequencies of the atomic inversion. These properties still resemble those of the doublets in the JC model. We have also explained why the Fock initial-field state in this model generally gives the collapse and revival phenomena in the Rabi oscillations of the atomic inversion.

The method of obtaining the eigenvalues and the Rabi and evolution operators, by establishing a similarity to the model describing a spin-$\frac{1}{2}$ particle in the presence of a magnetic field, is also of theoretical interest in its own right, since it provides a systematic and effective way to obtain these quantities quickly, not only for this particular model but also for any other solvable two-level models including those describing a multilevel atom interacting with multimode quantum fields, so long as they can be transformed into an effective two-level problem by either an exact transformation [11] or approximation methods such as the adiabatic elimination.

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## APPENDIX A

In this appendix, we prove the conclusion that the allowable values of an angular quantum number are $l=N,N-2,\ldots,(N-2\,\mathrm{Int}(N/2))$ for a fixed $N$, where $\mathrm{Int}(N/2)=N/2$ for even $N$ and $\mathrm{Int}(N/2)=(N-1)/2$ for odd $N$, and for any allowable value $l$, $m=0,\pm1,\pm2,\ldots,\pm l$.

First, we point out that the operators $N$, $L^2$, and $L_z$ are mutually commutative, and hence have common eigenvectors forming a complete set. Its typical member $|N,l,m\rangle$ is denoted as $|l,m\rangle$ throughout this paper for simplicity. Let $V$ denote the space spanned by the Fock states $|n_P,n_S,n_A\rangle$ with a fixed N $(=n_P+n_S+n_A)$. Obviously, it is also spanned by the vectors $|l,m\rangle$ with all allowable values of $l$ and $m$ for the fixed $N$, and its dimension is $(N+1)(N+2)/2$. In Sec. III, we explicitly constructed $(N+1)(N+2)/2$ independent vectors $|l,m\rangle$, with the above specified $l$ and $m$, all of which satisfy the restriction of a fixed $N$. Therefore, the above conclusion has actually been proved. However, we now present another kind of proof which does not need the concrete forms of $|l,m\rangle$ in terms of the Fock states. The proof of the allowable $m$ is simple. Any two of $(2l+1)$ vectors $|l,m\rangle$ $(m=0,\pm1,\pm2,\ldots,\pm l)$ are related by operator $L_+$ (or $L_-$), which does not change $N$. All of them will satisfy the condition of a fixed $N$ if one of them does.

Hence we consider the case of $m=-l$ to decide the alowable $l$. Using the expression for $L_z$ and $m=-l$ , we find that $2n_A=N-l-n_P$ and $2n_S=N+l-n_P$, implying $l\leqslant N$, and that $(N-l)$ and $n_P$ are either both even or both odd. An odd $(N-l)$ is impossible if we can show that $|\psi\rangle=0$ if $L_-|\psi\rangle=0$, where $|\psi\rangle$ contains only the Fock states of odd $n_P$ since $|l,m=-l\rangle$ satisfies $L_-|l,m=-l\rangle=0$ and is not a zero vector. We now prove this. Expanding $|\psi\rangle$ in terms of the Fock states as $|\psi\rangle=c|1,(N+l-1)/2,(N-l-1)/2\rangle+|\phi\rangle$, where $|\phi\rangle$ contains terms with $n_P\geqslant3$, we see that $L_-|\psi\rangle=0$ gives $c=0$ because the $a_Pa_S^\dagger$ term in $L_+$ causes the $c$ term to become a term with $n_P=0$, while $L_-|\phi\rangle$ contains only terms with $n_P\geqslant2$. By the same argument, we can show that $|\psi\rangle$ contains no term with $n_P=3$. Repeating this process, we finally prove $|\psi\rangle=0$. The last question is whether all the values $l=N,N-2,\ldots,(N-2\,\mathrm{Int}(N/2))$ can be taken. The answer is yes, since the total number of independent vectors $|l,m\rangle$ for these values of $l$ is $\Sigma(2l+1)=(N+1)(N+2)/2$ where summation is over all the values specified above. This would lead to the wrong conclusion that the space $V$ cannot be spanned by all the allowable $|l,m\rangle$ if some of the $l$ values in this range were not allowable. We have, now, finished our proof.

