Inverse scattering transform analysis of Stokes–anti-Stokes stimulated Raman scattering

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A system of Maxwell-Bloch type equations (MBE's), describing stimulated Raman scattering with both Stokes and anti-Stokes waves taken into account, is investigated. We introduce variables S_3 and $S_\pm$, which are bilinear in the electromagnetic fields, and prove that the corresponding equations possess Lax representation. This fact is used to obtain additional solutions for S_3, $S_\pm$ and for the MBE's which include solitons, periodical waves, and self-similarity solutions. The transient and bright threshold solitons are also analyzed. [S1050-2947(96)09909-X]

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I. INTRODUCTION

The so-called Maxwell-Bloch equations (MBE's) describe a wide variety of interactions of matter with light. These include self-induced transparency, resonance interactions in multilevel media, superfluorescence, amplification of laser pulses, etc., see the monographs [1–4] and papers [5–16]. As a rule these are large systems of nonlinear evolution equations, whose form is determined by the physical setup; most of them can be handled only numerically.

This situation changed with the development of the inverse scattering method (ISM), see [17–19]. This method was generalized also to handle nonisospectral deformations of the Lax operators, which allowed one to treat Maxwell-Bloch-type equations with pumpings, with dampings of special types, etc.; see [12,16], and references therein. In this respect we should mention paper [5], which showed, that this method can be applied to one of the versions of the MBE describing stimulated Raman scattering (SRS). As a result the N-soliton solutions of these equations were obtained [5]. The behavior of the pump and Stokes pulses in a Raman-active medium and the coherent effects on the transient pulses were analyzed in [6,7], where also the cnoidal wave solution of these equations was obtained. Today analysis of the SRS draws the interest of both theoreticians and experimentalists. Most of these papers (see [5,6,10,13–15,20–27]) considered a MBE with only pump and Stokes waves; in several others the presence of an anti-Stokes wave was also taken into account [11,28–31,33].

Our aim in this paper is to study, using the ISM, a special version of the MBE which describes the wave propagation in a Raman-active medium when Stokes E_s, anti-Stokes E_a, and pump E_p waves are present.

These type of MBE's in the slowly varying envelope approximation, and when the diffraction and the ground-state depletion of the material excitation can be ignored, have the form [11,28–31,33]

$$\frac{\partial E_p}{\partial \zeta} = -2iqE_p + \beta_a Q^* E_a - QE_s, \tag{1.1a}$$

$$\frac{\partial E_s}{\partial \zeta} = -2iqE_s + Q^* E_p, \tag{1.1b}$$

$$\frac{\partial E_a}{\partial \zeta} = -iqE_a - \beta_a Q E_p, \tag{1.1c}$$

$$\frac{\partial Q}{\partial \tau} + \tilde{g} Q = E_s^* E_p + \beta_a E_p^* E_a, \tag{1.1d}$$

where Q is the normalized effective polarization of the medium, $\zeta = z/L$ and $\tau = t - z/v$ are dimensionless space and retarded time coordinates, respectively, and v is the wave group velocity. T_2 is the natural damping time of the material excitation and $\tilde{g} = 1/T_2$. By β_a we denote the coupling coefficient which determines the number of anti-Stokes photons relative to number of Stokes photons, and its magnitude depends on the matrix element that describes the dipole transition [11,3]. In this paper we consider $\beta_a = 1$ [11], but some results are valid also for $\beta_a \neq 1$. In addition $q = (k_s + k_a - 2k_p)L$ is the scaled forward phase mismatch between the wave vectors k_p, k_s, and k_a of the pump, Stokes, and anti-Stokes waves. We will assume $q = 0$ for the present analysis.

For $\beta_a = 0$, i.e., when the anti-Stokes wave E_a is neglected, these equations describe the so-called transient stimulated Raman scattering equations, which possess Lax [17] representation when $\tilde{g} = 0$ [5]. The SRS soliton solutions, theoretically discovered by Chu and Scott [5], were experimentally observed by Drühl *et al.* in [27]. The SRS solitons (regarded as transient solitons [13,9] with a π phase jump at the Stokes frequency) have been extensively studied [2,4,13–15,20–27]. In later experiments by Duncan *et al.* [21], a careful comparison between theory and experiment showed good agreement. Shortly after this work [21], Hilfer and Menyuk [23] carried out simulations which indicate that in the highly depleted regime the solutions of transient SRS equations always tend toward a self-similar solution. This result [23,24] has been recovered by applying the ISM to the transient SRS equations; see [13]. An experiment to observe this solution was proposed in [23]. Kaup's theory [9] also

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indicates that the dissipation, which appears for finite T_2 plays a crucial role in soliton formation. The similarity solutions and other group invariant solutions of the SRS equations in the presence of dissipation are studied in [25].

Recently Claude and Leon [15] developed an alternative approach[1] to the transient SRS equations with inhomogeneous broadening, based on an equivalent $\bar{\partial}$ problem. They used it to interpret the experimental data in [27], and concluded, that the corresponding Zakharov-Shabat system on the infinite line with the initial condition $Q(\zeta,\tau)|_{\tau=0}=0$, does not contain discrete eigenvalues.

For the general system (1.1), with an anti-Stokes wave, phase mismatch, $\beta_a\neq 1$ and dissipation, transient π solitons were investigated by Scalora, Singh, and Bowden [31] using numerical methods. They predicted the formation of soliton-like pulses at the anti-Stokes frequency. Another type threshold bright 2π solitons, which have a Lorentzian form, were theoretically obtained in [33].

In Sec. II we introduce new variables S_3 and $S_\pm$, Eq. (2.1) which are quadratic in terms of E_p, E_s, and E_a. Then system (2.2) for S_3 and $S_\pm$, derived from Eq. (1.1) with $\beta_a=1$ and $\tilde{g}=0$, allows a Lax representation similar to the one used by Chu and Scott [5] for other physical quantities: the difference for the normalized Stokes-anti-Stokes local intensities and for normalized (complex) local Rabi frequency. We also introduce ''nonlinear time'' and renormalized dimensionless variables different from the ones used in [9]. Then we solve the inverse scattering problem (ISP) for system (2.16) with the ''nonlinear time'' τ' restricted to the finite interval $0\leqslant\tau'\leqslant 1$; i.e., we derive the corresponding Gel'fand-Levitan-Marchenko (GLM) equation using a more direct approach than [9] and obtaining results, compatible with [9].

Basically we show that, in terms of the bilinear variables S_3 and $S_\pm$ the nonlinear evolution equations (NLEE's) are formally the same as for the case when there is no anti-Stokes wave. There is, however, an important difference, namely the so-called ''nonlinear time'' has a different definition, which accounts for the new physical situation. Thus we are also able to translate the known results [5,9,13,26] for systems with an anti-Stokes wave present.

In Sec. III we obtain periodic, soliton, and self-similarity solutions of Stokes–anti-Stokes SRS equations without dissipation for both formulations (1.1) and (2.2), and briefly analyze their relation to the ones already known [6]. In addition, the transient solitons and the bright solitons of Kaplan *et al.* [33] are discussed.

In Sec. IV we propose an extension of the Stokes–anti-Stokes SRS equations for N Stokes and N anti-Stokes waves, and conjecture that it is also integrable by means of the ISM.

