

Nonclassical interferometry with intelligent light

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We study the phase sensitivity of SU(2) and SU(1,1) interferometers fed by two-mode field states which are intelligent states for Hermitian generators of the SU(2) and SU(1,1) groups, respectively. Intelligent states minimize uncertainty relations and this makes possible an essential reduction of the quantum noise in interferometers. Exact closed expressions for the minimum detectable phase shift are obtained in terms of the Jacobi polynomials. These expressions are compared with results for some conventional input states, and some known results for the squeezed input states are reviewed. It is shown that the phase sensitivity for an interferometer that employs squeezing-producing active devices (such as four-wave mixers) should be analyzed in two regimes: (i) fixed input state and variable interferometer, and (ii) fixed interferometer and variable input state. The behavior of the phase sensitivity is essentially different in these two regimes. The use of the SU(2) intelligent states allows us to achieve a phase sensitivity of order $1/\bar{N}$ (where \bar{N} is the total number of photons passing through the phase shifters of the interferometer) without adding four-wave mixers. This avoids the duality in the behavior of the phase sensitivity that occurs for the squeezed input. On the other hand, the SU(1,1) intelligent states have the property of achieving the phase sensitivity of order $1/\bar{N}$ in both regimes. [S1050-2947(96)04311-9]

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I. INTRODUCTION

Much attention has been recently paid to the improvement of measurement accuracy of interferometers, because this problem is of great importance in many areas of experimental physics. A very promising way to reduce quantum fluctuations in interferometers is based on the use of input light fields prepared in special quantum states. Therefore, with further development of technology, high-accuracy interferometry seems to have become one of the most important applications of nonclassical photon states whose properties are now extensively studied in the quantum optics literature.

The first steps in this area were taken by Caves [1] and Bondurant and Shapiro [2], who showed that the use of squeezed light can reduce the quantum noise in interferometers [3]. Yurke, McCall, and Klauder [4] used powerful group-theoretic methods for the study of interferometers employing passive and active optical devices. The interferometers considered in [1,2] employ passive lossless devices, such as beam splitters. Yurke, McCall, and Klauder [4] showed that such interferometers can be characterized by the SU(2) group. They also introduced a class of interferometers which employ active lossless devices, such as four-wave mixers, and are characterized by the SU(1,1) group. The actual problem of high-accuracy interferometry is the improvement of the phase sensitivity, i.e., the optimization of the minimum detectable phase shift $\delta\phi$ for a given mean total number \bar{N} of photons passing through phase shifters. This problem arises because of the back-action effect of the radiation pressure. It was shown [4] that SU(2) interferometers

can achieve a phase sensitivity $\delta\phi \sim 1/\bar{N}$ provided that light entering the input ports is prepared in a two-mode squeezed state. SU(1,1) interferometers can achieve this sensitivity even when the vacuum fluctuations enter the input ports [4]. Holland and Burnett [5] have considered the reduction of the uncertainty in the relative quantum phase of two field modes propagating in an SU(2) interferometer fed by two Fock states with equal numbers of photons. They considered [5] the specific ‘‘reduced’’ situation of the measurement with the sensitivity measure different from that used in Ref. [4].

In a separate line of research, considerable efforts have been devoted during the past few years to generalize the idea of squeezing to the SU(2) and SU(1,1) Lie groups. The usual squeezed states are the generalized coherent states of SU(1,1) [6], i.e., they are produced by the action of the group elements on the extreme state of the group representation Hilbert space. Another interesting class of states which has been considered is the class of the so-called intelligent states [7], which minimize the uncertainty relations for the Hermitian generators of the group. Squeezing properties of the SU(2) and SU(1,1) intelligent states have been widely discussed in the literature [8–20]. Recently, Nieto and Truax [14] showed that a generalization of squeezed states for an arbitrary dynamical symmetry group leads to the intelligent states for the group generators. Connections between the concepts of squeezing and intelligence were further investigated by Trifonov [17]. It turns out that the intelligent states for two Hermitian operators can provide an arbitrarily strong squeezing in either of these observables [17]. Some schemes for the experimental production of the SU(2) and SU(1,1) intelligent states in nonlinear optical processes have been suggested recently by a number of authors [12,16,19,21]. The most recent scheme, developed by Luis and Peřina [21], is of remarkable physical elegance and conceptual clarity and seems to be technically realizable.

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The group-theoretic analysis of interferometers and the group-theoretic generalization of squeezing (i.e., intelligence) were brought together by Hillery and Mlodinow [22], who proposed to use intelligent states of the two-mode light field for increasing the precision of interferometric measurements. They derived [22] approximate results for the phase sensitivity of an SU(2) interferometer fed with the SU(2) intelligent states. The possibility to improve further the accuracy of SU(1,1) interferometers by using specially prepared input states has been also studied recently [23]. It was shown [23] that the use of two-mode SU(1,1) coherent states which are simultaneously the SU(1,1) intelligent states can improve the measurement accuracy when the photon-number difference between the modes is large.

In the present work we consider in detail both SU(2) and SU(1,1) interferometers whose input ports are fed with intelligent light. We use powerful analytic methods that employ representations of intelligent states in the generalized coherent-state bases. Thus we are able to obtain exact analytic expressions for the phase sensitivity and examine them in various limits. These results are compared with those obtained in the cases when the input field is prepared in the usual coherent state, in the generalized coherent state and in the squeezed state. We show that the use of squeezing-producing active devices (such as four-wave mixers) introduces a duality in the behavior of the phase sensitivity. For example, when the squeezed input states are used, the interferometer can be operated in two regimes: with variable squeezing parameter and fixed coherent amplitude, and vice versa. The regime of variable squeezing leads to the phase sensitivity $\delta\phi \sim 1/\bar{N}$, whereas the technically preferable regime of variable coherent amplitude gives only $\delta\phi \sim 1/\bar{N}^{1/2}$ (the standard noise limit). The use of the SU(2) intelligent states avoids this dual behavior and leads to the phase sensitivity $\delta\phi \sim 1/\bar{N}$ without adding a four-wave mixer to the interferometer. The SU(1,1) intelligent states also allow us to obtain a significant improvement of the measurement accuracy. These states exhibit a very specific behavior providing phase sensitivity of order $1/\bar{N}$ in the two regimes: variable interferometer, and variable input state. We emphasize that the optimization of the phase sensitivity by the intelligent input states is a consequence of their remarkable squeezing properties.

II. SU(2) INTERFEROMETERS WITH CONVENTIONAL INPUT STATES

A. The interferometer

An SU(2) interferometer is described schematically in Fig. 1. Two light beams represented by the mode annihilation operators a_1 and a_2 enter the first beam splitter BS1. After leaving BS1, the beams accumulate phase shifts ϕ_1 and ϕ_2 , respectively, and then they enter the second beam splitter BS2. The photons leaving the interferometer are counted by detectors D1 and D2.

For the analysis of such an interferometer it is convenient to consider the Hermitian operators

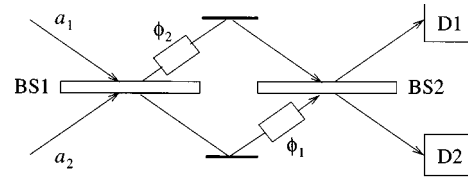


FIG. 1. An SU(2) interferometer. Two light modes a_1 and a_2 are mixed by beam splitter BS1, accumulate phase shifts ϕ_1 and ϕ_2 , respectively, and then they are again mixed by beam splitter BS2. The photons in the output modes are counted by detectors D1 and D2.

$$\begin{aligned} J_1 &= \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1), \\ J_2 &= \frac{1}{2i}(a_1^\dagger a_2 - a_2^\dagger a_1), \\ J_3 &= \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2). \end{aligned} \quad (2.1)$$

These operators form the two-mode boson realization of the su(2) Lie algebra:

$$\begin{aligned} [J_1, J_2] &= iJ_3, \\ [J_2, J_3] &= iJ_1, \\ [J_3, J_1] &= iJ_2. \end{aligned} \quad (2.2)$$

It is also useful to introduce the raising and lowering operators

$$\begin{aligned} J_+ &= J_1 + iJ_2 = a_1^\dagger a_2, \\ J_- &= J_1 - iJ_2 = a_2^\dagger a_1. \end{aligned} \quad (2.3)$$

The Casimir operator for any unitary irreducible representation of SU(2) is a constant

$$J^2 = J_1^2 + J_2^2 + J_3^2 = j(j+1), \quad (2.4)$$

and a representation of SU(2) is determined by a single number j that acquires discrete positive values $j = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. By using the operators of Eq. (2.1), one gets

$$J^2 = \frac{N}{2} \left(\frac{N}{2} + 1 \right), \quad (2.5)$$

where

$$N = a_1^\dagger a_1 + a_2^\dagger a_2 \quad (2.6)$$

is the total number of photons entering the interferometer. We see that N is an SU(2) invariant related to the index j via $j = N/2$. The representation Hilbert space is spanned by the complete orthonormal basis $|j, m\rangle$ ($m = -j, -j+1, \dots, j-1, j$) that can be expressed in terms of Fock states of two modes:

$$|j, m\rangle = |j+m\rangle_1 |j-m\rangle_2. \quad (2.7)$$

The actions of the interferometer elements on the vector $\mathbf{J}=(J_1, J_2, J_3)$ can be represented as rotations in the three-dimensional space [4]. BS1 acts on \mathbf{J} as a rotation about the 1st axis by the angle $\pi/2$. The transformation matrix of this rotation is

$$\mathbf{R}_1(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2.8)$$

The transformation matrix of BS2 is $\mathbf{R}_1(-\pi/2)$, i.e., the two beam splitters perform rotations in opposite directions. The phase shifters rotate \mathbf{J} about the third axis by an angle $\phi = \phi_2 - \phi_1$. The transformation matrix of this rotation is

$$\mathbf{R}_3(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.9)$$

The overall transformation performed on \mathbf{J} is

$$\mathbf{J}_{\text{out}} = \mathbf{R}_1(-\pi/2)\mathbf{R}_3(\phi)\mathbf{R}_1(\pi/2)\mathbf{J}. \quad (2.10)$$

