Superradiant laser with partial atomic cooperativity

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We further develop the theory of the superradiant laser by allowing for submaximal atomic collectivity and spontaneous emission. In the absence of spontaneous emission we obtain a set of stationary solutions for different degrees of atomic cooperativity and investigate their fluctuation properties and interrelations with respect to the underlying U(3) symmetry. We use this set of solutions to discuss the breaking of U(3) by spontaneous emission. [S1050-2947(96)10510-2]

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I. INTRODUCTION

Superfluorescent pulses can be produced by N collectively radiating identical atoms [1–3] as these atoms decay from an initially excited state to the ground state. In contrast to such transient behavior would be the stationary output of the superradiant laser [4,5] to be discussed here. Collective behavior would be manifested in the proportionality of the output intensity to N^2 and of the linewidth to N^{-2}. Moreover, as was shown in [4,5], such a laser could display nearly perfect squeezing of the intensity fluctuations. While incoherent processes like pump fluctuations or spontaneous emission are in principle detrimental to the noise suppression of the fully cooperative and coherently pumped ideal case, a considerable degree of squeezing can survive in the presence of such perturbations [5].

The present paper extends the previous investigations of the superradiant laser in several aspects. Most importantly, we relax the restriction to the maximal cooperativity of N atoms which would arise in the absence of incoherent perturbations if the atoms were initially prepared in a fully symmetric state like the overall ground state. Since our model assumes identical three-level atoms coupled to resonator modes through the collective population and polarization operators $S_{ij}=\Sigma_{\mu=1}^{N}(|i\rangle\langle j|)^{\mu}$, with $|i\rangle^{\mu}$ the ith state of the μth atom, $i=0,1,2$, the time evolution has U(3) symmetry and stays in a fully symmetric subspace if it starts there. We here allow for partially symmetric subspaces and thus submaximal cooperativity. An interesting surprise is incurred in the semiclassical analysis: Depending on the values chosen for the conserved operators of U(3), $C_1=\Sigma_i S_{ii}=N$, $C_2=\Sigma_{i,j}S_{ij}S_{ji}$, $C_3=\Sigma_{i,j,k}S_{ij}S_{jk}S_{ki}$, the classical phase space (relevant for the limit $N\to\infty$) may have either four or six-dimensions. In the former case, a stationary regime is attained which differs from the one previously found for maximal cooperativity ($C_2=C_1^2$, $C_3=C_1^3$) only by some rescaling: Roughly speaking, submaximal cooperativity of N atoms is equivalent to maximal cooperativity of a smaller number of atoms. In the case of a six-dimensional atomic phase space more complicated behavior arises. In particular, the frequencies of the radiated modes are shifted away from the bare resonances. In no case have we found partial cooperativity to fully undo the noise suppression characteristic of maximal cooperativity.

We then take up spontaneous emission once more. The partial-cooperativity solution pertaining to the four-dimensional atomic phase space in the absence of spontaneous emission turns out to play an important role for the treatment of spontaneous emission as well, even though this perturbation breaks the U(3) symmetry of the atomic dynamics.

The paper is organized as follows. In Sec. II we describe in detail the model of the superradiant laser, the Heisenberg equations of motion which govern its evolution, and the corresponding constants of motion. In Sec. III we determine, in the semiclassical limit $N\gg 1$, the admissible values of the constants of motion. There we also discuss the construction of solutions with partial atomic cooperativity, based on a particular invariance property [related to U(3)] of the equations of motion. In Sec. IV we find the stationary solutions of the superradiant laser with partial atomic cooperativity and investigate their stability and fluctuation properties. In Sec. V we establish the stationary regime corresponding to a six-dimensional atomic phase space. In Sec. VI we employ a bit of group theory to investigate the interrelations between the solutions with partial atomic cooperativity and the underlying U(3) symmetry of the superradiant laser. Finally, in Sec. VII we discuss the breaking of the U(3) symmetry by spontaneous emission using the set of solutions with partial atomic cooperativity.

II. DYNAMICS OF THE SUPERRADIANT LASER AND ITS CONSTANTS OF MOTION

As in Refs. [4,5], we consider the simplest model of a superradiant laser which accounts for N three-level atoms (see Fig. 1) placed inside a resonator. We assume a pump process $0\to 2$ consisting of a classical resonant (two-photon) excitation. A coupling of the atoms to a cavity mode is assumed in tune with the transition $2\leftrightarrow 1$. We shall refer to this mode as to the ''active'' one. Finally, a certain relaxation process $1\to 0$ has to be included to recycle the atoms back to the influence of the pump. If spontaneous emission between

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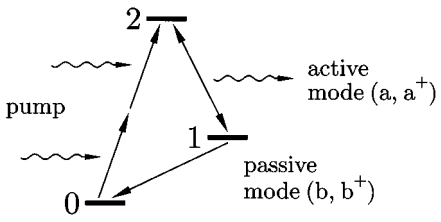

FIG. 1. Scheme of three-level superradiant laser.

levels 1 and 0 serves that purpose one has the so-called Raman laser whose quantum noise properties were recently investigated by Ritsch, Marte, and Zoller in Ref. [6].

We require a different type of relaxation by coupling the atoms to another cavity mode resonant with the transition $1\rightarrow 0$. With respect to the latter mode the resonator does not need to have a high finesse. In fact, in order to simplify the model we assume that this mode is damped sufficiently strongly so that it can be eliminated adiabatically. That is why we refer to this mode as the ''passive'' mode. The situation in consideration would, in the absence of level 2 and with level 1 populated initially, entail the no-ringing limit of superfluorescence on the transition $1\rightarrow 0$, as observed in Ref. [3]. As we have shown in Refs. [4,5], such a scheme with a third level and a stationary coherent pump yields stationary superradiance on both transitions, $2\leftrightarrow 1$ and $1\leftrightarrow 0$.

As long as we neglect spontaneous emission and other incoherent processes, the dynamics of the atoms is fully collective and can be described in terms of global operators $S_{ij}=\Sigma_{\mu=1}^{N}S_{ij}^{\mu}=\Sigma_{\mu}(|i\rangle\langle j|)^{\mu}$. There are nine such S_{ij}; those with $i\neq j$ refer to polarizations while each ''diagonal'' one, S_{ii}, measures the global occupation of level i; they obey $S_{ij}^{\dagger}=S_{ji}$ and $[S_{ij},S_{kl}]=\delta_{jk}S_{il}-\delta_{il}S_{kj}$. By choosing suitable linear combinations [namely, $i(S_{ij}+S_{ji})$ and $S_{ij}-S_{ji}$] one obtains nine anti-Hermitian generators of the Lie algebra of the group U(3). Denoting the creation and annihilation operators of photons in the active (passive) mode by a and $a^{\dagger}$ (b and $b^{\dagger}$), we have the interaction Hamiltonian

$$H_0=i\hbar g_a(a^{\dagger}S_{12}-aS_{21})+i\hbar\Omega(S_{20}-S_{02})+i\hbar g_b(b^{\dagger}S_{01}-bS_{10}). \tag{2.1}$$

In order to secure a ring of allowed transitions we assume that the pump process consists of a two-photon absorption; the pump strength Ω is proportional to the product of the amplitudes of the two classical pump waves and is chosen real; $g_a(g_b)$ is the coupling constant of the active (passive) mode with the corresponding atomic transition.

When neglecting the spatial dependence of the atomic operators in the Hamiltonian (2.1) we assume either the dimension of the atomic system much smaller than the wavelength of the laser light, or a running-wave setup with suitable phase matching.

The Heisenberg equations of motion generated by the Hamiltonian (2.1) must be complemented by the damping terms for the mode amplitudes and the corresponding Langevin forces which provide the conservation of the Bose commutators $[a(t),a^{\dagger}(t)]=[b(t),b^{\dagger}(t)]=1$ at all times. Assuming that the damping rate κ_b of the passive mode is much larger than κ_a of the active one and eliminating adiabatically the passive mode, we arrive at the following set of equations of motion for the superradiant laser [5]:

$$\begin{aligned}
\dot{S}_{02}&=-g_aaS_{01}-\Omega(S_{22}-S_{00})+\gamma S_{12}S_{01}-\sqrt{2\gamma}S_{12}\eta_b,\\
\dot{S}_{12}&=g_aa(S_{22}-S_{11})+\Omega S_{10}-\gamma S_{10}S_{02}+\sqrt{2\gamma}\eta_b^{\dagger}S_{02},\\
\dot{S}_{01}&=g_aa^{\dagger}S_{02}-\Omega S_{21}+\gamma(S_{11}-S_{00})S_{01}-\sqrt{2\gamma}(S_{11}-S_{00})\eta_b,\\
\dot{S}_{00}&=-\Omega(S_{02}+S_{20})+2\gamma S_{10}S_{01}-\sqrt{2\gamma}(S_{10}\eta_b+\eta_b^{\dagger}S_{01}),\\
\dot{S}_{11}&=g_a(aS_{21}+a^{\dagger}S_{12})-2\gamma S_{10}S_{01}+\sqrt{2\gamma}(S_{10}\eta_b+\eta_b^{\dagger}S_{01}),\\
\dot{S}_{22}&=\Omega(S_{02}+S_{20})+g_a(aS_{21}+a^{\dagger}S_{12}),\\
\dot{a}(t)&=g_aS_{12}-\kappa_aa(t)+\sqrt{2\kappa_a}\eta_a(t).
\end{aligned} \tag{2.2}$$

Here $\gamma=g_b^2/\kappa_b$ is a rate constant for the collective atomic relaxation $1\rightarrow 0$. The quantum Langevin noise forces η_{α} with $\alpha=a,b$ are taken as independent with Gaussian statistics and white spectra according to

$$\begin{aligned}
[\eta_{\alpha}(t),\eta_{\beta}^{\dagger}(t')]&=\langle\eta_{\alpha}(t)\eta_{\beta}^{\dagger}(t')\rangle=\delta_{\alpha\beta}\delta(t-t'),\\
\langle\eta_{\alpha}(t)\rangle&=\langle\eta_{\alpha}^{\dagger}(t)\eta_{\beta}(t')\rangle=\langle\eta_{\alpha}(t)\eta_{\beta}(t')\rangle=0,
\end{aligned} \tag{2.3}$$

and may be considered as representatives of the vacuum fluctuations of the electromagnetic field outside the resonators.