II. LAX REPRESENTATION AND GLM EQUATION

A. Lax representation

Let us introduce the following variables:

$$S_3=\tfrac{1}{2}(|E_s|^2-|E_a|^2), \tag{2.1a}$$

$$S_+=\frac{i}{2}(E_s^*E_p+E_p^*E_a), \tag{2.1b}$$

$$S_+=S_-^*. \tag{2.1c}$$

In terms of the new quadratic variables (2.1), the initial system with $\tilde{g}=0,\beta_a=1$ is rewritten as

$$\frac{\partial S_3}{\partial\zeta}=-iQ^*S_++iQS_-, \tag{2.2a}$$

$$\frac{\partial S_+}{\partial\zeta}=-iQS_3, \tag{2.2b}$$

$$\frac{\partial Q}{\partial\tau}=-2iS_+, \tag{2.2c}$$

or, in matrix form,

$$\frac{\partial q}{\partial\tau}=\frac{1}{\sqrt{2}}[\sigma_3,S(\zeta,\tau)], \tag{2.3a}$$

$$\frac{\partial S}{\partial\zeta}=\frac{1}{\sqrt{2}}[q(\zeta,\tau),S(\zeta,\tau)], \tag{2.3b}$$

where

$$q(\zeta,\tau)=\begin{pmatrix}0 & Q\\ -Q^* & 0\end{pmatrix}, \tag{2.4a}$$

$$S(\zeta,\tau)=\begin{pmatrix}S_3 & -i\sqrt{2}S_+\\ i\sqrt{2}S_- & -S_3\end{pmatrix}. \tag{2.4b}$$

Equations (2.2) and (2.3) can be written down as the compatibility condition

$$\partial_\tau U-\partial_\zeta V+[U,V]=0 \tag{2.5}$$

of the following linear systems:

$$L(\lambda)F(\zeta,\tau,\lambda)\equiv\frac{\partial F}{\partial\zeta}-U(\zeta,\tau,\lambda)F(\zeta,\tau,\lambda)$$
$$=F(\zeta,\tau,\lambda)C(\lambda), \tag{2.6}$$

$$M(\lambda)F(\zeta,\tau,\lambda)\equiv\frac{\partial F}{\partial\tau}-V(\zeta,\tau,\lambda)F(\zeta,\tau,\lambda)=0, \tag{2.7}$$

with

$$U(\zeta,\tau,\lambda)=-\frac{i}{\lambda}\sigma_3+\frac{1}{\sqrt{2}}q(\zeta,\tau), \tag{2.8a}$$

$$V(\zeta,\tau,\lambda)=\frac{\lambda}{2i}S(\zeta,\tau). \tag{2.8b}$$

The matrix $C(\lambda)$ will be fixed up below to our convenience; this is possible, because $C(\lambda)$ in fact does not appear in the compatibility condition (2.5).

[1]The approach of [15] has a close relationship to the one, based on the expansions over the ''squared solutions,'' compare with [38].

From the physical point of view [5,27] the initial value problem associated with the system (1.1) is specified by

$$Q(\zeta,0)=0, \tag{2.9a}$$

$$E_p(0,\tau)=E_{p0}(\tau), \tag{2.9b}$$

$$E_s(0,\tau)=E_{s0}(\tau), \tag{2.9c}$$

$$E_a(0,\tau)=E_{a0}(\tau). \tag{2.9d}$$

Then the problem consists of determining the output quantities $E_p(L,\tau)$, $E_s(L,\tau)$ and $E_a(L,\tau)$, where L is the total length of the beam path in the Raman cell. Analogically the initial value problem for the system (2.2) and (2.3) is

$$Q(\zeta,0)=0, \tag{2.10a}$$

$$S_3(0,\tau)=S_{30}(\tau), \tag{2.10b}$$

$$S_+(0,\tau)=S_{+0}(\tau). \tag{2.10c}$$

We follow the main idea of [9], namely, that as a Lax operator one should consider the operator $M(\lambda)$ in Eq. (2.7), and solve the inverse scattering problem for it. Then we will use the operator $L(\lambda)$ in Eq. (2.6) to determine the ζ dependence of the corresponding scattering data. However there will be substantial differences in the details.

It is well known how to solve the ISP for system (2.7) considered on the whole τ-line $-\infty\leqslant\tau\leqslant\infty$ and with boundary conditions of ferromagnetic type, i.e., $\lim_{\tau\to\pm\infty}S(\zeta,\tau)=\sigma_3$; see [19]. We will make use of these ideas, adopting them to our case. First we have to take into account that the eigenvalues of our $S(\zeta,\tau)$ differ from ± 1 and are generically τ dependent. In order to calculate them, it is enough to know that $\mathrm{tr}S(\zeta,\tau)=0$ and

$$\begin{aligned}-\det S(\zeta,\tau)&=S_3^2+2S_+S_-\\&=\frac{1}{4}(|E_s|^2-|E_a|^2)^2+\frac{1}{2}|E_p^*E_a+E_pE_s^*|^2\\&=K^4(\tau).\end{aligned} \tag{2.11}$$

Using the evolution equations (1.1), we check that

$$\frac{dK^4}{d\zeta}=0. \tag{2.12}$$

From Eq. (2.11) we also conclude that $K^2(\tau)$ is real-valued function. Then we can introduce a ‘‘nonlinear time'' τ' by

$$d\tau'=K^2(\tau)d\tau, \tag{2.13}$$

and the following dimensionless variables:

$$\tau'=\int_0^\tau K^2(\tau'')d\tau''/T_\infty, \tag{2.14a}$$

$$T_\infty=\int_0^\infty K^2(\tau)d\tau, \tag{2.14b}$$

$$\zeta'=\zeta T_\infty \tag{2.14c}$$

$$E_p'=\frac{E_p}{K(\tau)}, \tag{2.14d}$$

$$E_s'=\frac{E_s}{K(\tau)}, \tag{2.14e}$$

$$E_a'=\frac{E_a}{K(\tau)}, \tag{2.14f}$$

$$Q'=\frac{Q}{T_\infty}. \tag{2.14g}$$

The primed variables introduced above satisfy the same system (1.1) of nonlinear evolution equations provided $\tilde{g}'=\tilde{g}T_\infty/K^2$; in what follows below we put $\tilde{g}=0$. Note that the transformation $\{E_p,E_s,E_a,Q\}\to\{E_p',E_s',E_a',Q';K(\tau)\}$ is one to one and invertible. In order to obtain the evolution of $\{E_p,E_s,E_a,Q\}$, one must first determine the evolution of $\{E_p',E_s',E_a',Q'\}$ and then use the given function $K(\tau)$ to return to the original variable set.

The nonlinear time τ' is introduced in analogy to the one in [9,26]; the difference is that now $K^2(\tau)$ cannot be interpreted as the total-energy density:

$$\mathcal{E}^2(\tau)=|E_p(\zeta,\tau)|^2+|E_s(\zeta,\tau)|^2+|E_a(\zeta,\tau)|^2. \tag{2.15}$$

Note that the constancy of $\mathcal{E}(\tau)$ corresponds to pointwise conservation of the photon intensity. Since all physical solutions are with finite energies, we conclude that each of the terms in Eq. (2.15) must be integrable functions of τ. In particular, each of these functions must vanish for $\tau\to\infty$. As a consequence of this fact and Eq. (2.11), we find that $\mathcal{E}^2(\tau)$ must have the same properties. Therefore for this class of solutions we have $T_\infty<\infty$ and in terms of τ' we obtain the system

$$M'(\lambda)F(\zeta,\tau',\lambda)\equiv\frac{\partial F}{\partial\tau'}-\frac{\lambda}{2i}S'(\zeta,\tau')F(\zeta,\tau',\lambda)=0, \tag{2.16a}$$

$$S'(\zeta,\tau')=\frac{S(\zeta,\tau)}{K^2(\tau)}, \tag{2.16b}$$

where $S'(\zeta,\tau')$ satisfies $\mathrm{tr}S'=0$ and $\det S'=-1$. As a result the eigenvalues of $S'(\zeta,\tau')$ are equal to ± 1, and there exists a nondegenerate matrix-valued function $g(\zeta,\tau')$ such that

$$S'(\zeta,\tau')=g(\zeta,\tau')\sigma_3 g^{-1}(\zeta,\tau'). \tag{2.17}$$

From Eq. (1.1) it is also easy to derive the following important relation:

$$\frac{1}{2}\frac{\partial}{\partial\tau}|Q|^2+\tilde{g}|Q|^2+\frac{\partial}{\partial\zeta}S_3=0, \tag{2.18}$$

from which, for $\tilde{g}=0$, we find that $\int|Q|^2d\zeta$ is an integral of motion if $S_3(0,\tau)-S_3(L,\tau)=0$. This will be fulfilled if E_a and E_s satisfy quasiperiodic boundary conditions, i.e., if $E_{a,s}|_{\tau'=0}=e^{i\phi_{a,s}}E_{a,s}|_{\tau'=1}$ with some $\phi_{a,s}$.