The information on the phase shift ϕ is inferred from the photon statistics of the output beams. One should measure the difference between the number of photons in the two output modes, $(N_d)_{\text{out}}$, or, equivalently, the operator $J_{3\text{out}} = \frac{1}{2}(N_d)_{\text{out}}$. Since there are fluctuations in $J_{3\text{out}}$, a phase shift is detectable only if it induces a change in $\langle J_{3\text{out}} \rangle$ which is larger than $\Delta J_{3\text{out}}$. Therefore, the minimum detectable phase shift (i.e., the uncertainty of the phase measurement) is determined by

$$(\delta\phi)^2 = \frac{(\Delta J_{3\text{out}})^2}{|\partial\langle J_{3\text{out}} \rangle / \partial\phi|^2}. \quad (2.11)$$

The value of $\delta\phi$ characterizes the accuracy of the interferometer. The expression for $J_{3\text{out}}$ can be easily found by using Eq. (2.10):

$$J_{3\text{out}} = -(\sin\phi)J_1 + (\cos\phi)J_3. \quad (2.12)$$

B. Standard noise limit

We consider some typical input states for which the phase sensitivity of an SU(2) interferometer is restricted by the so-called standard noise limit (SNL). Let the input state be $|j, m\rangle = |j+m\rangle_1 |j-m\rangle_2$ (an eigenstate of J_3 with eigenvalue m). The phase sensitivity for this input state is obtained from Eq. (2.11) by a straightforward calculation:

$$(\delta\phi)^2 = \frac{j^2 - m^2 + j}{2m^2}, \quad \phi \neq 0 \pmod{\pi}. \quad (2.13)$$

In this situation the best phase sensitivity is obtained for $m = \pm j$. Thus for the input state $|j, j\rangle = |2j\rangle_1 |0\rangle_2$, one gets [4]

$$(\delta\phi)_{\text{SNL}}^2 = 1/(2j) = 1/N, \quad \phi \neq 0 \pmod{\pi}. \quad (2.14)$$

This means that the phase sensitivity $\delta\phi$ of the interferometer goes as $1/\sqrt{N}$. The phase sensitivity (2.14) is usually referred to as the standard noise limit [4].

It follows from Eq. (2.13) that for the input state $|j, m\rangle$ with $m=0$ (i.e., when the interferometer is fed by two Fock states with equal numbers of photons), the phase measurement is absolutely uncertain [under the condition $\phi \neq 0 \pmod{\pi}$]. This result is in accordance with qualitative arguments of Yurke, McCall and Klauder (see Fig. 2 of Ref. [4]). However, it has been shown by Holland and Burnett [5] that this input state can be used in an SU(2) interferometer with the specific ‘‘reduced’’ situation of the measurement of the relative quantum phase between two field modes. In the Holland-Burnett situation the use of the simplified sensitivity measure (2.11) is excluded.

In what follows we assume, for the sake of simplicity, $\phi=0$. This can be achieved by controlling ϕ_2 with a feedback loop which maintains $\phi = \phi_2 - \phi_1 = 0$ [4]. Then Eq. (2.11) with $J_{3\text{out}}$ given by (2.12) can be simplified to the form

$$(\delta\phi)^2 = \frac{(\Delta J_3)^2}{\langle J_1 \rangle^2}, \quad \langle J_1 \rangle \neq 0. \quad (2.15)$$

Consider now the input state $|\alpha\rangle_1 |\alpha'\rangle_2$, where

$$|\alpha\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (2.16)$$

is the familiar Glauber coherent state. A simple calculation yields

$$(\Delta J_3)^2 = (|\alpha|^2 + |\alpha'|^2)/4, \quad (2.17)$$

$$\langle J_1 \rangle = |\alpha||\alpha'| \cos(\theta + \theta'), \quad (2.18)$$

where $\alpha = |\alpha|e^{i\theta}$, $\alpha' = |\alpha'|e^{i\theta'}$. For the optimal choice $\theta + \theta' = 0$, we get

$$(\delta\phi)^2 = \frac{|\alpha|^2 + |\alpha'|^2}{4|\alpha|^2|\alpha'|^2}. \quad (2.19)$$

The total number of photons is $N = |\alpha|^2 + |\alpha'|^2$. Hence the best phase sensitivity is obtained for $|\alpha|^2 = |\alpha'|^2 = N/2$ and it achieves the standard noise limit of Eq. (2.14).

We also consider the SU(2) generalized coherent states that are defined by [6]

$$\begin{aligned} |j, \zeta\rangle &= \exp(\zeta J_+ - \zeta^* J_-) |j, -j\rangle = \frac{\exp(\zeta J_+)}{(1 + |\zeta|^2)^j} |j, -j\rangle \\ &= (1 + |\zeta|^2)^{-j} \sum_{m=-j}^j \left[\frac{(2j)!}{(j+m)!(j-m)!} \right]^{1/2} \zeta^{j+m} |j, m\rangle, \end{aligned} \quad (2.20)$$

where $\zeta = (\xi/|\xi|)\tan|\xi|$. Expectation values of the SU(2) generators can be easily calculated for the $|j, \zeta\rangle$ states:

$$(\Delta J_3)^2 = 2j|\zeta|^2/(1 + |\zeta|^2)^2, \quad (2.21)$$

$$\langle J_1 \rangle = 2j(\text{Re}\zeta)/(1 + |\zeta|^2). \quad (2.22)$$

Then Eq. (2.15) reads

$$(\delta\phi)_{\text{coh}}^2 = \frac{|\zeta|^2}{2j(\text{Re}\zeta)^2}. \quad (2.23)$$

This phase uncertainty is minimized when ζ is real. Then $(\delta\phi)_{\text{coh}}^2$ achieves the standard noise limit of Eq. (2.14). We see that the use of the Glauber coherent states and of the SU(2) generalized coherent states does not improve the measurement accuracy over the standard noise limit.

C. Squeezed input states, the role of active devices and a duality of the phase sensitivity

There have been attempts to surpass the standard noise limit by using squeezed input states [1,2,4]. We reconsider here the scheme proposed by Yurke, McCall, and Klauder [4]. They considered the SU(2) interferometer outlined in Fig. 1, whose input ports are fed by the output beams b_1 and b_2 of a four-wave mixer (see Fig. 5 of Ref. [4]). The transformation caused by the four-wave mixer on the light beams a_1 and a_2 entering its input ports is an SU(1,1) transformation [4,24]:

$$\begin{pmatrix} b_1 \\ b_2^\dagger \end{pmatrix} = \begin{pmatrix} \cosh(\beta/2) & \sinh(\beta/2) \\ \sinh(\beta/2) & \cosh(\beta/2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix}. \quad (2.24)$$

The parameter β is related to the reflectivity r of the four-wave mixer (when it is used as a phase-conjugating mirror) via $\sinh^2(\beta/2) = r$ [25]. In the scheme considered here the Glauber coherent state $|\alpha\rangle$ enters one input port of the four-wave mixer and the vacuum state $|0\rangle$ enters the other. Since the transformation (2.24) is a squeezing Bogoliubov transformation, the output state of the four-wave mixer is the two-mode squeezed state.

The generator J_3 representing the photon-number difference between the two modes is invariant under the transformation (2.24). Therefore one finds

$$(\Delta J_3)^2 = |\alpha|^2/4. \quad (2.25)$$

The generator J_1 at the output of the four-wave mixer is given by

$$\begin{aligned} J_1 &= \frac{1}{4} \sinh\beta (a_1^\dagger + a_1^2 + a_2^\dagger + a_2^2) \\ &\quad - \frac{i}{4} \sinh\beta (a_1^\dagger - a_1^2 + a_2^\dagger - a_2^2) \\ &\quad + \frac{1}{2} \cosh\beta (a_1^\dagger a_2 + a_2^\dagger a_1). \end{aligned} \quad (2.26)$$

Its expectation value for the input state $|\alpha\rangle_1|0\rangle_2$ is

$$\langle J_1 \rangle = \frac{1}{2} |\alpha|^2 \sinh\beta \cos 2\theta \quad (2.27)$$

where $\alpha = |\alpha|e^{i\theta}$. The phase uncertainty of Eq. (2.15) is minimized when $\theta = 0$. Then one obtains [4]

$$(\delta\phi)_{\alpha,\beta}^2 = \frac{1}{|\alpha|^2 \sinh^2\beta}. \quad (2.28)$$

The measurement accuracy can be improved in two ways: (i) by increasing the parameter β of the four-wave mixer, or (ii) by increasing the coherent-state intensity $|\alpha|^2$. The first way can be viewed as related to the interferometer (including the four-wave mixer), while the second is related to the input state. Therefore, when we consider the phase sensitivity $\delta\phi(N)$, we should distinguish between the sensitivity for fixed input state ($\alpha = \text{const}$) and the sensitivity for fixed interferometer ($\beta = \text{const}$). This distinction seems formal at first look, but it has a crucial physical importance for an interferometer employing active devices because they do not conserve the total number of photons. Indeed, when the four-wave mixer is applied, the total number of photons is not constant any more. The mean total number \bar{N} of photons passing through the phase shifters depends on both α and β . In the scheme presented here \bar{N} is the mean total number of photons emitted by the four-wave mixer:

$$\bar{N} = \langle b_1^\dagger b_1 + b_2^\dagger b_2 \rangle = (|\alpha|^2 + 1) \cosh\beta - 1. \quad (2.29)$$