## APPENDIX B

In this appendix, we give the eigenvectors of the Hamiltonian and the orbital angular momentum for total photon number $N=1$, 2, and 3. Using Eq. (36) [or Eqs. (25) and (27)], we obtain the expressions of $|N,l,m\rangle$ in terms of Fock states $|n_P,n_S,n_A\rangle$ as follows (note that we use $|l,m\rangle$ for simplicity in the main text, while we use $|N,l,m\rangle$ here since we consider different values of $N$ in this appendix):

$$|N=1,l=1,m=0\rangle=|1,0,0\rangle, \tag{B1a}$$

$$|N=1,l=1,m=-1\rangle=|0,1,0\rangle,$$

$$|N=1,l=1,m=1\rangle=|0,0,1\rangle, \tag{B1b}$$

$$|N=2,l=2,m=-2\rangle=|0,2,0\rangle,$$

$$|N=2,l=2,m=2\rangle=|0,0,2\rangle, \tag{B2a}$$

$$|N=2,l=2,m=-1\rangle=|1,1,0\rangle,$$

$$|N=2,l=2,m=1\rangle=|1,0,1\rangle, \tag{B2b}$$

$$|N=2,l=2,m=0\rangle=\sqrt{\tfrac{2}{3}}|2,0,0\rangle+\sqrt{\tfrac{1}{3}}|0,1,1\rangle, \tag{B2c}$$

$$|N=2,l=0,m=0\rangle=\sqrt{\tfrac{2}{3}}|0,1,1\rangle-\sqrt{\tfrac{1}{3}}|2,0,0\rangle, \tag{B3}$$

$$|N=3,l=1,m=-1\rangle=\frac{2}{\sqrt{5}}|0,2,1\rangle-\frac{1}{\sqrt{5}}|2,1,0\rangle, \tag{B4a}$$

$$|N=3,l=1,m=0\rangle=\sqrt{\tfrac{2}{5}}|1,1,1\rangle-\sqrt{\tfrac{3}{5}}|3,0,0\rangle, \tag{B4b}$$

$$|N=3,l=1,m=1\rangle=\frac{2}{\sqrt{5}}|0,1,2\rangle-\frac{1}{\sqrt{5}}|2,0,1\rangle, \tag{B4c}$$

$$|N=3,l=3,m=-3\rangle=|0,3,0\rangle,$$

$$|N=3,l=3,m=3\rangle=|0,0,3\rangle, \tag{B5a}$$

$$|N=3,l=3,m=-2\rangle=|1,2,0\rangle,$$

$$|N=3,l=3,m=2\rangle=|1,0,2\rangle, \tag{B5b}$$

$$|N=3,l=3,m=-1\rangle=\frac{2}{\sqrt{5}}|2,1,0\rangle+\frac{1}{\sqrt{5}}|0,2,1\rangle, \quad \text{(B5c)}$$

$$|N=3,l=3,m=0\rangle=\sqrt{\tfrac{2}{5}}|3,0,0\rangle+\sqrt{\tfrac{3}{5}}|1,1,1\rangle, \quad \text{(B5d)}$$

$$|N=3,l=3,m=1\rangle=\frac{1}{\sqrt{5}}|0,1,2\rangle+\frac{2}{\sqrt{5}}|2,0,1\rangle. \quad \text{(B5e)}$$

Using the above equations and Eq. (24), one can express the energy eigenstates in terms of Fock states $|n_P,n_S,n_A\rangle$ and the atomic states $|\pm\rangle$ as follows:

$$|N=1,l=1,J_z=\tfrac{3}{2},m_y=1\rangle=|0,0,1\rangle\otimes|+\rangle, \quad \text{(B6a)}$$

$$|N=1,l=1,J_z=-\tfrac{3}{2},m_y=-1\rangle=-|0,1,0\rangle\otimes|-\rangle, \quad \text{(B6b)}$$

$$|N=1,l=1,J_z=\tfrac{1}{2},m_y=\pm1\rangle = \frac{1}{\sqrt{2}}|1,0,0\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{2}}|0,0,1\rangle\otimes|-\rangle, \quad \text{(B6c)}$$

$$|N=1,l=1,J_z=-\tfrac{1}{2},m_y=\pm1\rangle = \frac{1}{\sqrt{2}}|0,1,0\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{2}}|1,0,0\rangle\otimes|-\rangle, \quad \text{(B6d)}$$

$$|N=2,l=2,J_z=\tfrac{5}{2},m_y=1\rangle=|0,0,2\rangle\otimes|+\rangle, \quad \text{(B7a)}$$

$$|N=2,l=2,J_z=-\tfrac{5}{2},m_y=-1\rangle=-|0,2,0\rangle\otimes|-\rangle, \quad \text{(B7b)}$$