In Sec. II B we will use only renormalized quantities $S'(\zeta,\tau')$ and the ''nonlinear time'' τ' and for the simplicity of the notations will drop all primes.

B. GLM equation

We briefly sketch the derivation of the GLM equation related to the left end $\tau=0$ of the interval. Of course we also have to introduce slight modifications in order to take into account the fact that $S(\tau=0,\zeta)=S_0(\zeta)\neq\sigma_3$. The operator $M(\lambda)$ on finite interval generically possesses a purely discrete spectrum with an infinite number of simple discrete eigenvalues, see, e.g., [34]. As a consequence, the kernel of the GLM equation contains only a sum over the discrete spectrum. Skipping the details we write down the results.

Let the fundamental Jost solution of Eq. (2.16), normalized to the left end $\tau=0$ of the interval, be fixed up by

$$\lim_{\tau=0} F(\zeta,\tau,\lambda)=\lim_{\tau=0} F_0(\zeta,\tau,\lambda)=\mathbb{1}, \tag{2.19a}$$

$$F_0(\zeta,\tau,\lambda)=g_0 e^{\lambda\tau\sigma_3/2i} g_0^{-1}, \tag{2.19b}$$

$$\lim_{\tau\to 1} F(\tau,\zeta,\lambda)=T(\zeta,\lambda), \tag{2.20}$$

$$g_k(\zeta)=g(\zeta,\tau)\big|_{\tau=k}, \quad k=0,1. \tag{2.21}$$

Then its behavior at $\tau\simeq 1$ determines the scattering matrix $T(\zeta,\lambda)$ according to Eq. (2.20). It remains now to evaluate the ''evolution'' of T in ζ. In order to do this we have to calculate first $C(\lambda)$ in Eq. (2.6) by taking the limit of Eq. (2.19) for $\tau\to 0$ with the result $C(\zeta,\lambda)=U^{(0)}(\zeta,\lambda)$. Here and below, by $U^{(k)}(\zeta,\lambda)$, $k=0,1$ we denote $U(\zeta,\tau,\lambda)|_{\tau=k}$. Then we take the limit of Eq. (2.19) for $\tau\to 1$, which gives the following result for the evolution of $T(\zeta,\lambda)$:

$$\frac{dT}{d\zeta}=U^{(1)}(\zeta,\lambda)T(\zeta,\lambda)-T(\zeta,\lambda)U^{(0)}(\zeta,\lambda). \tag{2.22}$$

Formally this is a linear evolution equation for $T(\zeta,\lambda)$. However it is not trivial to solve since $T(\zeta,\lambda)$ is a nonlinear functional of $q(\zeta,\tau)$. We can determine $q^{(0)}(\zeta)=q(\zeta,\tau)|_{\tau=0}$ from the initial conditions, but in order to compute $q^{(1)}(\zeta)$ we must have the solution of the problem itself.

Let us now describe the solution of the ISP for the $M(\lambda)$ operator. It will be convenient to use the notations $S^{(k)}(\zeta)=S(\zeta,\tau=k)$, $k=0,1$ and let them be diagonalizable in the form

$$S^{(k)}(\zeta)=g_k(\zeta)\sigma_3 g_k^{-1}(\zeta), \tag{2.23a}$$

$$g_k(\zeta)=g(\zeta,\tau=k). \tag{2.23b}$$

We introduce the transformation operator which relates the fundamental Jost solution $F(\zeta,\tau,\lambda)$ to its asymptotics $F_0(\zeta,\tau,\lambda)$ [Eq. (2.19)];

$$F(\zeta,\tau,\lambda)=F_0(\zeta,\tau,\lambda)+\frac{\lambda}{2i}\int_0^\tau \Gamma_-(\tau,z;\zeta)F_0(z,\zeta,\lambda)dz. \tag{2.24}$$

We also have to keep in mind that all solutions and the scattering matrix of the system (2.16) are meromorphic functions of λ.

Then we obtain that $\Gamma_-(\tau,y;\zeta)$ must satisfy the following GLM-type equation:

$$\Gamma_-(\tau,y;\zeta)+S^{(0)}(\zeta)K(\tau+y;\zeta) + \int_0^\tau \Gamma_-(\tau,z;\zeta)K'(z+y;\zeta)dz=0, \tag{2.25}$$

where the kernel $K(\tau;\zeta)$ and its derivative $K'=dK(\tau;\zeta)/d\tau$ are determined by

$$K(\tau;\zeta)=g_0(\zeta)\begin{pmatrix} 0 & k \\ -k^* & 0\end{pmatrix} g_0^{-1}(\zeta), \tag{2.26a}$$

$$k(\tau;\zeta)=-\sum_{\lambda_j\in\mathcal{S}} \frac{m_j(\zeta)}{\lambda_j} e^{i\lambda_j\tau/2}. \tag{2.26b}$$

Here $\mathcal{S}$ and λ_j are the discrete spectrum and the discrete eigenvalues of $M(\lambda)$, and $m_j(\zeta)$ is related to the norm of the corresponding Jost solution of Eq. (2.16); generically λ_j may also depend on ζ.

The corresponding potential of Eq. (2.16) is recovered from the solution $\Gamma_-(\tau,y;\zeta)$ of Eq. (2.25) through

$$S(\tau,\zeta)=B_-(\tau,\zeta)S^{(0)}(\zeta)B_-^{-1}(\tau,\zeta), \tag{2.27a}$$

$$B_-(\tau,\zeta)=\mathbb{1}+\Gamma_-(\tau,\tau,\zeta)S^{(0)}(\zeta). \tag{2.27b}$$

The complete solution of the problem also requires the calculation of the ζ dependence of the scattering data, in our case $m_j(\zeta)$ and λ_j.

We finish this section with a brief discussion on the gauge transformations and the compatibility of our results to the ones obtained in [9,13,26]. It is well known that the Lax representation and the compatibility condition (2.5) are invariant with respect to the group of gauge transformations of the form

$$L\to\tilde{L}\tilde{F}\equiv\frac{\partial\tilde{F}}{\partial\zeta}-\tilde{U}\tilde{F}, \tag{2.28a}$$

$$M\to\tilde{M}\tilde{F}\equiv\frac{\partial\tilde{F}}{\partial\tau}-\tilde{V}\tilde{F}, \tag{2.28b}$$

where

$$\tilde{F}=G^{-1}F, \tag{2.29}$$

$$\tilde{U}=G^{-1}UG-G^{-1}G_\zeta=\tilde{U}_0+\frac{1}{\lambda}\tilde{U}_1, \tag{2.30}$$

$$\tilde{V}=G^{-1}VG-G^{-1}G_\tau=\tilde{V}_0+\lambda\tilde{V}_1. \tag{2.31}$$

Here $G(\tau,\zeta)$ is a generic element of the group $SL(2)$, and it can be fixed up to our convenience. Let us then fix it up so, that

$$G^{-1}VG\equiv\frac{\lambda}{2i}G^{-1}SG=\frac{\lambda}{2i}\sigma_3. \quad (2.32)$$

This is always possible due to the facts that $\mathrm{tr}S=0$ and $\det S\neq 0$.

Such a choice in a similar setup was used in the proof of the gauge equivalence between the nonlinear Schrödinger (NLS) equation and the Heisenberg ferromagnet equation [19]. However, in this case both operators L and M depended polynomialy on λ, and their fundamental solution F taken for $\lambda=0$ is a well-defined function.

As it is in our case, the operator $\tilde{L}$ is singular for $\lambda\to 0$, and thus we would be unable to define the common fundamental solution $\tilde{F}$ for $\lambda=0$ rigorously. That is why we preferred to attack the ISP for M directly, adapting to our purposes the well known results from [19].