The system of equations (2.2) possesses a set of constants of the motion. First, it is easy to verify invariance under a phase shift ϕ of the form

$$\begin{aligned}
a\rightarrow e^{i\phi}a,\quad S_{12}\rightarrow e^{i\phi}S_{12},\quad S_{10}\rightarrow e^{i\phi}S_{10},\quad \eta_a\rightarrow e^{i\phi}\eta_a,\\
\eta_b\rightarrow e^{-i\phi}\eta_b.
\end{aligned} \tag{2.4}$$

Moreover, the three operators

$$C_1=\sum_i S_{ii},\quad C_2=\sum_{i,j}S_{ij}S_{ji},\quad C_3=\sum_{i,j,k}S_{ij}S_{jk}S_{ki} \tag{2.5}$$

commute with the Hamiltonian (2.1). The operators C_2 and C_3 are the Casimir invariants of the group U(3) which commute with all operators S_{ij}, $[C_k,S_{ij}]=0$. In Sec. VI we shall calculate the spectra of these Casimir operators and discuss the corresponding irreducible representations of U(3).

As in [4,5] we shall confine ourselves to the semiclassical limit, $N\gg 1$. Each of the eleven operators $S_{ij},a,a^{\dagger}$ in the Heisenberg equations (2.2) can then be represented as a sum of a dominant ''classical'' term $\overline{X}\sim N$ and a ''small'' operator-valued fluctuation δX,

$$X=\overline{X}+\delta X. \tag{2.6}$$

To find the dominant part $\overline{X}$ from the Heisenberg equations (2.2) we drop the Langevin noise forces and degrade each operator X to a c number $\overline{X}$.

The solution of the classical equations of motion depend on the expectation values of the three constants of motion

C_1, C_2, C_3. To within corrections of relative order $1/N$ we may factorize the expectation values of operator products as, e.g., $\overline{S_{ij}(t)S_{kl}(t)} = S_{ij}(t)S_{kl}(t)$, and obtain

$$\overline{C_1} = \sum_i \overline{S_{ii}} = N, \quad \overline{C_2} = \sum_{i,j} \overline{S_{ij}}\,\overline{S_{ji}} = c_2 N^2,$$

$$\overline{C_3} = \sum_{i,j,k} \overline{S_{ij}}\,\overline{S_{jk}}\,\overline{S_{ki}} = c_3 N^3. \tag{2.7}$$

Clearly, the mean value of C_1 is just the number of atoms that is preserved in our scheme. The two other quantities depend upon the atomic initial state. We shall refer to their scaled versions c_2 and c_3 as to cooperativity parameters.

III. ADMISSIBLE COOPERATIVITY PARAMETERS

To be physically acceptable, the solutions of the classical equations of motion have to satisfy two requirements of quantum-mechanical origin. First, the mean values $\overline{S_{ij}}$ must form a nonnegative Hermitian matrix. Second, the diagonal elements sum up to the number of atoms. These restrictions imply three Schwartz' inequalities, $\overline{S_{ii}}\,\overline{S_{jj}} - \overline{S_{ij}}\,\overline{S_{ji}} \geqslant 0$ and $0 \leqslant \overline{S_{ii}}/N \leqslant 1$. As further consequences we now propose establishing minimum and maximum values of c_3 for given c_2 and N.

We can obviously write c_2 and c_3 as the traces of powers of the Hermitian and non-negative matrix $\overline{S_{ij}}$,

$$N^2 c_2 = \mathrm{tr}\overline{S^2}, \quad N^3 c_3 = \mathrm{tr}\overline{S^3}. \tag{3.1}$$

The eigenvalues $l_i, i = 1,2,3$, of $\overline{S_{ij}}$ must be real and non-negative. From (3.1) we have $c_2 = l_1^2 + l_2^2 + l_3^2$ and $c_3 = l_1^3 + l_2^3 + l_3^3$. Observing that $(S_{00} + S_{11} + S_{22})/N = l_1 + l_2 + l_3 = 1$ we immediately obtain

$$\frac{1}{3} \leqslant c_2 \leqslant 1, \quad \frac{1}{9} \leqslant c_3 \leqslant 1. \tag{3.2}$$

Using Newton's formulas [7] we express the coefficients of the characteristic polynomial of the matrix $\overline{S_{ij}}$ via its traces and arrive at

$$(l - l_1)(l - l_2)(l - l_3) = l^3 - l^2 + \frac{1}{3}(1 - r^2)l - \frac{1}{3}(c_3 - r^2), \tag{3.3}$$

where we introduced the participation parameter

$$r = \sqrt{\frac{3c_2 - 1}{2}}, \quad 0 \leqslant r \leqslant 1. \tag{3.4}$$

The three roots of the third-order polynomial (3.3) are real only if its discriminant

$$D = -\frac{1}{9}r^6 + 9\left(\frac{c_3}{2} - \frac{r^2}{3} - \frac{1}{18}\right)^2 \tag{3.5}$$

is smaller than zero [7], which gives

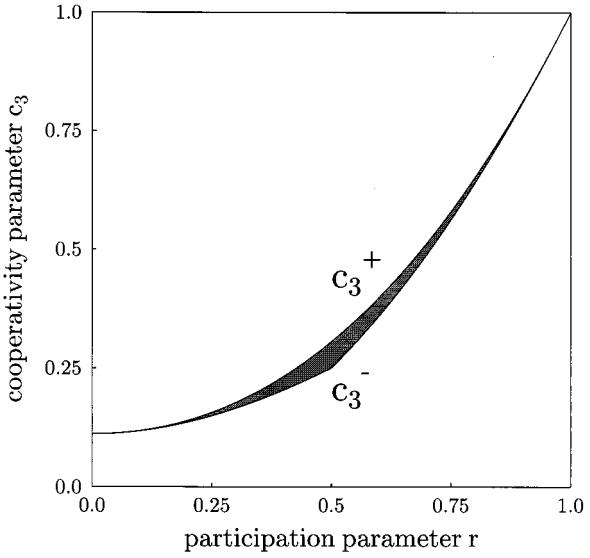

FIG. 2. The region of admissible cooperativity parameters c_2 and c_3 according to (3.9). In the text we refer to curve $c_3^+(r)$ as the upper boundary and to the two segments of $c_3^-(r)$ as the lower $(0 \leqslant r \leqslant 1/2)$ and right $(1/2 \leqslant r \leqslant 1)$ boundaries.

$$\frac{1 + 6r^2 - 2r^3}{9} \leqslant c_3 \leqslant \frac{1 + 6r^2 + 2r^3}{9}. \tag{3.6}$$

Applying the Hurwitz criterion [8] to the third-order polynomial (3.3) we find that all roots $l_i, i = 1,2,3$ are non-negative only if

$$r^2 \leqslant c_3. \tag{3.7}$$

Hence we finally obtain the following relation between the admissible cooperativity parameters r and c_3:

$$c_3^-(r) \leqslant c_3 \leqslant c_3^+(r), \tag{3.8}$$

where

$$c_3^-(r) = \min\{r^2, (1 + 6r^2 - 2r^3)/9\} = \begin{cases} (1 + 6r^2 - 2r^3)/9, & 0 \leqslant r \leqslant 1/2, \\ r^2, & 1/2 \leqslant r \leqslant 1, \end{cases}$$

$$c_3^+(r) = (1 + 6r^2 + 2r^3)/9, \quad 0 \leqslant r \leqslant 1. \tag{3.9}$$

This region is shown in Fig. 2; $r = 1$ brings us back to the case of the full cooperativity, $c_2 = c_3 = 1$; minimal cooperativity is attained for $c_2 = 1/3$, $c_3 = 1/9$, i.e., $r = 0$; in this case the mean values of the atomic populations are all equal, $\overline{S_{ii}} = N/3$, and all stationary polarizations vanish, $\overline{S_{ij}} = 0$, $i \neq j$.

The system of Heisenberg equations of motion (2.2) has a particular symmetry. Suppose that we have found a time-dependent solution $S_{ij}(t), a(t)$ of the system (2.2) for $\widetilde{N}$ atoms in full cooperativity, i.e., $\widetilde{C}_1 = \widetilde{N}$, $\widetilde{C}_2 = \widetilde{N}^2$, $\widetilde{C}_3 = \widetilde{N}^3$. Using this solution we can construct two other solutions as follows:

$$S^e_{ii}(t)=\frac{1-er}{3r}\widetilde{C}_1+eS_{ii}(t),\quad a^e(t)=ea(t),\quad \text{for } e=\pm 1,$$

$$S^e_{02}(t)=S_{02}(t),\quad S^e_{12}(t)=S_{12}(t),\quad S^e_{01}(t)=S_{01}(t),\quad \text{for } e=+1, \tag{3.10}$$

$$S^e_{02}(t)=S^\dagger_{02}(t),\quad S^e_{12}(t)=-S^\dagger_{12}(t),\quad S^e_{01}(t)=S^\dagger_{01}(t),\quad \text{for } e=-1.$$

Here we have introduced a bivalued parameter $e=\pm 1$ which we will use throughout the paper. Using the fact that $\widetilde{C}_1$ commutes with all S_{ij} it is easy to verify that $S^e_{ij}(t)$ and $a^e(t)$ obey the same commutation relations as $S_{ij}(t)$ and $a(t)$. By direct substitution of these solutions into the evolution equations (2.2) we observe that the first one, i.e., the one with $e=+1$, satisfies the same equations as $S_{ij}(t),a(t)$ while the second, with $e=-1$, formally solves the time reversed system, $dS^e_{ij}/dt=-dS_{ij}/dt$, $da^e/dt=-da/dt$. Since time reversal, $t\rightarrow -t$, does not affect the stationary solution, we can use the transformation (3.10) to construct two new stationary solutions of the system (2.2) using a given solution $\overline{S_{ij}},\overline{a}$ for $\widetilde{N}$ atoms.