$$|N=2,l=2,J_z=\tfrac{3}{2},m_y=\pm1\rangle = \frac{1}{\sqrt{2}}|1,0,1\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{2}}|0,0,2\rangle\otimes|-\rangle, \quad \text{(B7c)}$$

$$|N=2,l=2,J_z=\tfrac{1}{2},m_y=\pm1\rangle = \frac{1}{\sqrt{3}}|2,0,0\rangle\otimes|+\rangle+\frac{1}{\sqrt{6}}|0,1,1\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{2}}|1,0,1\rangle\otimes|-\rangle, \quad \text{(B7d)}$$

$$|N=2,l=2,J_z=-\tfrac{1}{2},m_y=\pm1\rangle = \frac{1}{\sqrt{2}}|1,1,0\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{3}}|2,0,0\rangle\otimes|-\rangle\pm\frac{1}{\sqrt{6}}|0,1,1\rangle\otimes|-\rangle, \quad \text{(B7e)}$$

$$|N=2,l=2,J_z=-\tfrac{3}{2},m_y=\pm1\rangle = \frac{1}{\sqrt{2}}|0,2,0\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{2}}|1,1,0\rangle\otimes|-\rangle, \quad \text{(B7f)}$$

$$|N=2,l=0,J_z=\tfrac{1}{2},m_y=1\rangle = \frac{2}{\sqrt{3}}|0,1,1\rangle\otimes|+\rangle-\frac{1}{\sqrt{3}}|2,0,0\rangle\otimes|+\rangle, \quad \text{(B8a)}$$

$$|N=2,l=0,J_z=-\tfrac{1}{2},m_y=-1\rangle = -\frac{2}{\sqrt{3}}|0,1,1\rangle\otimes|-\rangle+\frac{1}{\sqrt{3}}|2,0,0\rangle\otimes|-\rangle, \quad \text{(B8b)}$$

$$|N=3,l=1,J_z=\tfrac{3}{2},m_y=1\rangle = \frac{2}{\sqrt{5}}|0,1,2\rangle\otimes|+\rangle-\frac{1}{\sqrt{5}}|2,0,1\rangle\otimes|+\rangle, \quad \text{(B9a)}$$

$$|N=3,l=1,J_z=-\tfrac{3}{2},m_y=-1\rangle = \frac{1}{\sqrt{5}}|2,1,0\rangle\otimes|-\rangle-\frac{2}{\sqrt{5}}|0,2,1\rangle\otimes|-\rangle, \quad \text{(B9b)}$$

$$|N=3,l=1,J_z=\tfrac{1}{2},m_y=\pm1\rangle = \frac{1}{\sqrt{5}}|1,1,1\rangle\otimes|+\rangle-\sqrt{\tfrac{3}{10}}|3,0,0\rangle\otimes|+\rangle\pm\sqrt{\tfrac{2}{5}}|0,1,2\rangle\otimes|-\rangle \mp\frac{1}{\sqrt{10}}|2,0,1\rangle\otimes|-\rangle, \quad \text{(B9c)}$$

$$|N=3,l=1,J_z=-\tfrac{1}{2},m_y=\pm1\rangle = \sqrt{\tfrac{2}{5}}|0,2,1\rangle\otimes|+\rangle-\frac{1}{\sqrt{10}}|2,1,0\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{5}}|1,1,1\rangle\otimes|-\rangle \mp\sqrt{\tfrac{3}{10}}|3,0,0\rangle\otimes|-\rangle, \quad \text{(B9d)}$$

$$|N=3,l=3,J_z=\tfrac{7}{2},m_y=1\rangle=|0,0,3\rangle\otimes|+\rangle, \quad \text{(B10a)}$$

$$|N=3,l=3,J_z=-\tfrac{7}{2},m_y=-1\rangle=-|0,3,0\rangle\otimes|-\rangle, \quad \text{(B10b)}$$

$$|N=3,l=3,J_z=\tfrac{5}{2},m_y=\pm1\rangle = \frac{1}{\sqrt{2}}|1,0,2\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{2}}|0,0,3\rangle\otimes|-\rangle, \quad \text{(B10c)}$$

$$|N=3,l=3,J_z=\tfrac{3}{2},m_y=\pm1\rangle = \sqrt{\tfrac{2}{5}}|2,0,1\rangle\otimes|+\rangle+\sqrt{\tfrac{1}{10}}|0,1,2\rangle\otimes|+\rangle\pm\sqrt{\tfrac{1}{2}}|1,0,2\rangle\otimes|-\rangle, \quad \text{(B10d)}$$