Kaup, in [9], solved the inverse scattering problem (ISP) for the operator M on a finite interval in the following way. First he applied the gauge transformation and obtained $\tilde{M}$ to be the Zakharov-Shabat (ZS) system but on a finite interval. He wrote GLM equations corresponding to $\tilde{M}$, and then made conclusions about the inverse scattering problem for M. The next step— the evaluation of the evolution of the corresponding spectral data—he overcame by other means.

Of course, as far as the inverse scattering problem is considered our results are compatible with the ones of Kaup [9]. Indeed, the GLM equation for $\tilde{M}(\lambda)$ considered on the whole τ axis is well known to be equivalent to the GLM equation for the ZS system; see [19]. The proof for the finite interval case is quite analogous. We use just the explicit form of the L operator, while Kaup makes use of the equations of motion. In this way he avoids the problem with abovementioned singularity of the gauge transformation.

Finally, let us compare our ζ dependence of $T(\zeta,\lambda)$ with the one derived in [9,26,13]. The interrelation between $T(\zeta,\lambda)$ and the scattering matrix $T_{\mathrm{K}}(\zeta,\lambda)$ used in these papers is given by

$$T_{\mathrm{K}}(\zeta,\lambda)=e^{-i\lambda\sigma_3}g_1^{-1}T(\zeta,\lambda)g_0. \quad (2.33)$$

From Eq. (2.3) we derive that $g_k(\zeta)$ satisfies

$$\frac{dg_k}{d\zeta}-\frac{1}{\sqrt{2}}q^{(k)}(\zeta)g_k(\zeta)=a_kg_k(\zeta), \quad (2.34a)$$

$$a_k\equiv a(\tau)_{\tau=k}=\text{ const},\quad k=0,1, \quad (2.34b)$$

where the constants a_k are arbitrary and reflect the arbitrariness in the choice of the gauge. From Eqs. (2.33), (2.34) and (2.22) we find

$$\frac{dT_{\mathrm{K}}}{d\zeta}=\frac{i}{\lambda}[T_{\mathrm{K}}(\zeta,\lambda)S^{(0)}(\zeta)-e^{i\lambda\sigma_3}S^{(1)}(\zeta)e^{-i\lambda\sigma_3}T_{\mathrm{K}}(\zeta,\lambda)] +a_1\sigma_3T_{\mathrm{K}}(\zeta,\lambda)-a_0T_{\mathrm{K}}(\zeta,\lambda)\sigma_3. \quad (2.35)$$

Obviously for $a_0=a_1=0$ this evolution coincides with the one derived by Kaup. A reduced version of this equation with $S^{(1)}(\zeta)=\sigma_3$ has been shown to have physical importance and solved in [9,13,26].

As we already noted, in order to apply the ISM to the solution of our problem we will first need to calculate not only $g_0(\zeta)$, which is determined from the initial conditions, but also $g_1(\zeta)$. The situation is greatly simplified if we impose quasiperiodic boundary conditions on the fields $E_{a,s,p}(\zeta,\tau)$ in such a way, that $S^{(0)}(\zeta)=S^{(1)}(\zeta)$. Then we obtain $U^{(0)}(\zeta,\lambda)=U^{(1)}(\zeta,\lambda)$, and the right-hand side of Eq. (2.22) becomes proportional to the commutator $[T,U^{(0)}(\zeta,\lambda)]$. The importance of this imposition can be seen from the fact that it immediately provides us with the hierarchy of conservation laws. The generating function of this hierarchy is $\mathrm{tr}T(\lambda)$, which is now ζ independent.

We end this section with the one-soliton solution of the system (2.7);

$$S_3=\frac{1}{2}\mathcal{E}^2(\tau)\left(1-\frac{2}{\cosh^2 Z}\right), \quad (2.36a)$$

$$S_+=\frac{i\sqrt{2}}{2}\mathcal{E}^2(\tau)\frac{\tanh Z}{\cosh Z}e^{i\phi}, \quad (2.36b)$$

$$S_+=S_-^*, \quad (2.36c)$$

$$Q=\frac{\sqrt{2}\,\eta e^{i\phi}}{\cosh Z}, \quad (2.36d)$$

$$Z=\eta\zeta-\frac{1}{\eta}\int_0^\tau\mathcal{E}^2(\tau')d\tau', \quad (2.36e)$$

where $\mathcal{E}(\tau)=\sqrt{2}K(\tau)$ is real, the soliton's eigenvalue is $i\eta$, and ϕ is a constant real phase. This solution satisfies the differential equations, but unfortunately fails to satisfy the boundary condition $Q(\zeta,0)=0$ in Eq. (2.10a). As a function of τ Eq. (2.36) coincides with the one-soliton solution of the Heisenberg ferromagnet on the whole τ axis, provided the the soliton parameters are adjusted, so that $S(\tau,\zeta)|_{\tau=0}=S_0(\zeta)$ as given by the boundary conditions.

In fact the class of soliton solutions of our system considered on the whole τ axis can also be used as a good approximation when restricted to a finite-τ interval. The substantial difference between the corresponding GLM equations consists of the fact that the kernel $K(\tau,\zeta)$ [Eq. (2.26)] always contains an infinite sum. In some cases, when the amplitudes (or equivalently, the imaginary parts of λ_j) of the first several solitons are much larger than that of the others, we can truncate the kernel and consider the remaining finite sum as a good overall approximation. In addition, since in our model $0<\tau<1$, we can have at most a finite number of such solitons. They move toward the edge of the interval which corresponds to the edge of the optical pulses, and after a finite propagation distance L they disappear; that is why they are called the transient solitons [9,13,26]. Our expressions for the soliton solution for bilinear variables coincides with the ones obtained in the papers just cited. If we forget for a moment about the anti-Stokes field and make use of Eq. (2.1) with $E_a=0$, we will recover the results of Kaup and Menyuk for the electromagnetic fields.

The presence of the anti-Stokes field E_a first changes the definition of the ''nonlinear time.'' In deriving the soliton solution, we directly solved the ISP for the system (2.7).

Finally, Eq. (2.36) combined with Eq. (2.1) produces solutions in terms of the fields E_i, $i=p,s,a$ which are different from the ones in [9,13,26].

The above analysis allows us to link our results to the results of [5,9,13,22,23,26,27]. Indeed, we can see that, in the presence of the anti-Stokes wave, one can also find three regimes: an initial regime, when the Stokes and anti-Stokes waves are small compared to the pump wave; a transient regime, where we find transient solitons both for the S and E fields [see formulas (2.36) and (3.42) and (3.43) below], and a self-similarity regime [see Eq. (3.18) below].

III. PERIODIC, SOLITON, AND SIMILARITY SOLUTIONS

In this section we generalize the results of [13,24] to describe the similarity solutions—solitons, periodic, and self-similar solutions for system (2.2). We also have not been able to resolve the fundamental problem, inherent to this type of NLEE. We have to solve the ISP for $M(\lambda)$ on a finite interval, and naturally the ζ dependence of the corresponding scattering data will depend on the boundary values of Q at both ends of this interval. Indeed, the initial conditions allow us to calculate all necessary quantities such as $S_3(\zeta,\tau=0)$, $S_\pm(\zeta,\tau=0)$ at $\tau=0$. In order to evaluate them at $\tau=1$, we have to solve the problem completely.

On the other hand, the initial conditions of such a physical system must uniquely determine its evolution. One way out of this problem is to impose certain boundary conditions on the operator $M(\lambda)$ — e.g., (quasi) periodic, which are difficult to obtain experimentally [32]. They will relate the values $S_3(\zeta,\tau=0)$, $S_\pm(\zeta,\tau=0)$ to $S_3(\zeta,\tau=1)$, $S_\pm(\zeta,\tau=1)$.