Requiring that the new solutions describe N atoms we immediately get from the transformation (3.10) that $\widetilde{N}=rN$. Inasmuch as all populations S^e_{ii} and S_{ii} must be positive the first new solution makes physical sense only as long as $0\leqslant(1-r)N/3\leqslant N/3$. It also follows for $e=+1$ that $S^e_{ii}\geqslant(1-r)N/3$, i.e., that all atomic levels are populated by at least $N(1-r)/3$ atoms. Upon semiclassically evaluating the Casimir constants for the case $e=+1$ we find the cooperativity parameter $c_2=(1+2r^2)/3$, $c_3=c_3^+(r)$, i.e., according to the upper boundary in Fig. 2. In conclusion, the transformation (3.10) with $e=+1$ reveals that all points on the upper boundary $c_3^+(r)$, $0\leqslant r\leqslant 1$, of the allowed range of c_3 and r actually correspond to stationary solutions of the superradiant laser. These solutions are all related to one another such that, summarily speaking, submaximal cooperativity corresponds to full cooperativity of a smaller number of atoms.

Similarly, for the second new solution ($e=-1$) to make physical sense both the S_{ii} and S^e_{ii} must be nonnegative. This gives the restrictions $0\leqslant(1-2r)N/3\leqslant S^e_{ii}\leqslant(1+r)N/3$. The participation parameter is thus confined to the reduced range $0\leqslant r\leqslant 1/2$. Moreover, none of the atomic levels can accommodate more than $(1+r)N/3\leqslant N/2$ atoms. Once more evaluating the Casimir constants we again find $c_2=(1+2r^2)/3$ but $c_3=(1+6r^2-2r^3)/9=c_3^-(r)$, the latter equality due to $0\leqslant r\leqslant 1/2$. We conclude that all points on the lowest boundary of the allowed range in the r,c_3 plane correspond to stationary states of the superradiant laser. These solutions connect continuously to the one of smallest cooperativity but not, in contrast to the one on the upper boundary $c_3^+(r)$, to the point of maximal cooperativity.

The upper and the lower boundaries, $c_3^\pm(r)$, are distinguished from all the other points within the region of admissible c_2 and c_3 by the fact that the Schwartz inequalities generalize to identities on these lines. Indeed, we have shown in Ref. [5] that for the point $r=1$, $c_3=1$ we have $|\overline{S_{ij}}|^2=S_{ii}S_{jj}$ and, in addition, $\psi\equiv\arg(S_{10}S_{02}S_{21})=0$. Using the transformation (3.10) we can express S_{ij} through S^e_{ij} and obtain the following relations between the mean values of polarizations and populations holding on the upper boundary, $c_3^+(r)$, $0\leqslant r\leqslant 1$,

$$|\overline{S^e_{ij}}|^2=\left(\overline{S^e_{ii}}-N\frac{1-r}{3}\right)\left(\overline{S^e_{jj}}-N\frac{1-r}{3}\right),\quad i\neq j,$$

$$\psi^e=0,\quad e=+1. \tag{3.11}$$

The corresponding relations for the lower boundary, $c_3^-(r)$, $0\leqslant r\leqslant 1/2$, read as

$$|\overline{S^e_{ij}}|^2=\left(\overline{S^e_{ii}}-N\frac{1+r}{3}\right)\left(\overline{S^e_{jj}}-N\frac{1+r}{3}\right),\quad i\neq j,$$

$$\psi^e=\pi,\quad e=-1. \tag{3.12}$$

As for the full collectivity investigated in Ref. [5], these relations offer an opportunity to simplify the analysis of the superradiant laser for the collectivity parameters $0\leqslant r\leqslant 1/2$, $c_3^-(r)$ and $0\leqslant r\leqslant 1$, $c_3^+(r)$. The simplification is based on the four arising constants of motion of the classical dynamical equations. Indeed, if the initial atomic configurations are chosen so that the conditions (3.11) or (3.12) hold at $t=0$, they will be preserved for all other times. The set of seven constants of the motion consisting of N,C_2,C_3 and (3.11) or (3.12) contains five independent ones. These reduce the number of independent atomic variables $\overline{S^e_{ij}}$ from nine to four. Solutions with r and c_3 inside the allowed range shown in Fig. 2 have two constants of the motion less.

A conclusion may be drawn which is worth a little emphasis. Quantum-mechanical U(3) dynamics (as executed here by a collection of N identical three-level atoms) may have a limiting classical dynamics (attained here as $N\rightarrow\infty$) taking place in a phase space of dimension either four or six, depending on the initial state. The dimension is six for pairs c_3,r inside the range shown in Fig. 2 as well as on the right boundary, $c_3=c_3^-(r)$, $1/2\leqslant r\leqslant 1$. The dimension four arises for pairs c_3,r on either the upper or the lower boundary. No such dimensional ambiguity is possible for U(2) dynamics while the number of different possible phase space dimensions increases for U(n) as n grows, $n=3,4,\ldots$.

IV. STATIONARY SOLUTIONS AND FLUCTUATION PROPERTIES OF THE SUPERRADIANT LASER WITH PARTIAL COOPERATIVITY

In this section we shall investigate the stationary solutions and the fluctuation spectra of the superradiant laser for the boundary cooperativity parameters $c_3^+(r)$, $0\leqslant r\leqslant 1$ and $c_3^-(r)$, $0\leqslant r\leqslant 1/2$. We confine ourselves, as before, to the semiclassical limit, $N\gg 1$.

To describe the stationary solution of the classical equations of motion we introduce the same dimensionless parameters as in [4], namely, a dimensionless coupling strength c, an effective pump strength p, and a time scale ratio ξ,

$$c=\frac{g_a^2}{\kappa_a\gamma},\quad p=\frac{\Omega}{N\gamma\sqrt{c}},\quad \xi=\frac{N\gamma}{\kappa_a}. \tag{4.1}$$

Note that the pump parameter p and the time scale ratio ξ depend on the number of atoms in the cavity.

Solving the equations of motion in the stationary regime we obtain two real valued solutions ($e=\pm 1$) which are expressed in terms of a single amplitude x as

$$\begin{aligned}
\overline{S^e_{01}}/N&=x,\quad \overline{S^e_{02}}/N=\frac{x^2}{p\sqrt{c}},\quad \overline{S^e_{12}}/N=e\frac{x}{\sqrt{c}},\\
\overline{S^e_{00}}/N&=\frac{1}{3}\left[1-e\left(p+(1-2c)\frac{x^2}{pc}\right)\right],\\
\overline{S^e_{11}}/N&=\frac{1}{3}\left[1-e\left(-2p+(1+c)\frac{x^2}{pc}\right)\right],\\
\overline{S^e_{22}}/N&=\frac{1}{3}\left[1-e\left(p+(c-2)\frac{x^2}{pc}\right)\right],\\
\overline{a^e}/N&=-\frac{e}{\sqrt{c}}x.
\end{aligned} \tag{4.2}$$

We would like to point out here that in the case of full cooperativity the alternative $e=\pm 1$ formally arises as well. However, the minus sign leads to a physically unacceptable solution in conflict with the Schwartz inequality, $|\overline{S_{ij}}|^2\leqslant\overline{S_{ii}}\,\overline{S_{jj}}$. In our previous publication [5], devoted to full cooperativity, we had employed the Schwinger representation $S_{ij}=z_i^\dagger z_j$ which enforces $|\overline{S_{ij}}|^2=\overline{S_{ii}S_{jj}}$. Therefore, the solution with the minus sign was automatically ruled out.

Inserting the solutions (4.2) into the expression for the mean value of the Casimir operator C_2 we obtain an equation for the amplitude x which entails

$$x=\sqrt{\frac{pc(r-p)}{1+c}}. \tag{4.3}$$

Surprisingly, we do not need the expectation value $\overline{C_3}$ to determine the stationary solutions but rather obtain the two corresponding collectivity parameters $c_3^{\pm}(r)=(1+6r^2\pm 2er^3)/9$ after inserting the stationary solutions (4.2) into C_3. We may thus infer that resonant stationary solutions can be achieved only when the collectivity parameters lie on the upper or lower boundaries of the allowed region in Fig. 2.