Then we find the phase sensitivity for fixed input state:

$$(\delta\phi)^2|_{\alpha} = \frac{(|\alpha|^2 + 1)^2}{|\alpha|^2 [(\bar{N} + 1)^2 - (|\alpha|^2 + 1)^2]}, \quad (2.30)$$

and for fixed interferometer:

$$(\delta\phi)^2|_{\beta} = \frac{\cosh\beta}{\sinh^2\beta} \frac{1}{(\bar{N} + 1 - \cosh\beta)}. \quad (2.31)$$

When $|\alpha|^2$ is close to 1 and \bar{N} is large, Eq. (2.30) yields

$$\delta\phi|_{\alpha} \approx \frac{2}{\bar{N}}. \quad (2.32)$$

This is much better than the standard noise limit, but there is a subtlety. Actually, for $|\alpha|^2 \sim 1$ the range of \bar{N} is restricted by available four-wave mixers. It is much more convenient for the experimenter to improve the measurement accuracy by increasing the intensity of the coherent state $|\alpha\rangle$. However, Eq. (2.31) shows that in this regime the standard noise limit cannot be surpassed. Therefore, when speaking about the phase sensitivity achieved with the squeezed input states, it is necessary to specify the regime of operation of the interferometer.

III. SU(2) INTERFEROMETERS WITH INTELLIGENT INPUT STATES

A. SU(2) intelligent states

It is known [22] that the standard noise limit for SU(2) interferometers can be surpassed by using the SU(2) intelligent states. However, an expression for $\delta\phi$ was found in Ref. [22] only for a special limiting case. We would like to derive an exact analytic expression for $\delta\phi$, that holds for a wide class of the SU(2) intelligent states. The commutation relation $[J_2, J_3] = iJ_1$ implies the uncertainty relation

$$(\Delta J_2)^2(\Delta J_3)^2 \geq \frac{1}{4} \langle J_1 \rangle^2. \quad (3.1)$$

Therefore, Eq. (2.15) reads

$$(\delta\phi)^2 \geq \frac{1}{4(\Delta J_2)^2}. \quad (3.2)$$

For intelligent states an equality is achieved in the uncertainty relation. Such J_2 - J_3 intelligent states with large values of ΔJ_2 would allow us to measure small changes in ϕ . The J_2 - J_3 intelligent states $|\lambda, \eta\rangle$ are determined by the eigenvalue equation

$$(\eta J_2 + i J_3)|\lambda, \eta\rangle = \lambda|\lambda, \eta\rangle, \quad (3.3)$$

where λ is a complex eigenvalue and η is a real parameter given by $|\eta| = \Delta J_3 / \Delta J_2$. For $|\eta| > 1$, these states are squeezed in J_2 , and for $|\eta| < 1$, they are squeezed in J_3 . In what follows we will consider only the region $|\eta| < 1$, that guarantees, as we will see, an improvement of the measurement accuracy. The states of Eq. (3.3) can be generated from the vacuum in two parametric down-conversion crystals with aligned idler beams after a measurement of the photon number in some of the modes [21]. For the J_2 - J_3 intelligent states, Eq. (3.2) reads

$$(\delta\phi)_{\text{int}}^2 = \frac{1}{4(\Delta J_2)^2} = \frac{\eta^2}{4(\Delta J_3)^2}. \quad (3.4)$$

Our aim is now to evaluate the variance $(\Delta J_3)^2$. In order to do that, we use the analytic representation of the intelligent states in the coherent-state basis $|j, \zeta\rangle$. This basis is overcomplete and any state in the Hilbert space can be expanded in it [6]. For example, the SU(2) intelligent state

$$|\lambda, \eta\rangle = \sum_{m=-j}^j C_m |j, m\rangle \quad (3.5)$$

is represented by the entire analytic function

$$\begin{aligned} \Lambda(j, \lambda, \eta; \zeta) &= (1 + |\zeta|^2)^j \langle j, \zeta^* | \lambda, \eta \rangle \\ &= \sum_{m=-j}^j C_m \left[\frac{(2j)!}{(j+m)!(j-m)!} \right]^{1/2} \zeta^{j+m}. \end{aligned} \quad (3.6)$$

The SU(2) generators act on $\Lambda(\zeta)$ as first-order differential operators [6]:

$$J_+ = -\zeta^2 \frac{d}{d\zeta} + 2j\zeta, \quad J_- = \frac{d}{d\zeta}, \quad J_3 = \zeta \frac{d}{d\zeta} - j. \quad (3.7)$$

Then Eq. (3.3) can be converted into a first-order linear homogeneous differential equation for $\Lambda(\zeta)$:

$$(\eta + 2\zeta + \eta\zeta^2) \frac{d\Lambda}{d\zeta} + 2(i\lambda - j - j\eta\zeta)\Lambda = 0. \quad (3.8)$$

The solution of this equation can be easily found to be

$$\Lambda(j, m_0, \eta; \zeta) = \mathcal{N}^{-1/2} (1 + \zeta/\tau)^{j+m_0} (1 + \tau\zeta)^{j-m_0}, \quad (3.9)$$

where \mathcal{N} is a normalization factor, and we have defined

$$\tau \equiv (1 - \sqrt{1 - \eta^2}) / \eta, \quad (3.10)$$

$$\lambda(m_0) \equiv i m_0 \sqrt{1 - \eta^2}. \quad (3.11)$$

The analyticity condition for the function $\Lambda(\zeta)$ requires that m_0 can take only the values:

$$m_0 = -j, -j+1, \dots, j-1, j. \quad (3.12)$$

Then Eq. (3.11) becomes a quantization condition which means that the operator $\eta J_2 + i J_3$ has a discrete spectrum, and the corresponding eigenstates and eigenvalues are characterized by the quantum number m_0 .

In the special cases $m_0 = \pm j$, the J_2 - J_3 intelligent states $|\lambda, \eta\rangle$ become the SU(2) generalized coherent states $|j, \zeta_0\rangle$ with $\zeta_0 = \tau^{\mp 1}$, respectively. Since η is real and $|\eta| < 1$, ζ_0 is also real. Thus we have an intersection between the intelligent and coherent states. The SU(2) coherent states which are simultaneously the J_2 - J_3 intelligent states allow us to achieve the standard noise limit (2.14) due to the fact that ζ_0 is real. It means that the states in the coherent-intelligent intersection lead to the best phase sensitivity among all the coherent states. However, the standard noise limit can be surpassed by using the intelligent states which are not the generalized coherent states.

The decomposition of the intelligent states $|\lambda, \eta\rangle$ over the orthonormal basis is obtained by expanding the function $\Lambda(j, m_0, \eta; \zeta)$ of Eq. (3.9) into a Taylor series in ζ . It is known [26,27] that a function of the form (3.9) is the generating function for the Lagrange polynomials:

$$\Lambda(j, m_0, \eta; \zeta) = \mathcal{N}^{-1/2} \sum_{n=0}^{\infty} g_n^{(-j-m_0, -j+m_0)} (-1/\tau, -\tau) \zeta^n. \quad (3.13)$$

Actually, this series is finite, because we have

$$g_n^{(-j-m_0, -j+m_0)} = 0 \quad \text{for } n > 2j. \quad (3.14)$$

The Lagrange polynomials are related to the Jacobi polynomials via [26]

$$g_n^{(\alpha, \beta)}(u, v) = (v-u)^n P_n^{(-\alpha-n, -\beta-n)} \left(\frac{u+v}{u-v} \right). \quad (3.15)$$

Using this relation, we can write

$$\begin{aligned} |\lambda, \eta\rangle &= \mathcal{N}^{-1/2} \sum_{m=-j}^j \left[\frac{(j+m)!(j-m)!}{(2j)!} \right]^{1/2} \\ &\quad \times P_{j+m}^{(m_0-m, -m_0-m)}(x) t^{(j+m)/2} |j, m\rangle, \end{aligned} \quad (3.16)$$

where we have defined

$$x \equiv (1 - \eta^2)^{-1/2},$$

$$t \equiv 4(1 - \eta^2) / \eta^2 = 4/(x^2 - 1). \quad (3.17)$$

The normalization factor is

$$\mathcal{N} = \sum_{n=0}^{2j} \frac{n!(2j-n)!}{(2j)!} [P_n^{(j+m_0-n, j-m_0-n)}(x)]^2 t^n. \quad (3.18)$$

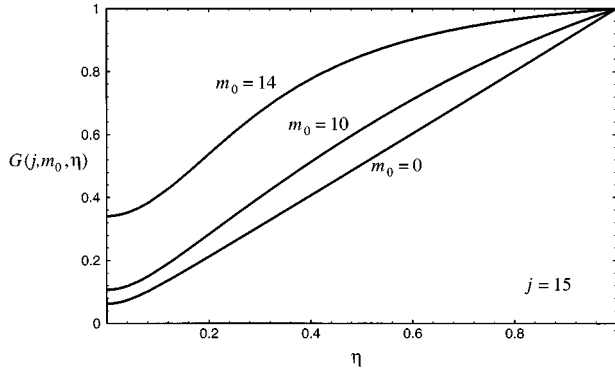


FIG. 2. The factor $G(j, m_0, \eta)$ of Eq. (3.25) versus η for $j = 15$ and various values of m_0 .