A. Periodic (cnoidal) and solitary waves

At first we will study the cnoidal wave similarity solutions which include solitons as a special limit. Indeed, from Eqs. (2.1) and (2.2) and using the transformed variable $\xi=\zeta-\tau/\alpha$, we find the system

$$\frac{dS_3}{d\xi}=-iQ^*S_++iQS_-, \tag{3.1a}$$

$$\frac{dS_+}{d\xi}=-iQS_3, \tag{3.1b}$$

$$\alpha\frac{dQ}{d\xi}=2iS_+, \tag{3.1c}$$

which has the following first integrals:

$$\tfrac{1}{2}S_3^2+S_+S_-=I_1, \tag{3.2a}$$

$$S_+Q^*+S_-Q=I_2, \tag{3.2b}$$

$$\frac{1}{\alpha}S_3+\frac{1}{2}|Q|^2=I_3. \tag{3.2c}$$

Introducing the real variables A_+, ϕ_+, $\tilde{Q}$, and ϕ by

$$S_+=e^{i\phi_+}A_+, \tag{3.3a}$$

$$S_-=e^{-i\phi_+}A_+, \tag{3.3b}$$

$$Q=\tilde{Q}e^{i\phi}, \tag{3.3c}$$

$$\tilde{\phi}=\phi-\phi_+, \tag{3.3d}$$

we rewrite Eqs. (3.1) and (3.2) as follows:

$$\frac{dS_3}{d\xi}=-2\tilde{Q}A_+\sin\tilde{\phi}, \tag{3.4a}$$

$$\frac{dA_+}{d\xi}=\tilde{Q}S_3\sin\tilde{\phi}, \tag{3.4b}$$

$$\alpha\frac{d\tilde{Q}}{d\xi}=2A_+\sin\tilde{\phi}, \tag{3.5a}$$

$$\frac{d\tilde{\phi}}{d\xi}=\left(\frac{2}{\alpha}\frac{A_+}{\tilde{Q}}+\frac{\tilde{Q}S_3}{A_+}\right)\cos\tilde{\phi}, \tag{3.5b}$$

$$I_1=\tfrac{1}{2}S_3^2+A_+^2, \tag{3.6a}$$

$$I_2=2\tilde{Q}A_+\cos\tilde{\phi}, \tag{3.6b}$$

$$I_3=\tfrac{1}{\alpha}S_3+\tfrac{1}{2}\tilde{Q}^2. \tag{3.6c}$$

Squaring the equation for S_3 and using the expression for I_k, $k=1$, 2, 3, we obtain

$$\left(\frac{dS_3}{d\xi}\right)^2=\frac{4}{\alpha}(S_3-Z_1)(S_3-Z_2)(S_3-Z_3), \tag{3.7}$$

where the constants Z_i are related to I_k by

$$Z_1+Z_2+Z_3=\alpha I_3, \tag{3.8a}$$

$$Z_1Z_2+Z_2Z_3+Z_3Z_1=-2I_1, \tag{3.8b}$$

$$Z_1Z_2Z_3=\alpha I_2^2/4-2\alpha I_1I_3. \tag{3.8c}$$

The solutions for S_3 may be written explicitly in terms of Jacobian sn function. We have the following periodic (cnoidal) waves.

For $\alpha>0$, $Z_1<0<Z_2<Z_3$,

$$S_3=Z_1+(Z_2-Z_1)\,\mathrm{sn}^2[p(\xi-\xi_0),k)], \tag{3.9a}$$

$$p=\left(\frac{Z_3-Z_1}{\alpha}\right)^{1/2}, \tag{3.9b}$$

$$k^2=\frac{Z_2-Z_1}{Z_3-Z_1}. \tag{3.9c}$$

For $\alpha<0$, $Z_1<Z_2<0<Z_3$,

$$S_3=Z_3+(Z_3-Z_2)\,\mathrm{sn}^2[p(\xi-\xi_0),k)], \tag{3.10a}$$

$$p=\left(\frac{Z_3-Z_1}{-\alpha}\right)^{1/2}, \tag{3.10b}$$

$$k^2=\frac{Z_3-Z_2}{Z_3-Z_1}. \tag{3.10c}$$

Here we point out particularly the subcase, when $I_2=0$. Then we are able to express the roots Z_k of Eq. (3.7) explicitly in terms of I_1 and I_3 with the results

$$Z_1=-\sqrt{2I_1}, \tag{3.11a}$$

$$Z_2=\alpha I_3, \tag{3.11b}$$

$$Z_3=\sqrt{2I_1}, \tag{3.11c}$$

$$p=\left(\frac{8I_1}{\alpha^2}\right)^{1/4}, \tag{3.11d}$$

$$k^2=\frac{\alpha I_3+\sqrt{2I_1}}{2\sqrt{2I_1}} \tag{3.11e}$$

for the case of Eq. (3.9), and

$$Z_1=-\sqrt{2I_1}, \tag{3.12a}$$

$$Z_2=\sqrt{2I_1}, \tag{3.12b}$$

$$Z_3=\alpha I_3, \tag{3.12c}$$

$$p=\left(I_3+\frac{\sqrt{2I_1}}{\alpha}\right)^{1/2}, \tag{3.12d}$$

$$k^2=\frac{\alpha I_3-\sqrt{2I_1}}{\alpha I_3+\sqrt{2I_1}} \tag{3.12e}$$

for Eq. (3.10). In the next particular cases we put $k=1$ to find the corresponding solitary waves

For $\alpha>0$, $Z_1<0<Z_2=Z_3$,

$$S_3=Z_2-\frac{Z_2-Z_1}{\cosh^2\left[\sqrt{\frac{(Z_2-Z_1)}{\alpha}}(\xi-\xi_0)\right]}. \tag{3.13}$$

For $\alpha<0$, $Z_1=Z_2<0<Z_3$,

$$S_3=Z_3+(Z_3-Z_2)\tanh^2\left[\left(\frac{Z_3-Z_2}{-\alpha}\right)^{1/2}(\xi-\xi_0)\right]. \tag{3.14}$$

Let us concentrate on the most important solutions, from the physical point of view soliton, i.e., case (iii) with the additional constraint $Z_2=Z_3=\alpha I_3=\sqrt{2I_1}$. The result of integration is

$$\tilde{Q}=\frac{2\sqrt{I_3}}{\cosh Z}, \tag{3.15a}$$

$$S_3=\alpha I_3(\tanh^2 Z-\operatorname{sech}^2 Z), \tag{3.15b}$$

$$A_+=\sqrt{2}\alpha I_3\frac{\tanh Z}{\cosh Z}, \tag{3.15c}$$

$$\phi_+-\phi=\pi/2, \tag{3.15d}$$

$$z=\sqrt{2I_3}(\xi-\xi_0), \tag{3.15e}$$

where ξ_0 is the arbitrary initial phase. We will return again to this solution in Sec. III D.

B. Self-similarity solutions

We prefer here to analyze the solutions of Eq. (2.2) with another self-similarity variable $\xi=2\sqrt{2\zeta\tau}$. If we choose

$$S_3=\cos[\beta(\xi)], \tag{3.16a}$$

$$S_+=\frac{i}{2}\sqrt{2}\sin[\beta(\xi)], \tag{3.16b}$$

we find that Eq. (2.2) goes into

$$\frac{d^2\beta(\xi)}{d\xi^2}+\frac{1}{\xi}\frac{d\beta(\xi)}{d\xi}+\sin[\beta(\xi)]=0. \tag{3.17}$$

Equation (3.17) was first derived in another context for the transient stimulated Raman scattering by Elgin and O' Hare [35]. This equation can be reduced to one of the standard forms of the Painleve (P_{III}) equation [36,25]. When $\xi\gg1$, we can use the asymptotic formula given by Novokshenov [36] to obtain

$$\beta(\xi)=\frac{\tilde{\alpha}}{\xi^{1/2}}\cos\left(\xi+\frac{\tilde{\alpha}^2}{16}\ln\xi+\psi\right), \tag{3.18}$$

where

$$\tilde{\alpha}^2=-\frac{16}{\pi}\ln[\cos(\beta_0/2)], \tag{3.19a}$$

$$\psi=\frac{2\ln 2}{\pi}\ln[\cos(\beta_0/2)]+\arg\Gamma\left(\frac{i\tilde{\alpha}^2}{16}\right)-\frac{\pi}{4}. \tag{3.19b}$$

Here $\Gamma(x)$ is the gamma function with a complex argument, $\arg[\Gamma(x)]$ indicates its phase, and $\beta_0=\beta(\xi=0)$. Similar expression can be obtained for Q from Eq. (2.18). In Figs. 1 and 2 we plot the functions $S_3(\xi)$ and the real part of the Rabi frequency $\Omega_R(\xi)=E_s^*E_p+E_p^*E_a$.