To see explicitly how the stationary solutions for partial cooperativity given by (4.2) are related to those for full cooperativity found in Ref. [5], we rewrite (4.2) in terms of the following rescaled parameters:

$$\widetilde{N}=Nr,\quad \widetilde{p}=p/r,\quad \widetilde{\xi}=r\xi. \tag{4.4}$$

Note that the rescaled pump strength $\widetilde{p}$ now is allowed to take on values from 0 to 1, i.e., as is p for full cooperativity. In terms of these parameters the amplitude x reads

$$x=r\sqrt{\frac{\widetilde{p}c(1-\widetilde{p})}{1+c}}, \tag{4.5}$$

and the stationary solutions (4.2) take on the form

$$\begin{aligned}
\overline{S^e_{01}}&=\sqrt{\frac{\widetilde{p}c(1-\widetilde{p})}{1+c}}\widetilde{N},\quad \overline{S^e_{02}}=\frac{c(1-\widetilde{p})}{(1+c)\sqrt{c}}\widetilde{N},\\
\overline{S^e_{12}}&=e\sqrt{\frac{\widetilde{p}(1-\widetilde{p})}{1+c}}\widetilde{N},\\
\overline{S^e_{00}}&=\widetilde{N}\left[\frac{1-er}{3r}+e\frac{c(1-\widetilde{p})}{1+c}\right],\\
\overline{S^e_{11}}&=\widetilde{N}\left[\frac{1-er}{3r}+e\widetilde{p}\right],\\
\overline{S^e_{22}}&=\widetilde{N}\left[\frac{1-er}{3r}+e\frac{1-\widetilde{p}}{1+c}\right],\\
\overline{a^e}&=-e\sqrt{\frac{\widetilde{p}(1-\widetilde{p})}{1+c}}\widetilde{N}.
\end{aligned} \tag{4.6}$$

These reflect precisely the transformation (3.10). On the other hand, all terms containing the pump strength $\widetilde{p}$ are the same as for full cooperativity with $\widetilde{N}$ atoms. The additional $\widetilde{p}$-independent terms in the atomic populations are necessary to keep the total number of atoms equal to N. However, it is now obvious from (4.6) that in the case of partial cooperativity with participation parameter r only rN of the total N atoms take part in the collective interaction.

The linear stability analysis for the partially cooperative stationary solutions is done analogously to the case of full collectivity, investigated in Ref. [5]. We linearize the Heisenberg equations (2.2) around the stationary state (4.2) with respect to small fluctuations δS^e_{ij} and δa^e and split these fluctuations into Hermitian real and imaginary parts. The linearized equations separate into two independent blocks, a seven-dimensional one for the real parts (amplitude fluctuations) and a four-dimensional one for the imaginary parts (phase fluctuations). Stability requires that all eigenvalues have positive real parts. With the help of the Hurwitz criterion we obtain the following stability conditions:

$$\begin{aligned}
e\widetilde{p}&<e\frac{c+e(1+c)/\widetilde{\xi}}{1+2c},\\
\frac{1-c+e(1-c^2)/(\widetilde{\xi}c)}{2}&<\widetilde{p}<\frac{3+c-e(1-c^2)/(\widetilde{\xi}c)}{2},
\end{aligned} \tag{4.7}$$

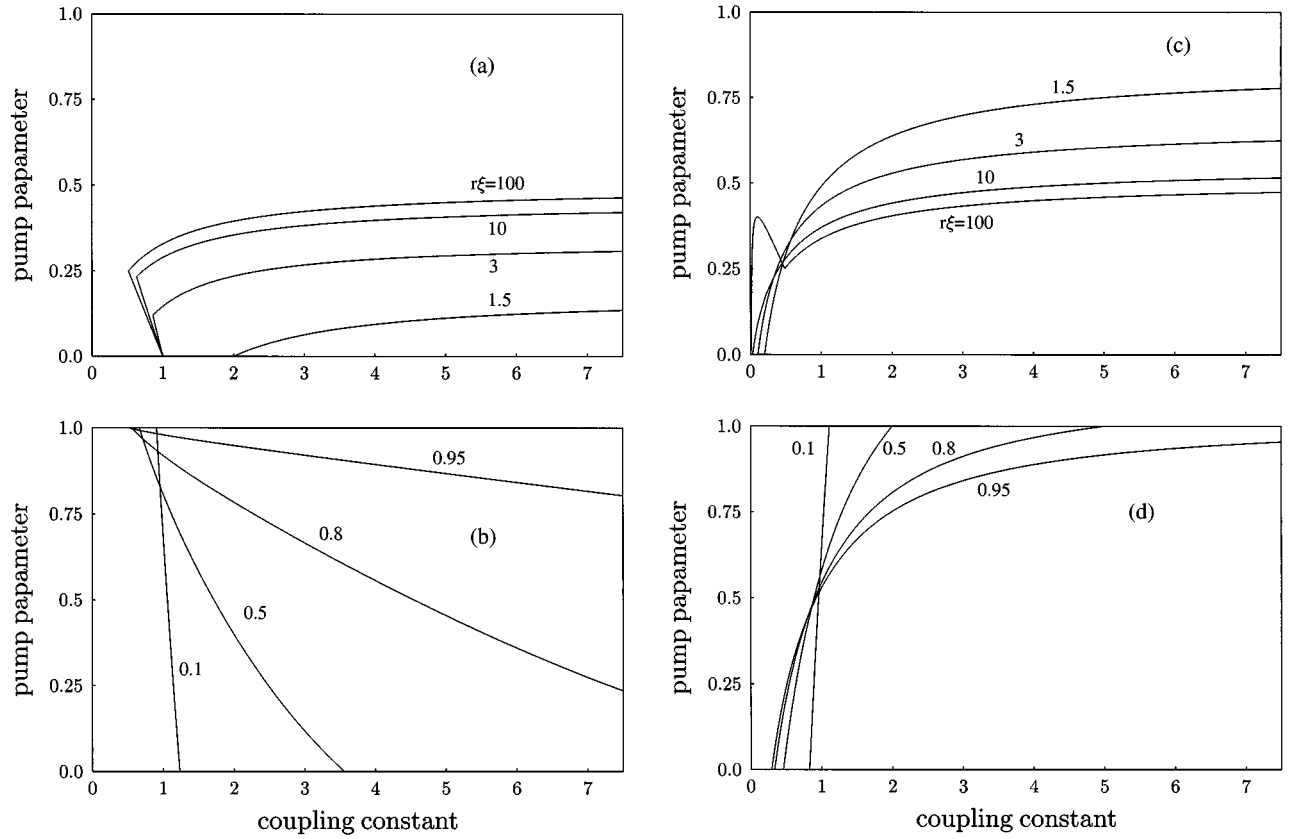

FIG. 3. Stability domains of the stationary solutions (4.6) according to (4.7) in the $(\tilde{p},c)$ plane for various values of the time scale ratio $\tilde{\xi}$. Cases (a), (b) pertain to $e=+1$; cases (c), (d) to $e=-1$. In (a) the range of stability lies upwards of the curve for the respective value of $\tilde{\xi}$ and in (b), (c), and (d) below.

$$0<e\left[-\frac{1-c}{\tilde{\xi}^2}+e\frac{\tilde{p}(1+2c)-2c}{\tilde{\xi}}-c\frac{[\tilde{p}(1+2c)-c](3-2\tilde{p}+c)}{(1+c)^2}\right].$$

In particular, for $e=+1$ we recover the stability conditions of full collectivity found in Ref. [5], now for properly rescaled parameters $\tilde{p}$ and $\tilde{\xi}$. It is important to realize that the two solutions obtained for $e=\pm 1$ cannot coexist since they arise for different values of c_3, i.e., $c_3^+(r)$ for $e=+1$ and $c_3^-(r)$ for $e=-1$. We thus incur no bistability.

In Fig. 3 we have plotted the regions of stability in the $\tilde{p},c$ plane for various time scale ratios $\tilde{\xi}$. As we know already from the case of full cooperativity, good squeezing arises for $p\ll 1$ and $c\approx 1$. Figure 3 reveals both solutions to be stable in this region when $\tilde{\xi}\ll 1$ or $\tilde{\xi}\gg 1$, i.e., in the bad- and good-cavity cases.

Beyond the fluctuations whose stability is secured by (4.7) we meet with six marginally stable eigenmodes, four from the amplitude block and two from the phase block. Four of these modes have vanishing eigenvalues and three of these are related to displacements δS_{ij}^e changing the values of the global constants of the motion C_1, C_2, and the phase ϕ appearing in the phase symmetry (2.4). We cannot count on C_3 as yielding the fourth vanishing eigenvalue since for the stationary state under discussion C_3 is not independent of C_1 and C_2.

In fact, infinitesimal displacements δS_{ij}^e changing global constants of the motion must lead to vanishing eigenvalues. This can be seen by considering such a constant $C(X)$ and an eigenmode $\delta X_i(t)=e^{-N\gamma\lambda t}\delta X_i(0)$ of the linearized evolution equations (with X_i a shorthand for the various observables and λ a dimensionless eigenvalue). We then rewrite the conservation law $C[X+\delta X(t)]=C[X+\delta X(0)]$ by keeping only first-order terms in the displacement δX from the stationary point X and differentiate with respect to the time. We thus obtain $\lambda\Sigma_i(\partial C/\partial X_i)_{X=\bar{X}}\delta X_i=0$. Indeed then, the eigenvalue λ must vanish for a δX not confined to the plane tangential to the surface $C(X)=C(\bar{X})$ in the fixed point $\bar{X}$.

Quite different is the fate of displacements away from a nonglobal constant of the motion (which yields only a single isolated invariant surface in phase space) [10] such as the three independent ones contained in (3.11) or (3.12). An eigenvalue associated with such a deviation δX may but need not vanish [10]. In our present case the three independent nonglobal constants of the motion actually still entail one eigenmode (the one associated with a change of the phase ψ^e) with vanishing eigenvalue. The other two eigenmodes come with breakings of the generalized Schwartz equalities in (3.11) or (3.12) and yield a pair of purely imaginary eigenvalues

$$\lambda=\pm i2r\sqrt{\tilde{p}c}. \tag{4.8}$$

The latter two eigenmodes obviously correspond to values of c_2 and c_3 away from the upper and lower boundaries [$c_3^+(r)$ with $0\leqslant r\leqslant 1$ and $c_3^-(r)$ with $0\leqslant r\leqslant 1/2$] of the allowed area in Fig. 2 and thus signal the behavior of the system, to be discussed in the next section, for c_2 and c_3 inside the allowed area.