It follows from Eq. (3.14) that the summation in (3.18) can be continued up to infinity. Then, by using the summation theorem for the Jacobi polynomials [26], we find the closed expression for the normalization factor:

$$\mathcal{N} = (-1)^{j-|m_0|} S_+^{j+m_0} S_-^{j-m_0} \frac{(j-m_0)!(j+m_0)!}{(2j)!} \times P_{j-|m_0|}^{(-2j-1, 0)} \left(1 - \frac{2t}{S_+ S_-} \right), \quad (3.19)$$

where

$$S_{\pm} \equiv 1 + (x \pm 1)^2 t/4. \quad (3.20)$$

The expression (3.18) of \mathcal{N} as a power series in t is very convenient, because it enables us to write moments of the generator J_3 over the states $|\lambda, \eta\rangle$ as derivatives of \mathcal{N} with respect to t . By using the property $J_3|j, m\rangle = m|j, m\rangle$, we obtain

$$(\Delta J_3)^2 = \frac{t^2}{\mathcal{N}} \frac{\partial^2 \mathcal{N}}{\partial t^2} + \frac{t}{\mathcal{N}} \frac{\partial \mathcal{N}}{\partial t} - \left(\frac{t}{\mathcal{N}} \frac{\partial \mathcal{N}}{\partial t} \right)^2. \quad (3.21)$$

By using the formula

$$\frac{dP_n^{(\alpha, \beta)}(x)}{dx} = \frac{n + \alpha + \beta + 1}{2n} P_{n-1}^{(\alpha+1, \beta+1)}(x) \quad (3.22)$$

and the differential equation for the Jacobi polynomials, we obtain the exact analytic expression for the variance of J_3 :

$$(\Delta J_3)^2 = \frac{\eta^2 j}{2} \left[1 + \frac{(j + |m_0|)}{j} (1 - \eta^2) \frac{P_{j-|m_0|}^{(1, -2j)}(1 - 2\eta^2)}{P_{j-|m_0|}^{(0, -2j-1)}(1 - 2\eta^2)} \right]. \quad (3.23)$$

B. Phase sensitivity

Substituting the above expression for $(\Delta J_3)^2$ into Eq. (3.4), we find the phase sensitivity of the interferometer fed with the SU(2) intelligent states:

$$(\delta\phi)_{\text{int}}^2 = \frac{G(j, m_0, \eta)}{2j}, \quad (3.24)$$

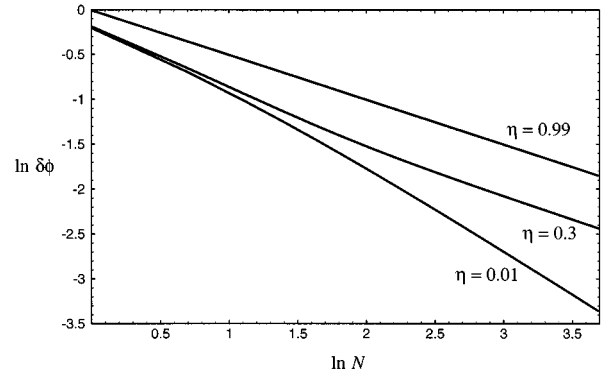


FIG. 3. $\ln \delta\phi$ as a function of $\ln N$ for an SU(2) interferometer using the SU(2) intelligent states with $m_0 = 0$ and various values of η .

where we have introduced the factor

$$G(j, m_0, \eta) \equiv \left[1 + \frac{(j + |m_0|)}{j} (1 - \eta^2) \times \frac{P_{j-|m_0|}^{(1, -2j)}(1 - 2\eta^2)}{P_{j-|m_0|}^{(0, -2j-1)}(1 - 2\eta^2)} \right]^{-1}. \quad (3.25)$$

In the case $m_0 = \pm j$, i.e., for a state in the coherent-intelligent intersection, we have $G(j, m_0, \eta) = 1$, so the phase sensitivity is at the standard noise limit $\delta\phi = 1/\sqrt{N}$. However, the use of the SU(2) intelligent states that do not belong to the coherent-intelligent intersection (i.e., with $|m_0| \neq j$) can yield a considerable improvement of the measurement accuracy in comparison with the standard noise limit. The quantitative measure of the improvement is the G -factor that can be expressed as the ratio between the intelligent phase uncertainty and the standard noise limit:

$$G(j, m_0, \eta) = (\delta\phi)_{\text{int}}^2 / (\delta\phi)_{\text{SNL}}^2. \quad (3.26)$$

It follows from the properties of the Jacobi polynomials that in the range considered here ($|\eta| < 1$) we always have $G(j, m_0, \eta) \leq 1$, so the measurement accuracy is improved for SU(2) interferometers fed with intelligent light.

Numerical results are presented in Figs. 2 and 3. The function $G(j, m_0, \eta)$ is plotted in Fig. 2 versus η for $j = 15$ and various values of m_0 . It is seen that for given η the smaller the value of m_0 , the smaller the G -factor. We also see that the minimum value of $G(j, m_0, \eta)$ (i.e., the best measurement accuracy) for given j and m_0 is achieved when $\eta \rightarrow 0$. On the other hand, when $\eta \rightarrow 1$, the G -factor approaches unity. The phase sensitivity, i.e., the dependence of the minimum detectable phase shift $\delta\phi$ on the number $N = 2j$ of photons passing through the interferometer is illustrated in Fig. 3 where $\ln \delta\phi$ is shown as a function of $\ln N$ for $m_0 = 0$ and various values of η . It is seen that for a given value of η the power law $\delta\phi \propto N^{-E}$ is a good approximation for large N . In order to express formally the slope of the curves in Fig. 3 for large N , we introduce the exponent

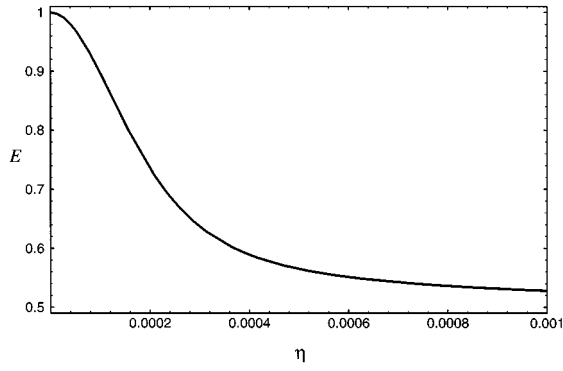


FIG. 4. The exponent E of Eq. (3.27) versus η for an SU(2) interferometer using the SU(2) intelligent states with $m_0=0$.

$$E = - \left. \frac{d(\ln \delta\phi)}{d(\ln N)} \right|_{N \rightarrow \infty}. \quad (3.27)$$

This quantity is plotted versus η in Fig. 4. For $\eta \rightarrow 0$ the exponent E approaches unity, which is the best available phase sensitivity. As η increases, the exponent E rapidly decreases to one half (the standard noise limit).

The dependence of the phase sensitivity on various parameters can be further studied by considering limiting values of the G -factor. We start from the limit $\eta \rightarrow 1$. Putting $\varepsilon = 1 - \eta^2$, we find, for $\varepsilon \ll 1$,

$$G(j, m_0, \eta) \approx [1 + 2\varepsilon(j^2 - m_0^2)]^{-1}, \quad (3.28)$$

It means that for η near 1 the phase sensitivity approaches the standard noise limit. It is also not difficult to see that

$$\lim_{\eta \rightarrow 0} G(j, m_0, \eta) = [1 + (j^2 - m_0^2)/j]^{-1}, \quad (3.29)$$

and in this case we recover the approximate result of Hillery and Mlodinow [22]:

$$(\delta\phi)_{\text{int}}^2 \approx \frac{1}{2(j^2 - m_0^2 + j)}, \quad (3.30)$$

that holds for η near zero. For $m_0=0$ the phase uncertainty is minimized:

$$(\delta\phi)_{\text{int}} \approx \frac{1}{\sqrt{2j(j+1)}}. \quad (3.31)$$

Because j is just half the total number N of photons passing through the interferometer, the phase sensitivity is of order $1/N$. We note that this sensitivity is achieved for the SU(2) intelligent states without adding an active device to the interferometer. Therefore, the total number of photons depends only on the value of j for the input state. This allows us to avoid the duality in the behavior of the phase sensitivity that occurs for the squeezed input.

We also consider a subtlety that is concerned with the limit $\eta \rightarrow 0$. It follows from the eigenvalue equation (3.3) that for $\eta=0$ the J_2 - J_3 intelligent state $|\lambda, \eta\rangle$ transforms into the state $|j, m_0\rangle$ (an eigenstate of J_3 with eigenvalue m_0). However, this transition should be treated with a great

care, because it does not preserve some basic properties of the intelligent states. In Eq. (3.4), that defines the phase sensitivity for the J_2 - J_3 intelligent states, we have used the relation

$$(\Delta J_2)^2 = (\Delta J_3)^2 / \eta^2. \quad (3.32)$$

This property holds for any intelligent state with arbitrarily small $|\eta|$. Therefore, we can take the limit $\eta \rightarrow 0$ for the phase uncertainty $(\delta\phi)_{\text{int}}^2$ calculated with the use of the relation (3.32). But we see that the result (3.30) obtained in this way is quite different from the phase uncertainty (2.13) for the input states $|j, m\rangle$. The reason for this discrepancy is that the ‘‘intelligent’’ relation (3.32) does not exist for the states $|j, m\rangle$. In other words, the result depends on the order in which we use the relation (3.32) and take the limit $\eta \rightarrow 0$. It means that the intelligent states $|\lambda, \eta\rangle$ with arbitrarily small $|\eta|$ and the states $|j, m_0\rangle$ may lead to different results.