From Fig. 1 we see the oscillating energy exchange whose amplitude decreases with ξ. Such self-similarity solutions in the case when the anti-Stokes wave is neglected are known as accordions [24,25].

C. Discussion

To obtain the bright solitons and to compare our results with the ones of Kaplan *et al.* [33] we slightly generalize Eqs. (1.1) (see for example [1,3]). The Raman quantum transition between the lower (ground) and upper (excited) level, i.e., two-level atom is described by a 2×2 Hermitian density matrix ρ, and the generalized Bloch equations [1,3]

$$\frac{\partial Q}{\partial\tau}=-\hat{\Omega}_R\Delta, \tag{3.20a}$$

$$\frac{\partial\Delta}{\partial\tau}=\mathrm{Re}(Q\hat{\Omega}_R^*). \tag{3.20b}$$

The system of equations

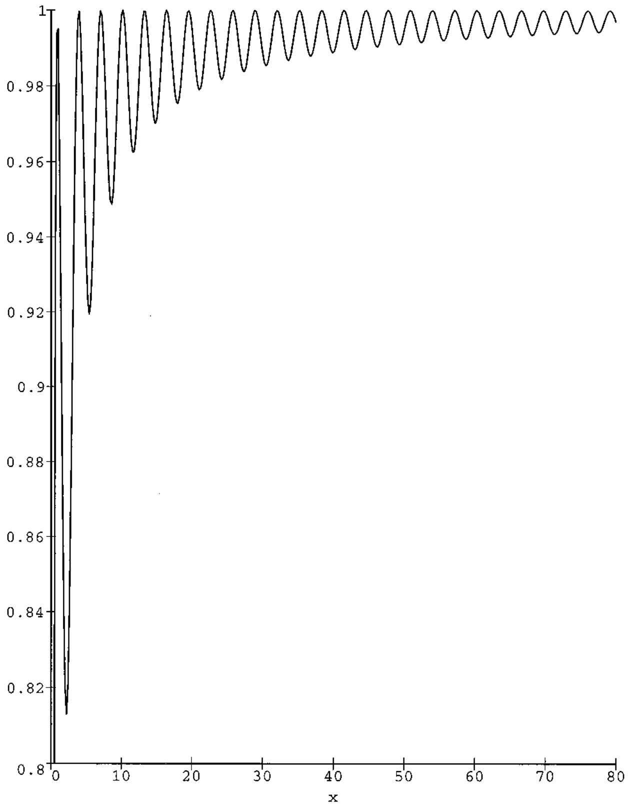

FIG. 1. The self-similar solution $S_3(\xi)=\cos\beta(\xi)$, $\xi=2\sqrt{2\zeta\tau}$ with the initial condition $\beta(0)=1.15$.

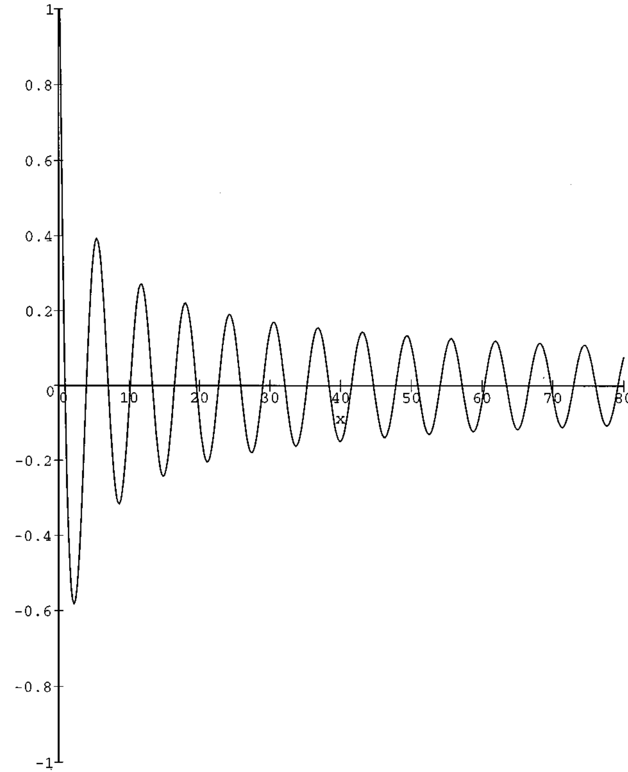

FIG. 2. The real part of the Rabi frequency $\Omega_R(\xi)=\mathrm{Re}(E_s^*E_p+E_p^*E_a)=\sqrt{2}\sin\beta(\xi)$, $\xi=2\sqrt{2\zeta\tau}$ with the initial condition $\beta(0)=1.15$.

$$\delta_p\frac{\partial\widetilde{E}_p}{\partial\zeta}=\beta_a Q^*\widetilde{E}_a-Q\widetilde{E}_s, \tag{3.21a}$$

$$\delta_s\frac{\partial\widetilde{E}_s}{\partial\zeta}=Q^*\widetilde{E}_p, \tag{3.21b}$$

$$\delta_a\frac{\partial\widetilde{E}_a}{\partial\zeta}=-\beta_a^*Q\widetilde{E}_p, \tag{3.21c}$$

completed with the corresponding equations for $Q\sim\rho_{12}$ [33] and $\Delta=\rho_{11}-\rho_{22}$ from Eq. (3.20), generalizes Eq. (1.1) with $q=0$ and $\widetilde{g}=0$. Here the following notations are introduced:

$$\widetilde{E}_i=w_{s,p}\sqrt{n_ic/2\hbar\omega_i}E_i, \tag{3.22a}$$

$$\beta_a=\frac{w_{p,a}^*}{w_{s,p}^*}, \tag{3.22b}$$

$$\hat{\Omega}_R=\frac{4}{\pi N_0}(\widetilde{E}_s^*\widetilde{E}_p+\beta_a\widetilde{E}_p^*\widetilde{E}_a), \tag{3.22c}$$

$$w_{s,p}=\frac{N_0\pi}{c}\alpha_{s,p}\left(\frac{\omega_s\omega_p}{n_sn_p}\right)^{1/2}, \tag{3.23a}$$

$$w_{p,a}=\frac{N_0\pi}{c}\alpha_{p,a}\left(\frac{\omega_p\omega_a}{n_pn_a}\right)^{1/2}, \tag{3.23b}$$

where $\alpha_{s,p}$ and $\alpha_{p,a}$ are Raman polarizability (see for example [3]), $n_i=n(\omega_i)$, $j=s,p,a$ is the refractive index at frequencies ω_i. By N_0 we have denoted the density of Raman particles. Using arguments analogous to the ones in [11] we find that one can expect such physical systems to be described by system (1.1) with $\beta_a\simeq 1$. In addition, $\hat{\Omega}_R$ is the generalized local Rabi frequency [1,3].