In Sec. VII we shall see that under the influence of spontaneous emission four of the six marginally stable eigenmodes acquire negative real parts for the eigenvalues, i.e., become stable, due to the breaking of U(3). Only two vanishing eigenvalues will then survive since only two globally conserved quantities remain, the phase ϕ from the phase symmetry (2.4) and the number of atoms N.

Similarly to stationary solutions and stability analysis, the fluctuation spectra of the superradiant laser with partial cooperativity can be obtained from those for full cooperativity by rescaling $p\rightarrow\tilde{p}$, $\xi\rightarrow\tilde{\xi}$. In Ref. [5] we had obtained a particularly simple result for the amplitude fluctuation spectrum in the good-cavity limit, $\xi\gg 1$. Now this result holds true for $\tilde{\xi}\gg 1$ and reads as follows:

$$\langle \delta u_{\rm out}(\omega)\,\delta u_{\rm out}(\omega')\rangle=(1/4)\,\delta(\omega+\omega')\left\{1-\frac{S_0(\tilde{p},c)}{1+\omega^2\tau_a^2}\right\}, \tag{4.9}$$

with the squeezing strength $S_0(\tilde{p},c)$,

$$S_0(\tilde{p},c)=\frac{1}{2}+\frac{2c}{(1+c)^2}-\frac{\tilde{p}^2}{2(1-\tilde{p})^2}, \tag{4.10}$$

and the width

$$\frac{1}{\tau_a}=4\kappa_a\frac{(1-\tilde{p})(1+c)}{3+c-2\tilde{p}}. \tag{4.11}$$

Positive values of the squeezing strength indicate noise reduction below the vacuum level. Ideal squeezing is incurred at $\tilde{p}=0$, $c=1$. Since zero pumping is admittedly not a particularly interesting working point of a laser it is important that the squeezing strength S_0 has a rather flat maximum at that point so that good squeezing prevails for weak nonvanishing pumping.

Another quantity of interest is the low-frequency asymptotic version of the fluctuation spectrum of the phase quadrature since it gives the linewidth $\Delta\nu_a$ of the laser output through $\langle \delta v_{\rm out}(\omega)\,\delta v_{\rm out}(\omega')\rangle\rightarrow\delta(\omega+\omega')(\overline{a^e})^2\Delta\nu_a/\omega^2$ for $\omega\rightarrow 0$. The low-frequency divergence of $\langle \delta v_{\rm out}(\omega)\,\delta v_{\rm out}(\omega')\rangle$ as $\sim 1/\omega^2$ is characteristic of phase diffusion (see also [9]). In the limit $\xi\gg 1$ we obtain the linewidth

$$\Delta\nu_a^0=\frac{\kappa_a}{(\overline{a^e})^2}\,\frac{\tilde{p}^2(1+c)^2+(1-\tilde{p})^2(1-c)^2}{c-1+2\tilde{p}}. \tag{4.12}$$

It may be well to point out that the low-frequency divergence of $\langle \delta v_{\rm out}(\omega)\,\delta v_{\rm out}(\omega')\rangle$ as $\sim 1/\omega^2$ is related to the vanishing eigenvalue associated with the phase symmetry (2.4). Interestingly, the other five eigenvalues with zero real parts do not cause an analogous divergency in either amplitude or phase quadratures. This is because the corresponding eigenmodes are not driven by Langevin forces and thus have zero diffusion coefficients.

V. SHIFTED-FREQUENCY SOLUTIONS

Up to this point we have been concerned with stationary solutions of the classical equations of motion pertaining to special values of the collectivity parameters, i.e., $c_3=c_3^-(r)$, $0\leqslant r\leqslant 1/2$ or $c_3=c_3^+(r)$, $0\leqslant r\leqslant 1$. Our numerical simulations of the classical versions of the evolution equations (2.2) indicate that for c_3 not on the lower or upper boundary the laser field exhibits temporally periodic behavior after some transient. To find this periodic solution we recall the phase symmetry (2.4) and introduce new atomic and field variables $\tilde{S}_{ij},\tilde{a},\tilde{b}$ as

$$\tilde{a}(t)=a(t)e^{i\Delta t},\quad \tilde{b}(t)=b(t)e^{-i\Delta t},$$

$$\tilde{S}_{12}(t)=S_{12}(t)e^{i\Delta t},\quad \tilde{S}_{01}(t)=S_{01}(t)e^{-i\Delta t},$$

$$\tilde{S}_{02}(t)=S_{02}(t),\quad \tilde{S}_{ii}(t)=S_{ii}(t), \tag{5.1}$$

where Δ is a frequency shift to be determined. This transformation allows for frequency shifts of the polarizations S_{12} and S_{01} as $\tilde{\omega}_{12}=\omega_{12}+\Delta$ and $\tilde{\omega}_{01}=\omega_{01}-\Delta$ (and similarly for the mode amplitudes) such as to keep intact the sum frequency $\tilde{\omega}_{02}=\omega_{02}$. When the so transformed observables approach constant values, $\tilde{X}(t)\rightarrow\tilde{X}(\infty)$, one may of course speak of a stationary state with respect to a suitable ''rotating frame.''

When again eliminating adiabatically the passive mode we have to assume that the damping rate κ_b of the passive mode is much larger than Δ. We obtain a closed system of equations of motion analogous to (2.2) for the new atomic variables and the amplitude of the active mode. The frequency shift Δ now is an additional unknown quantity so that we must invoke the conservation of c_3 to fully specify a stationary solution.

While we have not succeeded in finding closed-form solutions for general values of r and c_3 inside the allowed range, the behavior for r and c_3 close to the upper and lower boundaries, $c_3^+(r)$ with $0\leqslant r\leqslant 1$ and $c_3^-(r)$ with $0\leqslant r\leqslant 1/2$, is accessible perturbatively. The ''boundary layers'' in question can be defined by a small positive parameter δ as

$$c_3=c_3^+(r)-\delta\quad\text{for}\quad 0\leqslant r\leqslant 1$$

or

$$c_3=c_3^-(r)+\delta\quad\text{for}\quad 0\leqslant r\leqslant 1/2, \tag{5.2}$$

with $0\leqslant\delta\ll 1$. For the sake of simplicity we also restrict ourselves to the good-cavity limit $\xi\gg 1$ and $\Delta\ll(1-c)\kappa_a$. Under these assumptions we may use perturbation expansions in powers of the small parameter

$$\epsilon=\frac{\Delta}{(1-c)\kappa_a} \tag{5.3}$$

for the deviations from the former stationary solutions, given by (4.2). Including second-order terms in ϵ we obtain (we drop the ''tilde'' sign for brevity so that the overbar now distinguishes the new stationary state)

$$\overline{S_{01}^e}/N=x, \quad \overline{S_{12}^e}/N=\frac{ex}{\sqrt{c}}\left[1-ic\epsilon+\frac{1}{2}(1-c)^2\epsilon^2\right],$$

$$\overline{S_{02}^e}/N=\frac{x^2}{p\sqrt{c}}\left[1+i\epsilon\frac{p^2c-x^2}{x^2}+\frac{\epsilon^2}{2}\left(\frac{p^2c-x^2}{x^2}\right)^2\right],$$

$$\overline{S_{00}^e}/N=\frac{1}{3}\left[1+-e\left(1+\frac{\epsilon^2}{2}\right)\left(p+(1-2c)\frac{x^2}{pc}\right)\right],$$

$$\overline{S_{11}^e}/N=\frac{1}{3}\left[1-e\left(1+\frac{\epsilon^2}{2}\right)\left(-2p+(1+c)\frac{x^2}{pc}\right)\right], \quad (5.4)$$

$$\overline{S_{22}^e}/N=\frac{1}{3}\left[1-e\left(1+\frac{\epsilon^2}{2}\right)\left(p+(c-2)\frac{x^2}{pc}\right)\right],$$

$$\overline{a^e}/N=-\frac{e}{\sqrt{c}}x(1-i\epsilon),$$

with

$$x=\sqrt{\frac{pc(r-p)}{1+c}}\left[1-\frac{18cp^2-3cpr(5-c)+r^2(1+c)}{4(1+c)(r-p)}\epsilon^2\right]. \quad (5.5)$$

The parameter ϵ or, equivalently, the frequency shift Δ is determined by the deviation δ of c_3 from the boundary value $c_3^{\pm}(r)$ as

$$\delta=\frac{3cp[2p-r(1-c)]^2}{(1-c^2)^2}\left(\frac{\Delta}{\kappa_a}\right)^2. \quad (5.6)$$

It follows that the assumption of a small frequency shift, $\Delta/(1-c)\kappa_a \ll 1$, is consistent with $\delta \ll 1$, i.e., for collectivity parameters close to the upper and lower boundary of the domain in Fig. 2. Moreover, we find two frequency shifts $\Delta_{\pm}$ with different signs, i.e., two different nonresonant stationary solutions with symmetric frequency shifts.

To save space we do not present a full semiclassical analysis of these frequency-shifted solutions. We only point out that the characteristic polynomial of the linearized equations of motion is a function of $\Delta_{\pm}^2$. We may therefore infer that the domains of stability of the two solutions, determined by $\Delta_{\pm}$, coincide.