This phenomenon arising for the SU(2) intelligent states in the limit $\eta \rightarrow 0$ can be made more familiar if we recall a similar situation that occurs for the canonical squeezed states. Consider two canonically conjugate field quadratures, $Q = (a^\dagger + a)/2$ and $P = i(a^\dagger - a)/2$, which satisfy the uncertainty relation $\Delta Q \Delta P \geq 1/4$. It is well known that this uncertainty relation is minimized by the canonical squeezed states $|\zeta, \alpha\rangle$, which satisfy the eigenvalue equation $(\eta Q + iP)|\zeta, \alpha\rangle = \lambda|\zeta, \alpha\rangle$. Here α and ζ are displacement and squeezing amplitudes, respectively, and $\eta = (1 - \zeta)/(1 + \zeta)$, $\lambda = (\alpha - \zeta\alpha^*)/(1 + \zeta)$. It is seen that the states $|\zeta, \alpha\rangle$ can be regarded as the intelligent states for the Weyl-Heisenberg group. For instance, the relation $\Delta Q = \Delta P/|\eta|$ does hold for the Q - P intelligent states with arbitrarily small values of $|\eta|$. However, for $\eta=0$ the Q - P intelligent states transform into the eigenstates of the ‘‘momentum’’ operator P , and the above relation does not exist. Therefore, properties of the canonical squeezed states calculated using this relation may be different in the limit $\eta \rightarrow 0$ from corresponding properties of the momentum eigenstates.

C. Quasi-intelligent states

The standard noise limit can also be surpassed by using two-mode states which are not exactly intelligent, but are close to optimizing the uncertainty relation (3.1). We will call such states ‘‘quasi-intelligent.’’ For example, we can imagine a state for which the uncertainty product $(\Delta J_2)^2(\Delta J_3)^2$ is equal to its minimum $\langle J_1 \rangle^2/4$ times a numerical factor of order 1. For such a state we will get $(\delta\phi)^2 = \nu/[4(\Delta J_2)^2]$, where ν is the numerical factor. If this state is squeezed in J_3 and swelled in J_2 (i.e., $\Delta J_2 \sim j$ for $j \gg 1$), then $\delta\phi$ will be of order $1/N$. An example of such a quasi-intelligent state was given by Yurke, McCall, and Klauder [4] who considered the input state $(|j, 0\rangle + |j, 1\rangle)/\sqrt{2}$. A simple calculation yields

$$(\Delta J_3)^2 = \frac{1}{4}, \quad (3.33)$$

$$(\Delta J_2)^2 = \frac{1}{2}j(j+1) - \frac{1}{4}, \quad (3.34)$$

$$\langle J_1 \rangle = \frac{1}{2}[j(j+1)]^{1/2}. \quad (3.35)$$

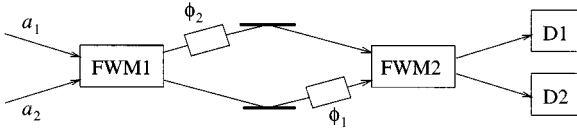


FIG. 5. An $SU(1,1)$ interferometer. Two light modes a_1 and a_2 are mixed by four-wave mixer FWM1, accumulate phase shifts ϕ_1 and ϕ_2 , respectively, and then are again mixed by four-wave mixer FWM2. The photons in the output modes are counted by detectors D1 and D2.

We see that for $j \gg 1$ this state gives the uncertainty product $(\Delta J_2)^2(\Delta J_3)^2 \approx j(j+1)/8$ that is greater than its minimum $\langle J_1 \rangle^2/4 = j(j+1)/16$ only by the factor $\nu=2$. Then one obtains

$$(\delta\phi)^2 \approx \frac{1}{2(\Delta J_2)^2} \approx \frac{1}{j(j+1)}. \quad (3.36)$$

Therefore the phase sensitivity is $\delta\phi \approx 2/N$ that differs from $(\delta\phi)_{\text{int}} \approx \sqrt{2}/N$ only by the factor $\sqrt{2}$. This example shows that the optimization of the phase sensitivity is intimately related to the optimization of the uncertainty relation (i.e., the intelligence) and to the corresponding $SU(2)$ squeezing.

IV. $SU(1,1)$ INTERFEROMETERS WITH CONVENTIONAL INPUT STATES

A. The interferometer

In $SU(1,1)$ interferometers four-wave mixers are employed instead of beam splitters. The application of active optical devices, that do not preserve the total number of photons, makes it possible to achieve high measurement accuracy, especially when intelligent light is used. On the other hand, this leads to the dual behavior of the phase sensitivity, as in the case of the squeezed input. Mathematical descriptions of $SU(2)$ and $SU(1,1)$ interferometers are rather similar, but the noncompactness of the $SU(1,1)$ Lie group leads to important physical distinctions between interferometers employing passive and active devices.

An $SU(1,1)$ interferometer is described schematically in Fig. 5. Two light beams represented by mode annihilation operators a_1 and a_2 enter the input ports of the first four-wave mixer FWM1. After leaving FWM1, the beams accumulate phase shifts ϕ_1 and ϕ_2 , respectively, and then they enter the second four-wave mixer FWM2. The photons leaving the interferometer are counted by detectors D1 and D2.

For the analysis of such an interferometer it is convenient to consider the Hermitian operators

$$\begin{aligned} K_1 &= \frac{1}{2}(a_1^\dagger a_2^\dagger + a_1 a_2), \\ K_2 &= \frac{1}{2i}(a_1^\dagger a_2^\dagger - a_1 a_2), \\ K_3 &= \frac{1}{2}(a_1^\dagger a_1 + a_2 a_2^\dagger). \end{aligned} \quad (4.1)$$

These operators form the two-mode boson realization of the $su(1,1)$ Lie algebra:

$$\begin{aligned} [K_1, K_2] &= -iK_3, \\ [K_2, K_3] &= iK_1, \\ [K_3, K_1] &= iK_2. \end{aligned} \quad (4.2)$$

It is also useful to introduce raising and lowering operators

$$\begin{aligned} K_+ &= K_1 + iK_2 = a_1^\dagger a_2^\dagger, \\ K_- &= K_1 - iK_2 = a_1 a_2. \end{aligned} \quad (4.3)$$

The Casimir operator for any unitary irreducible representation is a constant

$$K^2 = K_3^2 - K_1^2 - K_2^2 = k(k-1). \quad (4.4)$$

Thus a representation of $SU(1,1)$ is determined by a single number k that is called the Bargmann index. For the discrete-series representations [28] the Bargmann index acquires discrete values $k = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. By using the operators of Eq. (4.1), one gets

$$K^2 = \frac{1}{4} N_d^2 - \frac{1}{4}, \quad (4.5)$$

where

$$N_d = a_1^\dagger a_1 - a_2^\dagger a_2 \quad (4.6)$$

is the photon-number difference between the modes. We see that N_d is an $SU(1,1)$ invariant related to the Bargmann index k via $k = \frac{1}{2}(N_d + 1)$. The representation Hilbert space is spanned by the complete orthonormal basis $|k, n\rangle$ ($n=0, 1, 2, \dots$) that can be expressed in terms of Fock states of two modes:

$$|k, n\rangle = |n + 2k - 1\rangle_1 |n\rangle_2. \quad (4.7)$$

The actions of the interferometer elements on the vector $\mathbf{K} = (K_1, K_2, K_3)$ can be represented as Lorentz boosts and rotations in the $(2+1)$ -dimensional space-time [4]. FWM1 acts on \mathbf{K} as a Lorentz boost along the negative direction of the second axis with the transformation matrix

$$\mathbf{L}_2(-\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & -\sinh\beta \\ 0 & -\sinh\beta & \cosh\beta \end{pmatrix}. \quad (4.8)$$

As mentioned above, β is related to the reflectivity r of the four-wave mixer (when it is used as a phase-conjugating mirror) via $\sinh^2(\beta/2) = r$ [25]. The transformation matrix of FWM2 is $\mathbf{L}_2(\beta)$, i.e., the two four-wave mixers perform boosts in opposite directions. Phase shifters rotate \mathbf{K} about the third axis by an angle $\phi = -(\phi_1 + \phi_2)$. The transformation matrix of this rotation is $\mathbf{R}_3(\phi)$ of Eq. (2.9). The overall transformation performed on \mathbf{K} is

$$\mathbf{K}_{\text{out}} = \mathbf{L}_2(\beta) \mathbf{R}_3(\phi) \mathbf{L}_2(-\beta) \mathbf{K}. \quad (4.9)$$

The information on ϕ is once again inferred from the photon statistics of the output beams. One should measure the total number of photons in the two output modes, N_{out} , or, equivalently, the operator $K_{3\text{out}} = \frac{1}{2}(N_{\text{out}} + 1)$. Fluctuations in $\langle K_{3\text{out}} \rangle$ restrict the accuracy of the phase measurement. The phase uncertainty that determines the minimum detectable phase shift is given by

$$(\delta\phi)^2 = \frac{(\Delta K_{3\text{out}})^2}{|\partial \langle K_{3\text{out}} \rangle / \partial \phi|^2}. \quad (4.10)$$

From Eq. (4.9), we find

$$K_{3\text{out}} = (\sinh\beta \sin\phi)K_1 + \sinh\beta \cosh\beta (\cos\phi - 1)K_2 + (\cosh^2\beta - \sinh^2\beta \cos\phi)K_3. \quad (4.11)$$

B. Vacuum and coherent input states

We consider here some typical cases when the input field is prepared in the vacuum state, in the generalized coherent state and in the Glauber coherent state. If only vacuum fluctuations enter the input ports, then Eq. (4.10) with $K_{3\text{out}}$ of Eq. (4.11) reduces to the known result [4]

$$(\delta\phi)_{\text{vac}}^2 = \frac{\sin^2\phi + \cosh^2\beta(1 - \cos\phi)^2}{\sin^2\phi \sinh^2\beta}, \quad \phi \neq 0. \quad (4.12)$$