Equations (3.20) and (3.21), rewritten for the self-similarity variable $\xi=\zeta-\tau/\alpha$, coincide with Eq. (6) from Kaplan *et al.* [33]. The direct comparison of S_+ and $\hat{\Omega}_R$ shows that the physical interpretation of S_+ is the normalized local Rabi frequency. From these equations we also obtain that the quantity

$$\Delta^2+|Q|^2=I_4(\zeta) \tag{3.24}$$

is conserved in τ, and that the equations

$$\delta_p\Phi_p+\delta_s\Phi_s+\delta_a\Phi_a=I(\tau), \tag{3.25a}$$

$$\delta_i=\left(\frac{1}{v_{gi}}-\frac{1}{\widetilde{v}_g}\right),\quad i=p,s,a, \tag{3.25b}$$

$$2\frac{\partial}{\partial\zeta}(\delta_a\Phi_s-\delta_a\Phi_s)-\pi N_0\frac{\partial}{\partial\tau}\Delta=0, \tag{3.26a}$$

$$\Phi_i=|\widetilde{E}_i|^2 \tag{3.26b}$$

hold. Using the ansatz [33]

$$\Phi_p=|a_p|^2\Phi_\Sigma, \tag{3.27a}$$

$$\Phi_s=|a_s|^2\Phi_\Sigma\,, \tag{3.27b}$$

$$\Phi_a=|a_a|^2\Phi_\Sigma\,, \tag{3.27c}$$

$$|a_p|^2=\frac{\gamma_3^2}{W}, \tag{3.27d}$$

$$|a_s|^2=\frac{1}{\delta_s^2 W}, \tag{3.27e}$$

$$|a_a|^2=\frac{|\beta_a|^2}{\delta_a^2 W}, \tag{3.27f}$$

we find the system

$$\frac{d\Phi_\Sigma}{d\xi}=\alpha_2(Q^*\Omega_R+Q\Omega_R^*), \tag{3.28a}$$

$$\frac{d\Omega_R}{d\xi}=\alpha_1 Q\Phi_\Sigma\,, \tag{3.28b}$$

$$\alpha\frac{dQ}{d\xi}=\frac{4}{\pi N_0}\Omega_R\Delta, \tag{3.28c}$$

$$\alpha\frac{d\Delta}{d\xi}=-\frac{2}{\pi N_0}(Q^*\Omega_R+Q\Omega_R^*), \tag{3.28d}$$

$$\Omega_R=\tilde{E}_s^*\tilde{E}_p+\beta_a E_p^*\tilde{E}_a\,, \tag{3.28e}$$

$$\alpha_1=\left(\frac{1}{\delta_s}-\frac{|\beta_a|^2}{\delta_a}\right)|a_p|^2+\frac{1}{\delta_p}(|\beta_a|^2|a_a|^2-|a_s|^2), \tag{3.28f}$$

$$\alpha_2=\frac{1}{\delta_s|a_s|^2-\delta_a|a_a|^2}, \tag{3.28g}$$

which has the following first integrals:

$$\frac{1}{2\alpha_2}\Phi_\Sigma^2-\frac{1}{\alpha_1}|\Omega_R|^2=I_1\,, \tag{3.29a}$$

$$\Omega_R Q^*-\Omega_R^* Q=I_2\,, \tag{3.29b}$$

$$\frac{1}{\alpha\alpha_2}\Phi_\Sigma+\frac{\pi N_0}{2}\Delta=I_3\,, \tag{3.29c}$$

$$\Delta^2+|Q|^2=I_4\,. \tag{3.29d}$$

Introducing the real variables $\tilde{\Omega}_R$, ϕ_R, $\tilde{\Phi}_\Sigma$, $\tilde{Q}$, and ϕ by

$$\Omega_R=e^{i\phi_R}\tilde{\Omega}_R\,, \tag{3.30a}$$

$$Q=\tilde{Q}e^{i\phi}, \tag{3.30b}$$

$$\tilde{\phi}=\phi-\phi_R\,, \tag{3.30c}$$

we rewrite Eq. (3.28) and (3.29) as follows:

$$\left(\frac{d\Phi_\Sigma}{d\xi}\right)^2=\frac{4}{\alpha_3^2}(\Phi_\Sigma-Z_1)(\Phi_\Sigma-Z_2)(\Phi_\Sigma-Z_3)(\Phi_\Sigma-Z_4), \tag{3.31a}$$

$$\alpha_3=-\frac{\alpha_2(\pi N_0\alpha)^2}{2\alpha_1}, \tag{3.31b}$$

where the constants Z_i are related to I_k by

$$Z_1+Z_2+Z_3+Z_4=2\alpha_2\alpha I_3\,, \tag{3.32a}$$

$$Z_1Z_2+Z_2Z_3+Z_3Z_1+Z_1Z_4+Z_2Z_4+Z_3Z_4 =-\alpha_2^2(\pi N_0\alpha)^2 I_4/4+\alpha_2^2\alpha^2 I_3^2-2\alpha_2 I_1\,, \tag{3.32b}$$

$$Z_1Z_2Z_4+Z_1Z_2Z_3+Z_2Z_3Z_4+Z_1Z_3Z_4=-4\alpha_2^2\alpha I_3 I_1, \tag{3.32c}$$

$$Z_1Z_2Z_3Z_4=\frac{8\alpha_2\alpha_1 I_2^2}{(\pi N_0\alpha)^2}+\alpha_2^3(\pi N_0\alpha)^2 I_4 I_1-2\alpha_2^3\alpha^2 I_3^2 I_1\,, \tag{3.32d}$$

where Φ_Σ has different forms (cnoidal wave, soliton, Lorentzian etc.). The integration procedure is analogous to that used in Sec. III A.

In the particular case when the anti-Stokes wave is neglected, $\beta_a=0$, and $\tilde{E}_a=0$, and for the real quantities $\tilde{E}_p$, $\tilde{E}_s$, ϕ, and $\tilde{\Omega}_R$ from Eqs. (3.20) and (3.21) rewritten for the self-similarity variable $\xi=\zeta-\tau/\alpha$, we have

$$\delta_p\frac{d\tilde{E}_p^2}{d\xi}=2\,\tilde{\Omega}_R\sin\phi, \tag{3.33a}$$

$$\delta_s\frac{d\tilde{E}_s^2}{d\xi}=-2\,\tilde{\Omega}_R\sin\phi, \tag{3.33b}$$

$$\phi=\frac{4}{\pi N_0\alpha}\int_{-\infty}^{\xi}\Omega_R(\xi')d\xi', \tag{3.33c}$$

$$Q=\sin\phi, \tag{3.33d}$$

$$\Delta=-\cos\phi. \tag{3.33e}$$

After integration of Eqs. (3.33) we obtain the cnoidal solutions of [6]; see also [7]. The solutions of Eq. (3.33) are substantially different from the cnoidal solutions in Sec III A because they are related to a fourth-order polynomial rather than to a third order one as in Eq. (3.7).

Next we will consider the simplest solution (Lorenzian soliton). This class of solutions satisfies the boundary conditions (2.9). Inserting Eq. (3.27) into $I=0$, from Eq. (3.25) we obtain

$$\gamma_3^2=-\left(\frac{1}{\delta_s\delta_p}+\frac{|\beta_a|^2}{\delta_a\delta_p}\right). \tag{3.34}$$

Equation (3.26) has the form of a conservation law with conserved density $\delta_s\Phi_s-\delta_a\Phi_a$ and conserved flux $-\pi N_0\Delta$. Then

$$J=2(\delta_s\Phi_s-\delta_a\Phi_a)-\pi N_0\Delta=\pm N_0\,, \tag{3.35a}$$

$$\Delta^2+|Q|^2=I_4=1, \tag{3.35b}$$

$$\Phi_\Sigma=\Phi_0 S(\xi). \tag{3.35c}$$

Here

$$\Phi_0=\frac{\pm\pi N_0 W}{(1/\delta_s-|\beta_a|^2/\delta_a)}, \tag{3.36}$$

where $-$ indicates that the molecules (atoms) are initially at the equilibrium and $+$ that the population difference is inverted. Let us also introduce

$$\Delta=\pm[1-2S(\xi)], \tag{3.37a}$$

$$Q(\xi)=-4\gamma_3\xi S(\xi), \tag{3.37b}$$

$$S(\xi)=(1+4\gamma_3^2\xi^2)^{-1}, \tag{3.37c}$$

where $S(\xi)$ may have Lorentzian form (Lorentzian soliton) [33]. Finally from the normalization condition $|a_p|^2+|a_s|^2+|a_a|^2=1$, we have

$$W=\frac{1}{\delta_s\delta_{s,p}}-\frac{|\beta_a|^2}{\delta_a\delta_{p,a}}, \tag{3.38}$$

where

$$\delta_{s,p}=\frac{1}{\delta_s}-\frac{1}{\delta_p}, \tag{3.39a}$$

$$\delta_{p,a}=\frac{1}{\delta_p}-\frac{1}{\delta_a}. \tag{3.39b}$$

Recently these bright solitons [Eq. (3.37)], in a more general physical situation, cascade SRS [11,3], have been used to predict generation of subfemtosecond coherent pulses in SRS experiments [33]. From the above analysis it is clear that bright solitons are obtained in the case of finite group velocity dispersion parameters δ_i [Eq. (3.26)] [11,33] and nonzero population difference Δ.