VI. ANALYSIS OF THE ATOMIC HILBERT SPACE

In this section we shall interpret the results obtained above using the theory of group representations. Group theory comes in due to the fact that the operators C_2, C_3 are the Casimir operators of the group U(3), i.e., they commute with all the S_{ij}. Consequently, the atomic Hilbert space, consisting of 3^N product states of the form $|k_1\rangle_1|k_2\rangle_2\cdots|k_N\rangle_N$ with $k_i=0,1,2$, may be split into orthogonal eigenspaces of the Casimir operators. These eigenspaces are invariant under the application of the collective atomic operators and are called invariant subspaces or irreducible-representation spaces of the group U(3). For example, for two atoms the sextet of fully symmetric states $(1/\sqrt{2})(|i_1\rangle|i_2\rangle+|i_2\rangle|i_1\rangle)$ and the triplet of antisymmetric states $(1/\sqrt{2})(|i_1\rangle|i_2\rangle-|i_2\rangle|i_1\rangle)$ form two orthogonal invariant subspaces of the nine-dimensional atomic Hilbert space. Due to the exponential increase of the dimension of the Hilbert space with N, the explicit construction of invariant subspaces becomes a cumbersome task for large N. Nevertheless, the theory of group representations allows us to construct the basis states of any arbitrary invariant subspace in the form of Young tableaux [11,12].

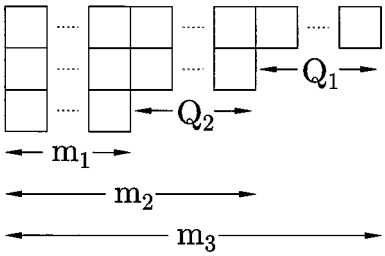

FIG. 4. Young frame for an irreducible representation with the highest weights m_1, m_2, m_3.

Provided each of the N atoms is represented by a box, each invariant subspace is uniquely specified by arranging the boxes into a diagram, a Young frame (see Fig. 4) which consists of at most three rows. The lengths m_1, m_2, m_3 of the rows, called highest weights, monotonically decrease from above, $m_1 \geqslant m_2 \geqslant m_3$, and sum up to N, $m_1+m_2+m_3=N$. The basis states of a given invariant subspace are represented by Young tableaux which are constructed from a Young frame by inserting one of the numbers 0,1,2 (the possible values of the level index) into each box, following two rules: (α) numbers within each row are distributed in nondecreasing order from left to right, and (β) numbers in every column are strictly increasing from top to bottom. Figure 5 shows an example. In the space spanned by the 3^N states of N three-level atoms with levels $|0\rangle_i, |1\rangle_i$, and $|2\rangle_i$, $=1,\ldots,N$, a Young tableau corresponds to a linear combination of the product states of the form $|k_1\rangle_1|k_2\rangle_2\cdots|k_N\rangle_N$ with $k_i=0,1,2$, obtained by first symmetrizing with respect to the atoms in each row and then antisymmetrizing in each column.

To classify the invariant subspaces of our N three-level atoms it is convenient to introduce two non-negative integer

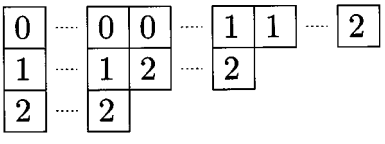

FIG. 5. Example of a Young tableau.

numbers $Q_1=m_1-m_2$ and $Q_2=m_2-m_3$. The weights can be expressed in terms of Q_1, Q_2, and N,

$$m_1=\frac{1}{3}(N+2Q_1+Q_2), \quad m_2=\frac{1}{3}(N-Q_1+Q_2),$$

$$m_3=\frac{1}{3}(N-Q_1-2Q_2). \tag{6.1}$$

The dimension d of the corresponding invariant subspace and the Casimir invariants C_2 and C_3 are given by [11]

$$d(Q_1,Q_2)=\frac{1}{2}(Q_1+1)(Q_2+1)(Q_1+Q_2+2), \tag{6.2}$$

$$C_2=\frac{1}{3}N^2+\frac{2}{3}(Q_1^2+Q_2^2+Q_1Q_2+3Q_1+3Q_2), \tag{6.3}$$

$$C_3=\frac{1}{9}N^3+\frac{2}{3}N(Q_1^2+Q_2^2+Q_1Q_2+3Q_1+3Q_2)$$
$$+\frac{1}{9}(2Q_1^3+3Q_1^2Q_2-3Q_1Q_2^2-2Q_2^3+18Q_1^2+9Q_1Q_2$$
$$+36Q_1+18Q_2). \tag{6.4}$$

Let us consider a representation with $m_3\neq 0$ (see Fig. 5). According to the rules for constructing Young tableaux there is only one possibility to fill the columns consisting of three boxes, as shown in Fig. 5. All states in that representation behave under the action of the polarizations S_{ij} with $i\neq j$ as well as of the population differences $S_{ii}-S_{jj}$ as if the three-box columns were absent, i.e., as if m_3 were zero with Q_1 and Q_2 unchanged. The reader can easily verify that statement, at least for some examples: The simplest case is the fully antisymmetric singlet state for $N=3$ which is annihilated by all polarizations and $S_{ii}-S_{jj}$; the triplet of states for $N=4$ with $m_1=2$, $m_2=m_3=1$ behaves like the fundamental triplet for $N=1$, etc. Since the atomic dynamics in our superradiant laser can be formulated in terms of the polarizations S_{ij}, $i\neq j$, and two population differences (note that the populations S_{ii} can be expressed in terms of the two inversions $S_{22}-S_{11}$, $S_{11}-S_{00}$ and the *conserved* N) we reach the following conclusion. If the initial state belongs to a representation with $m_3\neq 0$ the subsequent evolution may be said to take place among the boxes outside the three-box columns. In other words, only $N-3m_3=Q_1+2Q_2$ atoms are responsible for the evolution leading to changes in the state of the electromagnetic field since at each instant of time the occupation numbers of the states 0, 1, and 2 are at least equal to m_3. From this point of view the essentially different representations are characterized by different pairs Q_1,Q_2. The physical irrelevance of three-box columns, i.e., the possibility to add three-box columns without changing the dynamics beyond changing the number of atoms, is precisely the content of the transformation (3.10).

As follows from Eq. (6.2), the invariant subspaces characterized by Q_1, Q_2 and $\tilde{Q}_1=Q_2$, $\tilde{Q}_2=Q_1$ have the same dimension. However, there is no unitary transformation which maps one representation into the other. As a consequence, these two representations lead to different physical behavior of the system. As an example, let us compare the three-dimensional invariant subspace of two atoms with fully antisymmetric basis states,

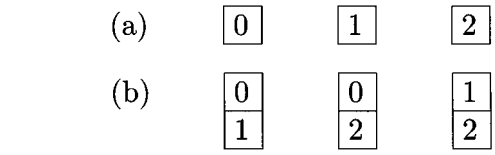

FIG. 6. Young tableaus for the two fundamental representations of U(3): (a) single atom states, and (b) antisymmetric two-atom states.

$$|0\rangle'=\frac{1}{\sqrt{2}}(|1\rangle|2\rangle-|2\rangle|1\rangle), \quad |1\rangle'=\frac{1}{\sqrt{2}}(|2\rangle|0\rangle-|0\rangle|2\rangle),$$

$$|2\rangle'=\frac{1}{\sqrt{2}}(|0\rangle|1\rangle-|1\rangle|0\rangle), \tag{6.5}$$

and the Hilbert space of a single atom (see Fig. 6). Applying the collective atomic operators S_{ij} to the states of a single atom we obtain $S_{ij}|k\rangle=\delta_{kj}|i\rangle$. For the antisymmetric states given by Eq. (6.5) the following relation takes place: $S_{ji}^{\dagger}|k\rangle'=-\delta_{ki}|j\rangle'$. Thus, for the antisymmetric two-atom states the atomic raising and lowering operators change their role. This transformation of operators is not unitary.

The two triplets of states $\{|i\rangle\}$ and $\{|i\rangle'\}$ form two ''fundamental representations'' of the group U(3). With the help of the fundamental representations all invariant subspaces or irreducible representations of the group U(3) can be constructed. In the case of U(3) there are two fundamental representations since there are two Casimir invariants C_2 and C_3. This is in distinction to the case of two-level atoms, corresponding to the symmetry group U(2), where only a single fundamental representation is needed to construct all invariant subspaces of the single Casimir operator $\vec{J}^2$, in quantum optics called squared Bloch vector.

We would like to mention an interesting analogy between the antisymmetric states (6.5) and the ''hole'' states in solid-state physics [13]. As we already mentioned, for these states the atomic raising and lowering operators change their role. Moreover, the new atomic operators obtained under such transformation satisfy the Heisenberg equations (2.2) with time reversed. Analogously, holes behave like electrons under time reversal. Finally, adding a third box to the column consisting of two boxes we obtain the state which does not show any time evolution and can be eliminated from the dynamics. This is analogous to annihilation of a hole put together with an electron. In elementary-particle physics the two fundamental representations of U(3) can often be associated with particles and their antiparticles; one then even speaks of triplet and antitriplet. We refrain from carrying over such jargon into our context of three-level atoms where the fundamental triplet is formed by single-atom states while the ''fundamental antitriplet'' is formed by two-atom states.