As $\phi \rightarrow 0$, these phase fluctuations are minimized, $(\delta\phi)_{\text{vac}}^2 \rightarrow 1/\sinh^2\beta$. We also consider a more complicated input state $|k, n\rangle = |n + 2k - 1\rangle_1 |n\rangle_2$. The corresponding phase uncertainty is obtained from Eq. (4.10) by a straightforward calculation:

$$(\delta\phi)^2 = \frac{\sin^2\phi + \cosh^2\beta(1 - \cos\phi)^2}{\sin^2\phi \sinh^2\beta} \frac{k + n(2k + n)}{2(k + n)^2}, \quad \phi \neq 0. \quad (4.13)$$

The vacuum state is obtained for $n=0$, $k=1/2$. Then Eq. (4.13) reduces to Eq. (4.12). The phase uncertainty (4.13) is minimized as $\phi \rightarrow 0$:

$$\lim_{\phi \rightarrow 0} (\delta\phi)^2 = \frac{k + n(2k + n)}{2\sinh^2\beta(k + n)^2}. \quad (4.14)$$

In what follows we take for simplicity $\phi=0$, as in the SU(2) case. Once again, the experimenter can control ϕ_2 with a feedback loop which maintains $\phi = -(\phi_1 + \phi_2) = 0$ [4]. Then Eq. (4.10) with $K_{3\text{out}}$ given by (4.11) can be simplified to the form

$$(\delta\phi)^2 = \frac{(\Delta K_3)^2}{\sinh^2\beta \langle K_1 \rangle^2}, \quad \langle K_1 \rangle \neq 0. \quad (4.15)$$

We next consider the SU(1,1) generalized coherent states. These states are defined by [6]

$$\begin{aligned} |k, \zeta\rangle &= \exp(\xi K_+ - \xi^* K_-) |k, 0\rangle \\ &= (1 - |\zeta|^2)^k \exp(\zeta K_+) |k, 0\rangle \\ &= (1 - |\zeta|^2)^k \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+2k)}{n! \Gamma(2k)} \right]^{1/2} \zeta^n |k, n\rangle, \end{aligned} \quad (4.16)$$

where $\zeta = (\xi/|\xi|) \tanh|\xi|$, so $|\zeta| < 1$. In the case of the two-mode boson realization, the SU(1,1) coherent states can be recognized as the well-known two-mode squeezed states with ξ being the squeezing parameter [8]. A simple calculation yields expectation values of the SU(1,1) generators over the $|k, \zeta\rangle$ coherent states [8]:

$$(\Delta K_3)^2 = 2k|\zeta|^2/(1 - |\zeta|^2)^2, \quad (4.17)$$

$$\langle K_1 \rangle = 2k(\text{Re}\zeta)/(1 - |\zeta|^2). \quad (4.18)$$

Then Eq. (4.15) reads

$$(\delta\phi)_{\text{coh}}^2 = \frac{|\zeta|^2}{2k \sinh^2\beta (\text{Re}\zeta)^2}. \quad (4.19)$$

This phase uncertainty is minimized when ζ is real. Then one gets [23]

$$(\delta\phi)_{k,\beta}^2 = \frac{1}{2k \sinh^2\beta}. \quad (4.20)$$

We see that this phase sensitivity depends only on the parameter β of the four-wave mixer and on the photon-number difference between the two input modes ($N_d = 2k - 1$). Therefore, ζ can be taken to be zero, i.e., one can choose an input state with a fixed number of photons in the one mode and the vacuum in the other. This is in accordance with the result (4.14) for the input state $|k, n\rangle$ with $n=0$.

The mean total number \bar{N} of photons passing through the phase shifters depends on both the input state and the four-wave mixer. For the interferometer considered here, \bar{N} is the total number of photons emitted by FWM1:

$$\bar{N} = 2\langle K'_3 \rangle - 1, \quad (4.21)$$

where $\mathbf{K}' = \mathbf{L}_2(-\beta)\mathbf{K}$, so we have

$$K'_3 = (\cosh\beta)K_3 - (\sinh\beta)K_2. \quad (4.22)$$

Calculating the expectation value for a coherent state with real ζ , we obtain

$$\bar{N} = 2k \frac{1 + \zeta^2}{1 - \zeta^2} \cosh\beta - 1. \quad (4.23)$$

Since $(\delta\phi)^2$ of Eq. (4.20) is independent of ζ , we may take $\zeta=0$; then $\bar{N} = 2k \cosh\beta - 1$. Once again, we have two ways for improving the measurement accuracy of the interferometer: (i) by increasing the parameter β of the four-wave mixer, or (ii) by increasing the photon-number difference $N_d = 2k - 1$ for the input state. In the first regime we obtain the phase sensitivity for fixed input state ($k = \text{const}$):

$$(\delta\phi)^2|_k = \frac{2k}{(\bar{N}+1)^2 - (2k)^2}. \quad (4.24)$$

For $k=1/2$ ($N_d=0$), we recover the result for the vacuum input [4]:

$$(\delta\phi)_{\text{vac}}^2 = \frac{1}{\bar{N}(\bar{N}+2)}. \quad (4.25)$$

We see that the phase sensitivity approaches $1/\bar{N}$. However, there is a problem with improvement of the measurement accuracy because the value of β is restricted by properties of available four-wave mixers. On the other hand, the phase sensitivity for fixed interferometer ($\beta=\text{const}$) is

$$(\delta\phi)^2|_{\beta} = \frac{\cosh\beta}{\sinh^2\beta} \frac{1}{(\bar{N}+1)}. \quad (4.26)$$

We see that the standard noise limit cannot be surpassed in this regime.

Next we consider the input state $|\alpha\rangle_1|\alpha'\rangle_2$ where $|\alpha\rangle$ and $|\alpha'\rangle$ are the Glauber coherent states. We easily find the following expectation values:

$$\langle K_3 \rangle = (|\alpha|^2 + |\alpha'|^2 + 1)/2, \quad (4.27)$$

$$(\Delta K_3)^2 = (|\alpha|^2 + |\alpha'|^2)/4, \quad (4.28)$$

$$\langle K_1 \rangle = |\alpha||\alpha'| \cos(\theta + \theta'), \quad (4.29)$$

$$\langle K_2 \rangle = |\alpha||\alpha'| \sin(\theta + \theta'), \quad (4.30)$$

where $\alpha = |\alpha|e^{i\theta}$, $\alpha' = |\alpha'|e^{i\theta'}$. For $\theta + \theta' = 0$ and $|\alpha| = |\alpha'|$, we obtain

$$(\delta\phi)_{\alpha,\beta}^2 = \frac{1}{2|\alpha|^2 \sinh^2\beta}, \quad (4.31)$$

$$\bar{N} = (2|\alpha|^2 + 1) \cosh\beta - 1. \quad (4.32)$$

These results are almost identical to Eqs. (2.28) and (2.29) for the SU(2) interferometer with squeezed input states; the only difference is the factor 2 before $|\alpha|^2$. We again have two regimes: (i) fixed input state ($\alpha=\text{const}$) and variable interferometer, or (ii) fixed interferometer ($\beta=\text{const}$) and variable input state. The first regime leads to the phase sensitivity of order $1/\bar{N}$, but is technically more complicated. The second regime is much more preferable from the technical point of view, but the phase sensitivity cannot be improved over the standard noise limit. This duality in the behavior of the phase sensitivity is a direct consequence of the fact that the SU(1,1) transformations performed by the four-wave mixers do not preserve the total number of photons.

V. SU(1,1) INTERFEROMETERS WITH INTELLIGENT INPUT STATES

A. SU(1,1) intelligent states

We would like to surpass the standard noise limit by using the SU(1,1) intelligent states. The commutation relation $[K_2, K_3] = iK_1$ implies the uncertainty relation

$$(\Delta K_2)(\Delta K_3) \geq \frac{1}{4} \langle K_1 \rangle. \quad (5.1)$$

Therefore, Eq. (4.15) can be written as

$$(\delta\phi)^2 \geq \frac{1}{4 \sinh^2\beta (\Delta K_2)^2}. \quad (5.2)$$

For intelligent states an equality is achieved in the uncertainty relation. Therefore, such K_2 - K_3 intelligent states with large values of ΔK_2 would allow us to measure small changes in ϕ . The K_2 - K_3 intelligent states $|\lambda, \eta\rangle$ are determined by the eigenvalue equation

$$(\eta K_2 + iK_3)|\lambda, \eta\rangle = \lambda|\lambda, \eta\rangle, \quad (5.3)$$

where λ is a complex eigenvalue and η is a real parameter given by $|\eta| = \Delta K_3 / \Delta K_2$. For $|\eta| > 1$, these states are squeezed in K_2 , and for $|\eta| < 1$, they are squeezed in K_3 . We consider here all the values of η . The scheme of Luis and Peřina [21] can be used for producing both the SU(2) and the SU(1,1) intelligent states. In particular, the K_2 - K_3 intelligent states of Eq. (5.3) can be generated in this scheme quite conveniently. For these states, Eq. (5.2) reads

$$(\delta\phi)_{\text{int}}^2 = \frac{1}{4 \sinh^2\beta (\Delta K_2)^2} = \frac{\eta^2}{4 \sinh^2\beta (\Delta K_3)^2}. \quad (5.4)$$