$$|N=3,l=3,J_z=\tfrac{1}{2},m_y=\pm1\rangle = \frac{1}{\sqrt{5}}|3,0,0\rangle\otimes|+\rangle+\sqrt{\tfrac{3}{10}}|1,1,1\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{10}}|0,1,2\rangle\otimes|-\rangle \pm\sqrt{\tfrac{2}{5}}|2,0,1\rangle\otimes|-\rangle, \quad \text{(B10e)}$$

$$|N=3,l=3,J_z=-\tfrac{1}{2},m_y=\pm1\rangle = \sqrt{\tfrac{2}{5}}|2,1,0\rangle\otimes|+\rangle+\sqrt{\tfrac{1}{10}}|0,2,1\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{5}}|3,0,0\rangle\otimes|-\rangle \pm\sqrt{\tfrac{3}{10}}|1,1,1\rangle\otimes|-\rangle, \quad \text{(B10f)}$$

$$|N=3,l=3,J_z=-\tfrac{3}{2},m_y=1\rangle$$

$$=\sqrt{\tfrac{1}{2}}|1,2,0\rangle\otimes|+\rangle\pm\sqrt{\tfrac{2}{5}}|2,1,0\rangle\otimes|-\rangle\pm\frac{1}{\sqrt{10}}|0,2,1\rangle\otimes|-\rangle, \quad \text{(B10g)}$$

$$|N=3,l=3,J_z=-\tfrac{5}{2},m_y=\pm 1\rangle$$

$$=\frac{1}{\sqrt{2}}|0,3,0\rangle\otimes|+\rangle\pm\frac{1}{\sqrt{2}}|1,2,0\rangle\otimes|-\rangle. \quad \text{(B10h)}$$

Note that $(N+1)(N+2)$ is the number of all the energy eigenvectors for a given $N$. Equation (B6) gives six $(2\times3)$ energy eigenvectors for $N=1$. Equations (B6) and (B7) give 12 $(3\times4)$ energy eigenvectors for $N=2$, while Eqs. (B6) and (B7) give 20 $(4\times5)$ energy eigenvectors for $N=3$. Therefore, the total number of the energy eigenvectors in this appendix is 38. It is easily checked by using the original Hamiltonian in Eq. (1) that they are correct energy eigenvectors, with the corresponding eigenvalues described by Eq. (8) in the main text.


[1] Liwei Wang, R. R. Puri, and J. H. Eberly, Phys. Rev. A **46**, 7192 (1992).

[2] H.-I. Yoo and J. H. Eberly, Phys. Rep. **118**, 239 (1985).

[3] S. M. Barnett, P. Filipwicz, J. Javanainen, P. L. Knight, and P. Meystre, in *Frontiers in Quantum Optics*, edited by E. R. Pike and S. Sarkar (Hilger, London, 1986), p. 485; E. A. Hinds, Adv. At. Mol. Opt. Phys. **28**, 237 (1990); P. Meystre, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1992), Vol. 30.

[4] B. Buck and C. V. Sukumar, Phys. Lett. **81A**, 132 (1980); C. V. Sukumar and B. Buck, *ibid.* **83A**, 211 (1983).

[5] P. L. Knight, Phys. Scr. **T12**, 51 (1986); S. J. D. Phoenix and P. L. Knight, J. Opt. Soc. Am. **B 7**, 116 (1990).

[6] C. C. Gerry and J. H. Eberly, Phys. Rev. A **42**, 6805 (1990).

[7] D. A. Cardimona, V. Kovanis, M. P. Sharma, and A. Gavrielides, Phys. Rev. A **42**, 3710 (1991).

[8] Fam Le Kien, G. M. Meyer, M. O. Scully, H. Walther, and S. Y. Zhu, Phys. Rev. A **49**, 1367 (1994).

[9] X. Li and N. Bei, Phys. Lett. **101A**, 169 (1984); N. N. Bogolubov, Jr., Fam Le Kien, and A. S. Shumovsky, *ibid.* **101A**, 201 (1984); A. S. Shumovsky, E. I. Aliskenderov, Fam Le Kien, and N. D. Vinh, J. Phys. A **19**, 3607 (1986).

[10] M. Alexanian and S. K. Bose, Phys. Rev. A **52**, 2218 (1995).

[11] Y. Wu, Phys. Rev. A **54**, 1586 (1996).

[12] Y. Wu and A. D. Stone (unpublished).

[13] C. K. Law and J. H. Eberly, Phys. Rev. A **47**, 3195 (1993).

[14] L. C. Andrews, *Special Functions for Engineers and Applied Mathematicians* (MacMillan, New York, 1985), Chap. 5.