In the semiclassical limit, $N\gg 1$, we put $Q_1=q_1N$, $Q_2=q_2N$, where $0\leq q_i\leq 1$, $i=1,2$. Up to $1/N$ corrections we obtain from (6.3) and (6.4)

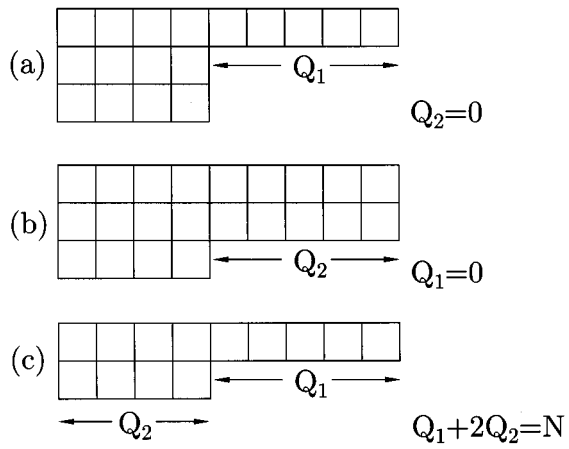

FIG. 7. Young frames for $c_3=c_3^+(r)$, $0\leqslant r\leqslant 1$ (a), $c_3=c_3^-(r)$, $0\leqslant r\leqslant 1/2$ (b), and $c_3=c_3^-(r)$, $1/2\leqslant r\leqslant 1$ (c).

$$c_2=\frac{C_2}{N^2}=\frac{1}{3}[1+2q_2q_1+2q_2^2+2q_1^2],$$

$$c_3=\frac{C_3}{N^3}=\frac{1}{9}[1+6(q_1^2+q_2^2+q_2q_1)+(2q_1^3-2q_2^3-3q_2^2q_1+3q_1^2q_2)]. \tag{6.6}$$

Now we can interpret the results obtained in the preceding sections in terms of symmetry properties of the invariant subspaces. Comparing (6.6) with (3.9) we observe that we recover the upper boundary $c_3^+(r)$ of the cooperativity parameter c_3 setting $q_1=r$ and $q_2=0$. The corresponding Young tableaux are given in Fig. 7(a) and show that in this case $q_1N=rN$ atoms are active as if in totally symmetric states. This result makes it particularly transparent why the semiclassically calculated stationary solutions and fluctuation spectra are simply obtained from the results for full cooperativity by rescaling as $p\rightarrow\tilde{p}$ and $\xi\rightarrow\tilde{\xi}$.

Conversely, for $q_1=0$, $q_2=r$, $0\leqslant r\leqslant 1/2$ we obtain the lower boundary $c_3^-(r)$ of the cooperativity parameter c_3. The corresponding Young tableaux are shown in Fig. 7(b). Now $2q_2N=2rN$ atoms are active as if in rN antisymmetric antitriplets (6.5). Since for a single antitriplet we need two atoms, the maximal number of antitriplets is $N/2$.

Finally, the right boundary $c_3^-(r)=r^2$ of the cooperativity parameter c_3 for $1/2\leqslant r\leqslant 1$ is recovered by $q_1=(1/6)\sqrt{12r^2-3}$, $q_2=(1/2)(1-q_1)$. The corresponding Young tableaux are shown in Fig. 7(c). To construct the atomic Hilbert space in this case we need both fundamental representations. Correspondingly, in the semiclassical case the atomic dynamics is described by six independent equations, two more than in the case when $c_3=c_3^+(r)$, $0\leqslant r\leqslant 1$ or $c_3=c_3^-(r)$ for $0\leqslant r\leqslant 1/2$ [see the above discussion of the additional conservation laws (3.11) and (3.12)]. As we have seen in the preceding section, this results in more complex behavior of the superradiant laser exemplified by ''nonresonant'' stationary solutions.

The atomic Hilbert space corresponding to the nonresonant stationary solutions found in the preceding section is constructed as follows. For $c_3=c_3^+(r)-\delta$ we start from a Young tableau shown in Fig. 8(a) with almost all atoms packed in a three-row block. A sequence of ''transformations'' $Q_1\rightarrow Q_1+3$, $Q_2\rightarrow Q_2$ indicated in Fig. 8(a) leads to the set of representations close to the upper boundary. The number of such transformations must, of course, be a finite fraction of N in order to be semiclassically perceptible. For $\delta\ll 1$ the resulting $q_i=Q_i/N$ must obey $q_2\ll q_1$. Similarly, we obtain the set of representations close to the lower boundary $c_3=c_3^-(r)+\delta$ by a succession of transformations indicated in Fig. 8(b) for which $Q_1\rightarrow Q_1$ and $Q_2\rightarrow Q_2+3$ such that in the end $q_1\ll q_2$.

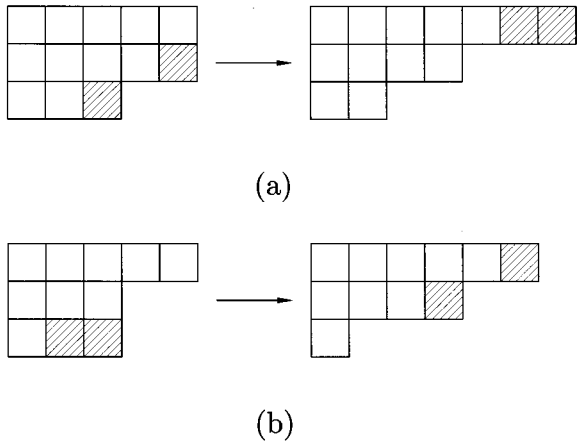

FIG. 8. Construction of the Young frames close to the upper boundary, $c_3=c_3^+(r)-\delta$ (a), and the lower boundary $c_3=c_3^-(r)+\delta$, $r\leqslant 1/2$ (b).

In conclusion, in this section we have employed the theory of group representations which gives a natural classification of solutions of the superradiant laser with partial cooperativity in terms of irreducible representations of U(3) group.

VII. SYMMETRY BREAKING BY SPONTANEOUS EMISSION

In Ref. [5] we had already addressed the question of spontaneous emission. In particular, we have shown that in the semiclassical limit, $N\gg 1$, even with spontaneous emission included, stationary superradiance is still possible and good squeezing available over some range of frequencies in the amplitude noise spectrum. However, in Ref. [5] we were not able to give the full explanation of the role of spontaneous emission since we did not have the set of solutions with partial cooperativity. In this section we want to interpret the results obtained in Ref. [5] using the notion of partial cooperativity and the symmetry properties of the invariant spaces of the Casimir operators C_2 and C_3.

For simplicity we restrict ourselves, as in Ref. [5], to spontaneous emission only from the upper level 2 to the intermediate level 1. (Taking into account spontaneous emission from the intermediate level 1 to the ground state 0 leaves the results qualitatively unchanged [14]. Note that $2\leftrightarrow 0$ is considered to be a two photon transition such that spontaneous emission on this transition can be neglected.)

We assume that the rate of spontaneous emission γ_s is much slower than the rate of collective relaxation γN and the

decay rate of the laser field κ_a,

$$\gamma_s \ll N\gamma, \kappa_a . \tag{7.1}$$

As already mentioned in Ref. [5], there is an important difference between the collective and noncollective relaxation terms in dynamical equations of the superradiant laser: The first ones scale with the number of atoms as N^2 while the second are proportional only to N. Therefore, for the semiclassical approximation, $N \gg 1$, we may confine ourselves to the limit $\gamma_s \to 0$ without incurring more than an error of relative order $1/N$ which is inherent in the semiclassical approximation anyway.

However, the limit $\gamma_s \to 0$ must not be taken blindly by setting $\gamma_s = 0$ in the dynamical equations from the outset. The reason for this is that the incoherent terms proportional to γ_s break the collectivity of the dynamics and destroy the conservation of the Casimir operators C_2 and C_3. Instead of being constants, both C_2 and C_3 now evolve in time with a rate of the order of γ_s,

$$\frac{d}{dt}\overline{C_2} = 4\gamma_s[\overline{S_{02}}\overline{S_{20}} + \overline{S_{21}}\overline{S_{12}} + \overline{S_{22}}(\overline{S_{11}} - \overline{S_{22}})], \tag{7.2}$$

$$\frac{d}{dt}\overline{C_3} = 6\gamma_s[(\overline{S_{22}} + \overline{S_{11}})\overline{S_{21}}\overline{S_{12}} + \overline{S_{01}}\overline{S_{12}}\overline{S_{02}} + \overline{S_{10}}\overline{S_{02}}\overline{S_{21}} + \overline{S_{20}}\overline{S_{02}}\overline{S_{00}} + \overline{S_{22}}(\overline{S_{11}^2} - \overline{S_{22}^2} + \overline{S_{01}}\overline{S_{10}})]. \tag{7.3}$$

To find a stationary solution we must set $d\overline{C_2}/dt = 0$ and $d\overline{C_3}/dt = 0$. Obviously, now, for $\gamma_s = 0$ both conditions are trivially fulfilled. For $\gamma_s \neq 0$ the square brackets in (7.2) and (7.3) must vanish and these conditions take the role previously played by the conservation of C_2 and C_3 in nailing down all stationary means. Only at this point, i.e., after dividing (7.2) and (7.3) by γ_s, we may let $\gamma_s \to 0$.