We will use the analytic representation in the basis of the SU(1,1) generalized coherent states $|k, \zeta\rangle$ [6]. This basis is overcomplete, and any state in the Hilbert space can be expanded in it. For example, the SU(1,1) intelligent state

$$|\lambda, \eta\rangle = \sum_{n=0}^{\infty} C_n |k, n\rangle \quad (5.5)$$

is represented by the function

$$\begin{aligned} \Lambda(k, \lambda, \eta; \zeta) &= (1 - |\zeta|^2)^{-k} \langle k, \zeta^* | \lambda, \eta \rangle \\ &= \sum_{n=0}^{\infty} C_n \left[\frac{\Gamma(2k+n)}{n! \Gamma(2k)} \right]^{1/2} \zeta^n, \end{aligned} \quad (5.6)$$

which is analytic in the unit disk $|\zeta| < 1$. The analytic representation of the SU(1,1) intelligent states was studied in Ref. [18]. The SU(1,1) generators act on $\Lambda(\zeta)$ as first-order differential operators [6]:

$$K_+ = \zeta^2 \frac{d}{d\zeta} + 2k\zeta, \quad K_- = \frac{d}{d\zeta}, \quad K_3 = \zeta \frac{d}{d\zeta} + k. \quad (5.7)$$

Then Eq. (5.3) can be converted into a first-order linear homogeneous differential equation for $\Lambda(\zeta)$:

$$(\eta + 2\zeta - \eta\zeta^2)\frac{d\Lambda}{d\zeta} + 2(i\lambda + k - k\eta\zeta)\Lambda = 0. \quad (5.8)$$

The solution of this equation can be easily found to be

$$\Lambda(k, l, \eta; \zeta) = \mathcal{N}^{-1/2} (1 + \zeta/\tau)^l (1 - \tau\zeta)^{-2k-l}, \quad (5.9)$$

where \mathcal{N} is a normalization factor, and we have defined

$$\tau \equiv (\sqrt{\eta^2 + 1} - 1)/\eta, \quad |\tau| < 1, \quad (5.10)$$

$$\lambda(l) = i(k+l)\sqrt{\eta^2 + 1}. \quad (5.11)$$

The analyticity condition for the function $\Lambda(k, \lambda, \eta; \zeta)$ requires that l can be only a positive integer or zero: $l = 0, 1, 2, \dots$. Then Eq. (5.11) becomes a quantization condition which means that the operator $\eta K_2 + iK_3$ has a discrete spectrum, and the corresponding eigenstates and eigenvalues are characterized by the quantum number l .

In the simplest case $l=0$, the function $\Lambda(k, l, \eta; \zeta)$ represents the K_2 - K_3 intelligent states which are simultaneously the SU(1,1) generalized coherent states $|k, \zeta_0\rangle$ with $\zeta_0 = \tau$. Since η is real, ζ_0 is also real. Hence there is an intersection between the intelligent and coherent states. States which belong to this intersection allow us to achieve the measurement accuracy (4.20) due to the fact that ζ_0 is real. Therefore, these states lead to the best phase sensitivity among all the coherent states. However, we will see that the noise level (4.20) can be surpassed by using the SU(1,1) intelligent states which are not the generalized coherent states.

As in Sec. III A, the function $\Lambda(k, l, \eta; \zeta)$ of Eq. (5.9) is expanded into a Taylor series in ζ as the generating function for the Lagrange polynomials [26,27]:

$$\Lambda(k, l, \eta; \zeta) = \mathcal{N}^{-1/2} \sum_{n=0}^{\infty} g_n^{(-l, 2k+l)} (-1/\tau, \tau) \zeta^n. \quad (5.12)$$

By using the relation (3.15) between the Lagrange and Jacobi polynomials, we obtain the decomposition of the SU(1,1) intelligent states over the orthonormal basis:

$$|\lambda, \eta\rangle = \mathcal{N}^{-1/2} \sum_{n=0}^{\infty} \left[\frac{n! \Gamma(2k)}{\Gamma(2k+n)} \right]^{1/2} P_n^{(l-n, -2k-l-n)}(x) t^{n/2} |k, n\rangle, \quad (5.13)$$

where we have defined

$$x \equiv (\eta^2 + 1)^{-1/2}, \quad (5.14)$$

$$t \equiv 4(\eta^2 + 1)/\eta^2 = 4/(1 - x^2).$$

By using the summation theorem for the Jacobi polynomials [26], we find the normalization factor:

$$\begin{aligned} \mathcal{N} &= \sum_{n=0}^{\infty} \frac{n! \Gamma(2k)}{\Gamma(2k+n)} [P_n^{(l-n, -2k-l-n)}(x)]^2 t^n \\ &= S_+^l S_-^{-2k-l} \frac{l! \Gamma(2k)}{\Gamma(2k+l)} P_l^{(2k-1, 0)} \left(1 + \frac{2t}{S_+ S_-} \right), \end{aligned} \quad (5.15)$$

where

$$S_{\pm} \equiv 1 - (x \pm 1)^2 t/4. \quad (5.16)$$

We can write moments of the generator K_3 over the states $|\lambda, \eta\rangle$ as derivatives of \mathcal{N} with respect to t . By using the property $K_3|k, n\rangle = (k+n)|k, n\rangle$, we obtain

$$(\Delta K_3)^2 = \frac{t^2}{\mathcal{N}} \frac{\partial^2 \mathcal{N}}{\partial t^2} + \frac{t}{\mathcal{N}} \frac{\partial \mathcal{N}}{\partial t} - \left(\frac{t}{\mathcal{N}} \frac{\partial \mathcal{N}}{\partial t} \right)^2. \quad (5.17)$$

By using formula (3.22), we find the exact analytic expression for the variance of K_3 :

$$(\Delta K_3)^2 = \frac{\eta^2 k}{2} \left[1 + \frac{(2k+l)(\eta^2+1)P_{l-1}^{(1, 2k)}(2\eta^2+1)}{k P_l^{(0, 2k-1)}(2\eta^2+1)} \right]. \quad (5.18)$$

B. Phase sensitivity

Substituting the above result for $(\Delta K_3)^2$ into Eq. (5.4), we find the phase sensitivity of the interferometer fed with the SU(1,1) intelligent states:

$$(\delta\phi)_{\text{int}}^2 = \frac{G(k, l, \eta)}{2k \sinh^2 \beta}, \quad (5.19)$$

where we have introduced the factor

$$G(k, l, \eta) \equiv \left[1 + \frac{(2k+l)(\eta^2+1)P_{l-1}^{(1, 2k)}(2\eta^2+1)}{k P_l^{(0, 2k-1)}(2\eta^2+1)} \right]^{-1}. \quad (5.20)$$

In the case of the coherent-intelligent intersection, $l=0$, and then $G(k, l, \eta) = 1$. Then the phase uncertainty is on the noise level (4.20). The use of the SU(1,1) intelligent states that do not belong to the coherent-intelligent intersection (i.e., with $l \neq 0$) can yield a great improvement of the measurement accuracy. The quantitative measure of the improvement is the G -factor that can be expressed as the ratio between the intelligent phase uncertainty and the SU(1,1) coherent noise level (4.20):

$$G(k, l, \eta) = \frac{(\delta\phi)_{\text{int}}^2}{(\delta\phi)_{k, \beta}^2} = 2k \frac{(\delta\phi)_{\text{int}}^2}{(\delta\phi)_{\text{vac}}^2}. \quad (5.21)$$

It follows from the properties of the Jacobi polynomials that the G -factor is always less than unity, so the measurement accuracy is improved for the SU(1,1) interferometers fed with intelligent light. Quantitative results are presented in Fig. 6, where the factor $G(k, l, \eta)$ is shown as a function of η for $k=1/2$ and different values of l . We see that for given η the larger the value of l , the smaller the G factor. The best measurement accuracy for given l and k is achieved when $\eta \rightarrow 0$. For large values of η , the G factor approaches a limiting value. By using the properties of the Jacobi polynomials, we find

$$\lim_{\eta \rightarrow 0} G(k, l, \eta) = [1 + l(2k+l)/k]^{-1}, \quad (5.22)$$

$$\lim_{\eta \rightarrow \infty} G(k, l, \eta) = (1 + l/k)^{-1}. \quad (5.23)$$

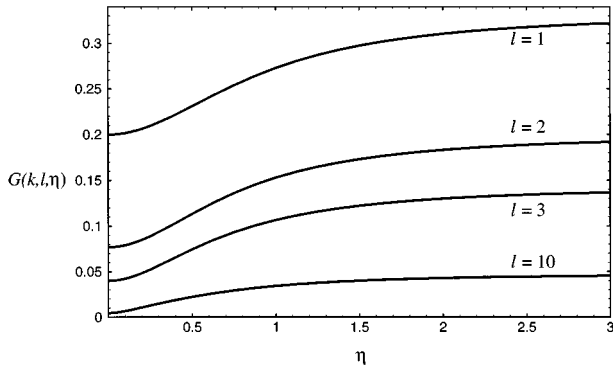


FIG. 6. The factor $G(k, l, \eta)$ of Eq. (5.20) versus η for $k=1/2$ and different values of l .