D. One-soliton solution

Here we will show that the auxiliary linear problem for the vector NLS equation with some additional reduction is equivalent to Stokes–anti-Stokes SRS equations without the last equation for Q. This formal equivalence allows us to recover E_p, E_s, and E_a from the potential Q, already obtained by the ISM of Sec. II with Eq. (2.36).

Indeed, we introduce

$$\frac{\partial}{\partial\zeta}\begin{pmatrix}\tilde\psi_1\\ \tilde\psi_2\\ \tilde\psi_3\end{pmatrix}=\begin{pmatrix}0 & q_1 & q_2\\ -q_1^* & 0 & 0\\ -q_2^* & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde\psi_1\\ \tilde\psi_2\\ \tilde\psi_3\end{pmatrix}, \tag{3.40}$$

and require that

$$q_1=-Q, \tag{3.41a}$$

$$q_2=Q^*, \tag{3.41b}$$

$$\psi_1=E_p, \tag{3.41c}$$

$$\psi_2=E_s, \tag{3.41d}$$

$$\psi_3=E_a, \tag{3.41e}$$

without the last equation for Q, which we may obtain from Sec. I. Using the well-known one-soliton solution of the vector nonlinear Schrödinger equation under reduction (3.41) and the solution of the linear problem (3.40) with potentials Q,Q^* we obtain in (2.36), i.e.,

$$Q=\frac{\sqrt{2}\,\eta e^{i\phi}}{\cosh Z}, \tag{3.42a}$$

$$Z=\eta\zeta-\frac{1}{\eta}\int_0^\tau \mathcal{E}^2(\tau')\,d\tau', \tag{3.42b}$$

where $\mathcal{E}(\tau)$ is real, the soliton's eigenvalue is $i\eta$ and ϕ is constant real phase. The direct integration of Eq. (3.40) with reduction (3.41) is given by

$$E_p=\sqrt{2}\mathcal{E}(\tau)\frac{\tanh Z}{\cosh Z}e^{i\phi}, \tag{3.43a}$$

$$E_s=\mathcal{E}(\tau)\tanh^2 Z, \tag{3.43b}$$

$$E_a=\frac{\mathcal{E}(\tau)}{\cosh^2 Z}e^{2i\phi}. \tag{3.43c}$$

These solutions are similar to transient SRS solitons obtained in [13] (see also [15]) and for $\mathcal{E}=1$, and $\phi=0$ coincide with the ones in [11]. If we now calculate S_3 and $S_\pm$ using the above expressions for E_p, E_s, E_a, we obtain precisely the soliton solution (2.36) and (3.15).

IV. *N*-COMPONENT GENERALIZATIONS

In this section we show that the extended model with N-Stokes and N-Stokes components is also integrable in the sense means of ISM. The considerations are formal from a physical point of view.

Let us consider the following equations:

$$\frac{\partial E_p}{\partial\zeta}=\sum_{i=1}^N(\beta_a Q^* E_a^{(i)}-QE_s^{(i)}), \tag{4.1a}$$

$$\frac{\partial E_s^{(i)}}{\partial\zeta}=Q^*E_p,\quad i=1,\dots,N, \tag{4.1b}$$

$$\frac{\partial E_a^{(i)}}{\partial\zeta}=-\beta_a Q E_p, \tag{4.1c}$$

$$\frac{\partial Q}{\partial\tau}+\tilde g Q=\sum_{i=1}^N(E_s^{*(i)}E_p^{(i)}+\beta_a E_p^{*(i)}E_a^{(i)}), \tag{4.1d}$$

with $\beta_a=1$. The equations for E_k^i can be written down as the same auxiliary linear problem, which solves the N-component vector NLS equation

$$\frac{\partial}{\partial \zeta}\mathbf{\Psi}=\mathbf{U_0}\mathbf{\Psi}, \tag{4.2}$$

with the spectral parameter $\lambda=0$. Here

$$\mathbf{\Psi}=\begin{pmatrix}\tilde{\psi}_1\\ \tilde{\psi}_2\\ \tilde{\psi}_3\\ \vdots\\ \tilde{\psi}_{2N}\\ \tilde{\psi}_{2N+1}\end{pmatrix}, \tag{4.3a}$$

$$\mathbf{U_0}=\begin{pmatrix}0 & q_1 & q_2 & \dots & q_{2N-1} & q_{2N}\\ -q_1^* & 0 & 0 & \dots & 0 & 0\\ -q_2^* & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -q_{2N-1}^* & 0 & 0 & \dots & 0 & 0\\ -q_{2N}^* & 0 & 0 & \dots & 0 & 0\end{pmatrix}, \tag{4.3b}$$

and

$$q_1=q_3=\cdots=q_{2N-1}=-Q, \tag{4.4a}$$

$$q_2=q_4=\cdots=q_{2N}=Q^*, \tag{4.4b}$$

$$\psi_1=E_p, \tag{4.4c}$$

$$\psi_{2k}=E_s^{(k)}, \quad k=1,2,\dots,N \tag{4.4d}$$

$$\psi_{2k+1}=E_a^{(k)}, \quad k=1,2,\dots,N. \tag{4.4e}$$

We again introduce bilinear variables

$$S_3=\frac{1}{2}\sum_{i=1}^{N}(|E_s^{(i)}|^2-|E_a^{(i)}|^2), \tag{4.5a}$$

$$S_+=S_-^*=\frac{i}{2}\sum_{i=1}^{N}(E_s^{*(i)}E_p^{(i)}+E_p^{*(i)}E_a^{i}), \tag{4.5b}$$

and show that if $E_{a,p,s}^{(i)}$ satisfies Eq. (4.1) then S_3 and $S_\pm$ satisfy the same equation (2.2). Therefore the Lax representation (2.5) and Kaup's method can be used as above for analysis of system (4.1). The procedure of solving Eq. (4.1) is analogous to the considerations of Sec III D. Clear physical interpretation and solutions of Eq. (4.1) will be given elsewhere.

V. CONCLUSION

In this paper we have generalized the method in [9,13,26] for solving the SRS problem with $\beta_a=1$ when both Stokes and anti-Stokes fields are present. For bright solitons, our results are valid also for $\beta\neq 1$. Several types of explicit (periodic, soliton, and self-similarity) solutions are obtained. The only difference between the mathematical description of the MBE without an anti-Stokes wave and our case in terms of the bilinear variables S_3 and $S_\pm$ consists of the definition of the ''nonlinear'' time τ. Of course, in terms of the electromagnetic fields E_i, $i=p,s,a$, both the physical interpretation and the form of the solutions are substantially different.

For bright and transient solitons our results are in agreement with these of Kaplan *et al.* [33] and Kaup and Menyuk [26,9]. In some cases we have constructed not only the bilinear variables S_3 and $S_\pm$, but also the fields E_p, E_s, and E_a themselves.

The ISM meets with difficulties when both initial and boundary conditions (2.9) on the potentials are imposed. These problems do not arise when one uses quasiperiodic boundary conditions, which, however, may be difficult to realize in experiment.

Normally Stokes–anti-Stokes scattering occurs when the wave number mismatch q is different from zero. Recent numerical studies in this direction show that in this case the solitons decay [31]. For simplicity we have considered only the exact resonance condition $q=0$, which also has an important physical meaning [11,33,37].

Looking at the Lax operator (2.16) we recognize the system of equations (2.2) as belonging to the Heisenberg ferromagnet hierarchy. One can try to apply the expansions over the ''squared solutions'' in the spirit of [38], and then treat the case $\tilde{g}\neq 0$ as a perturbation.

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