We proceed by parametrizing the stationary values $\overline{S_{ij}^e}$ in search as in (4.2). Entering with this ansatz into the square bracket in (7.2) and equating the latter to zero, we obtain a biquadratic equation for the parameter x,

$$2(1+c)x^4 + cp(e+cp)x^2 + p^3c^2(p-e) = 0, \tag{7.4}$$

which allows for the four solutions,

$$(x_{1,2}^e)^2 = \frac{cp(1+cpe)}{4(1+c)}\left\{ -e \pm \left(1 - \frac{8e(1+c)p(1-pe)}{(1+cpe)^2} \right)^{1/2} \right\}. \tag{7.5}$$

Here ''+'' corresponds to x_1^e and ''−'' to x_2^e. One of these four solutions, namely $(x_2^+)^2$, is negative and must be discarded as nonphysical. The other three are the solutions found in Ref. [5] which we had labeled ''alpha,'' ''beta,'' and ''gamma,''

$$x_1^- = x_\alpha, \quad x_2^- = x_\beta, \quad x_1^+ = x_\gamma . \tag{7.6}$$

On the other hand, these three stationary solutions may be viewed as former resonant stationary solutions with partial cooperativity given by (4.2) with some particular value of participation parameter r. To find that value for each of the three solutions we may substitute $x_{1,2}^e$ from (7.5) into the relation (4.3) between x and the participation parameter r. Thus we obtain r_a with $a = \alpha, \beta, \gamma$,

$$r_a = p + \frac{1+c}{pc}x_a^2 . \tag{7.7}$$

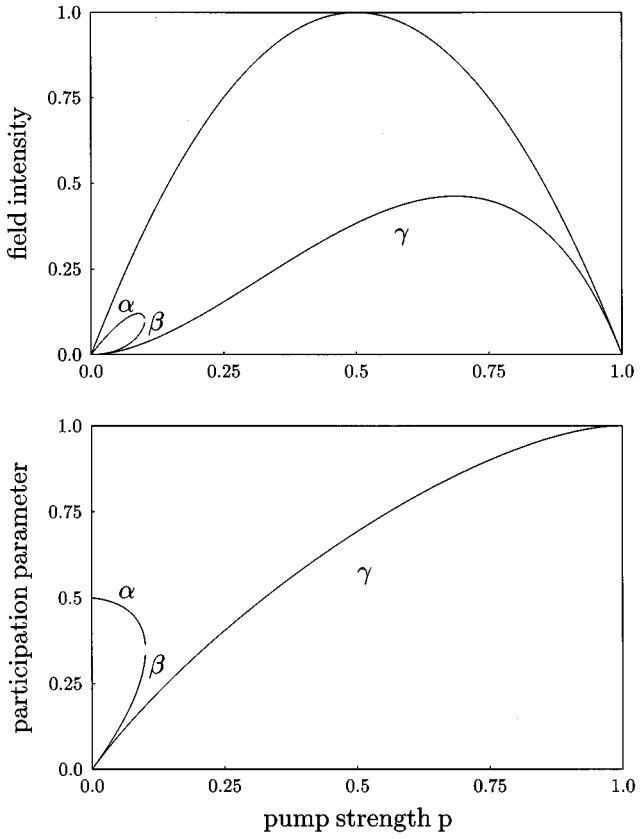

FIG. 9. Stationary intracavity field intensity and participation parameters vs pump-strength p without (uppermost curve for field intensities) and with (α, β, and γ) spontaneous emission. The coupling strength $c = 0.1$ and the limit $\gamma_s \to 0$.

From the way we have constructed the alpha, beta, and gamma solutions it is clear that upon substitution of these solutions into the expression for c_3 we obtain $c_3^{\alpha,\beta} = c_3^-(r)$ and $c_3^\gamma = c_3^+(r)$. These semiclassical values now have a different status, however: They are the stationary means of C_3/N^3 obtained in the limit $\gamma_s \to 0$ as time goes to $+\infty$, rather than constants of the motion determined by the initial state. In Fig. 9 we have plotted the participation parameter together with stationary intensities for the three solutions as a function of p for $c = 0.1$. Note that the participation parameter for alpha and beta solutions does not exceed the value 1/2 as it must be the case for the lower boundary $c_3^-(r)$ in accord with Schwartz's inequalities.

Figures 10(a) and 10(b) show the time development of the field amplitude, occupation numbers, and cooperativity parameters according to the classical equations of motion. As expected, the time development of the cooperativity parameters c_2 and c_3 takes place on a time scale γ_s^{-1}, i.e., much slower than that of the occupation numbers and field amplitude. From Fig. 10(b) we observe that in the good-cavity case even with spontaneous emission there is a possibility for

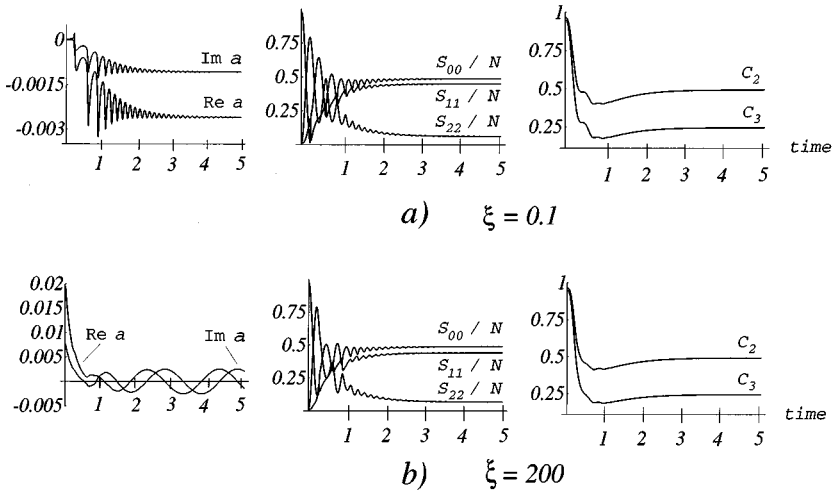

FIG. 10. Semiclassical time evolution of the field amplitude $a(t)$, populations of atomic levels $S_{ii}(t)/N$, and cooperativity parameters $c_2(t)$ and $c_3(t)$ for $p=0.05$, $c=0.02$, and $\gamma_s=10^{-3}N\gamma$; time is given in units of $1/\gamma_s$.

oscillating behavior of the laser field amplitude which corresponds to the nonresonant solution discussed in the preceding section.

Turning to the stability analysis we see that the stability conditions (4.7) remain unchanged for $\gamma_s\to 0$. However, as we mentioned in Sec. IV, some of the former marginally stable eigenvalues $\lambda_1,\ldots,\lambda_6$ now acquire real parts of order $\gamma_s/(N\gamma)$. Therefore, we have to check whether these real parts are positive. Since the zero eigenvalues λ_1 and λ_5 are related to the constant number of atoms and the conservation law (2.4), respectively, they remain zero also with spontaneous emission included. We have checked the other four eigenvalues in the limit of small $\tilde{p}$ and a bad-cavity, $\tilde{\xi}\ll 1$, when good squeezing is expected to arise. In this particular limit these eigenvalues are

$$\lambda_2=\frac{\gamma_s}{N\gamma}\frac{2}{1+c},\quad \lambda_{3,4}=\frac{\gamma_s}{N\gamma}\frac{1}{2\tilde{p}\tilde{\xi}}\pm ir\sqrt{4\tilde{p}c},$$

$$\lambda_6=\frac{\gamma_s}{N\gamma}\frac{c}{2\tilde{p}(1+c)}, \tag{7.8}$$

i.e., they do not give rise to additional stability conditions.

In Ref. [5] we have described the influence of spontaneous emission on the squeezing spectrum of the amplitude quadrature component. We have shown that spontaneous emission leads to the appearance of a high peak around zero frequency in the spectrum. That peak has a height $\propto N\gamma/\gamma_s$ and a width $\propto\gamma_s$, such that the area underneath remains constant in the limit $\gamma_s\to 0$, indicating an asymptotic δ peak. However, in a broad intermediate frequency range outside this peak squeezing persists and is not qualitatively changed from the one obtained for $\gamma_s=0$.

The situation is different for the linewidth. While without spontaneous emission the linewidth $\Delta\nu_a^0$ scales as $\propto 1/N^2$ [see Eq. (4.12)], with spontaneous emission it acquires an additional factor $N\gamma/\gamma_s$ and now is only proportional to $1/N$, $\Delta\nu_a\propto 1/N$.

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[10] Assume a real flow $\dot{X}_i=F_i(X)$ (where any global constants of the motion are already incorporated to reduce the dimension to, say, f) with a single nonglobal constant of the motion, $\varphi(X)=0$. Nonglobality means that the flow remains f dimensional unless the initial point $X(0)$ happens to lie on the surface $\varphi=0$. But if $X(0)$ does lie on $\varphi=0$, $X(t)$ will remain on that surface such that the dimension is reduced to $f-1$. We can always use $\varphi=0$ to eliminate one of the f variables X_i, say, the fth, and write $X_f=a(X_1,\ldots,X_{f-1})$. Compatibility of the original flow with $\varphi=0$ requires $F_f(X_1,\ldots,X_{f-1},a(X_1,\ldots,X_{f-1}))=\sum_{i=1}^{f-1}a_i F_i(X_1,\ldots,X_{f-1},a(X_1,\ldots,X_{f-1}))$ with $a_i=\partial a/\partial X_i$. Assume a fixed point X on $\varphi=0$ and consider fluctuations away from it. Upon

invoking the mentioned compatibility one finds the fluctuation out of $\varphi=0$ to obey $(d/dt)\delta\varphi=\Phi\,\delta\varphi$ with the real ''force constant'' $\Phi=\overline{F_{ff}}-\Sigma_{i=1}^{f-1}\overline{a_i}\overline{F_{if}}$, where $F_{ij}=(\partial F_{ij}/\partial X_i)_{X_f=a}$ and the overbar demands evaluation at the fixed point. In general, the force constant Φ has no reason to vanish and both signs are possible. By extending the argument to two nonglobal constants of the motion one also encounters the possibility of a pair of complex conjugate eigenvalues. For our superradiant laser where we face three nonglobal constants of the motion we find one vanishing eigenvalue and a pair of purely imaginary complex conjugate.

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