An interesting property of SU(1,1) interferometers is that for $l \neq 0$ the coherent noise level (4.20) is surpassed for any value of η . The phase uncertainty (5.19) for $\eta \rightarrow 0$ reads

$$\lim_{\eta \rightarrow 0} (\delta\phi)_{\text{int}}^2 = \frac{1}{2 \sinh^2 \beta [k + l(2k + l)]}. \quad (5.24)$$

We have seen for the SU(2) interferometer that there is a subtlety concerned with the limit $\eta \rightarrow 0$. A similar problem also arises for the SU(1,1) interferometer. It follows from the eigenvalue equation (5.3) that for $\eta = 0$ the K_2 - K_3 intelligent state $|\lambda, \eta\rangle$ transforms into the state $|k, l\rangle$ (an eigenstate of K_3 with eigenvalue $k+l$). However, this transition does not preserve the relation

$$(\Delta K_2)^2 = (\Delta K_3)^2 / \eta^2, \quad (5.25)$$

which has been used in Eq. (5.4) that defines the phase sensitivity for the intelligent states. The property (5.25) holds for any intelligent state with arbitrarily small $|\eta|$. Therefore, we can take the limit $\eta \rightarrow 0$ for the phase uncertainty $(\delta\phi)_{\text{int}}^2$ calculated with the use of the relation (5.25). But we see that the result (5.24) obtained in this way is different from the result (4.14) for the input states $|k, n\rangle$. This discrepancy occurs because the ‘‘intelligent’’ relation (5.25) does not exist for the states $|k, n\rangle$. Therefore, the intelligent states $|\lambda, \eta\rangle$ with arbitrarily small $|\eta|$ and the states $|k, l\rangle$ lead to different phase uncertainties.

We proceed by examining the phase sensitivity $\delta\phi(\bar{N})$ for the intelligent input. The mean total number \bar{N} of photons passing through the phase shifters is given by Eq. (4.21). It follows directly from the eigenvalue equation (5.3) that $\langle K_2 \rangle = (\text{Re}\lambda) / \eta$, $\langle K_3 \rangle = \text{Im}\lambda$. Then we use the quantization condition (5.11) and find

$$\langle K_2 \rangle = 0, \quad \langle K_3 \rangle = (k+l) \sqrt{\eta^2 + 1}. \quad (5.26)$$

Then \bar{N} is given by

$$\bar{N} = 2 \cosh \beta (k+l) \sqrt{\eta^2 + 1} - 1. \quad (5.27)$$

We see that \bar{N} depends on the parameters k, l, η of the input state and on the parameter β of the interferometer. The phase sensitivity for fixed input state is

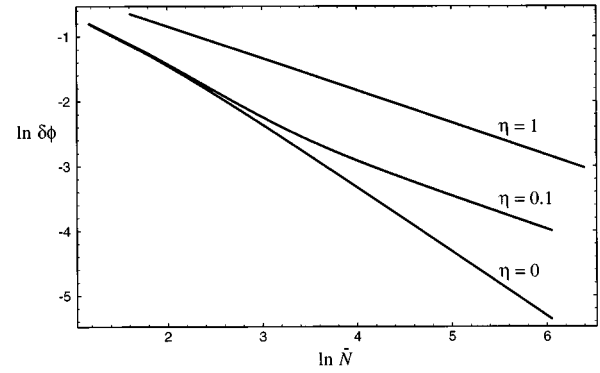


FIG. 7. $\ln \delta\phi$ as a function of $\ln \bar{N}$ for an SU(1,1) interferometer with $\sinh^2 \beta = 1$, using the SU(1,1) intelligent states with $k=1/2$ and various values of η . The values of $\delta\phi$ and \bar{N} are calculated for $l=1, 2, \dots, 150$.

$$(\delta\phi)^2|_{k,l,\eta} = \frac{1}{k} \frac{2(k+l)^2(\eta^2+1)G(k,l,\eta)}{(\bar{N}+1)^2 - 4(k+l)^2(\eta^2+1)}. \quad (5.28)$$

For $\eta \rightarrow 0$ this phase sensitivity is

$$(\delta\phi)^2|_{k,l} = \frac{2(k+l)^2}{(l^2 + 2kl + k)} \frac{1}{(\bar{N}+1)^2 - 4(k+l)^2}. \quad (5.29)$$

For $l=0$ and $k=1/2$, this result reduces to Eq. (4.25) for the vacuum input. For $k=1$, we obtain

$$(\delta\phi)^2|_l = \frac{2}{(\bar{N}+1)^2 - 4(l+1)^2}. \quad (5.30)$$

We see that this regime can yield a phase sensitivity of order $1/\bar{N}$, that, of course, depends on the available range of β .

In the regime of fixed interferometer ($\beta = \text{const}$) and variable input state, the phase sensitivity depends on the three parameters of the state: k, l , and η . We study the dependence $\delta\phi(\bar{N})$ numerically: for fixed $\sinh^2 \beta = 1$ and some values of k and η , we evaluate numerically $(\delta\phi)^2$ of Eq. (5.19) and \bar{N} of Eq. (5.27) for $l=1, 2, \dots, 150$. These results are presented in Fig. 7 where $\ln \delta\phi$ is plotted versus $\ln \bar{N}$ for $k=1/2$ and various values of η . In the region of large \bar{N} (small phase uncertainty), a good approximation is the power law $\delta\phi \propto \bar{N}^{-E}$. We introduce the exponent

$$E = - \left. \frac{d(\ln \delta\phi)}{d(\ln \bar{N})} \right|_{\bar{N} \rightarrow \infty}, \quad (5.31)$$

which expresses the slope of the curves in Fig. 7 for large \bar{N} . This quantity is plotted in Fig. 8 versus η . It is seen that E approaches unity for $\eta \rightarrow 0$ and rapidly decreases to one half as η increases.

Using the limit (5.22), we find the phase sensitivity for fixed interferometer ($\beta = \text{const}$) and the input state with $\eta \rightarrow 0$, fixed k and variable l :

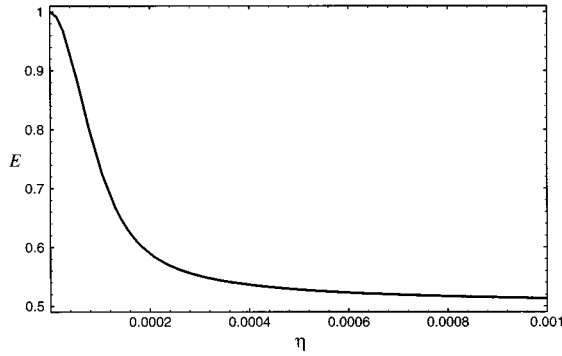


FIG. 8. The exponent E of Eq. (5.31) versus η for an SU(1,1) interferometer with $\sinh^2\beta=1$, using the SU(1,1) intelligent states with $k=1/2$.

$$(\delta\phi)^2|_{k,\beta} \approx \frac{2 \coth^2\beta}{(\bar{N}+1)^2 - 4(k^2 - k)\cosh^2\beta}. \quad (5.32)$$

This phase sensitivity is optimized for $k=1/2$:

$$(\delta\phi)^2|_{\beta} \approx \frac{2 \coth^2\beta}{(\bar{N}+1)^2 + \cosh^2\beta}. \quad (5.33)$$

Thus we see that the interferometer operated in the regime of fixed β can achieve a phase sensitivity of order $1/\bar{N}$. It means that SU(1,1) interferometers with intelligent input states can surpass the standard noise limit in both regimes: for fixed input state, and for fixed interferometer. This remarkable property distinguishes the intelligent states from the other states discussed above.

VI. DISCUSSION AND CONCLUSIONS

In this paper we considered in detail the phase sensitivity of passive and active interferometers characterized by the SU(2) and SU(1,1) groups respectively, for various types of input states. A usual method to reduce the quantum noise in interferometers is by the application of squeezing-producing active devices. We showed that the use of such active devices (e.g., four-wave mixers) leads to a duality in the behavior of the phase sensitivity. If the total number \bar{N} of photons passing through the phase shifters is determined by means of the squeezing parameter β of the four-wave mixer, the interferometer can achieve the phase sensitivity $\delta\phi \sim 1/\bar{N}$. However, if \bar{N} is determined by changing parameters of an input state (e.g., the intensity of a coherent laser beam), the phase sensitivity cannot generally surpass the

standard noise limit $\delta\phi \sim 1/\bar{N}^{1/2}$. We showed that these limitations can be overcome by the use of the remarkable squeezing properties of the intelligent states. On the one hand, the SU(2) intelligent states can lead to the phase sensitivity $\delta\phi \sim 1/\bar{N}$ in SU(2) interferometers without additional squeezing-producing devices. This avoids the duality mentioned above. On the other hand, SU(1,1) interferometers fed with the SU(1,1) intelligent states achieve phase sensitivity of order $1/\bar{N}$ in both regimes: for variable squeezing parameter β of the four-wave mixer, and for variable intensity of the input state.

The quantum noise is formally expressed via the uncertainty relations. Quantum states which optimize the uncertainty relations lead to minimum noise. This property is called intelligence and it can be manifested in arbitrarily strong squeezing achieved by the intelligent states. It means that while the uncertainty of a quantum observable is dramatically reduced, the uncertainty of a conjugate observable is increased as little as allowed by quantum theory. The quantum noise in SU(2) and SU(1,1) interferometers is expressed via the uncertainty relations for the Hermitian generators of the corresponding groups. Therefore, the best way to reduce this noise is by using the SU(2) and SU(1,1) intelligent states respectively, which are highly squeezed for an appropriate group generator (J_3 and K_3 , respectively, in the schemes considered here). The production of these states by means of advanced experimental techniques looks quite realistic in the near future. We also note that the powerful analytic method used for calculations with the intelligent states can be of considerable interest to workers in quantum optics.

In the present paper we adopted an ideal assumption that the input two-mode state has a definite total number of photons $N=2j$ [for an SU(2) interferometer] or a definite photon-number difference $N_d=2k-1$ [for an SU(1,1) interferometer]. In other words, we considered input states belonging to irreducible representations of SU(2) and SU(1,1). A more realistic assumption should deal with an input state which is a superposition of the intelligent states with different values of j or k . Properties of such a superposition state will depend on the photon-number sum and difference distribution in the SU(2) and the SU(1,1) case, respectively.

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