Random walks, continuum limits, and Schrödinger's equation

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By considering the simple binary symmetric random walk on a discrete lattice in 1+1 dimensions, we show that the discrete analog of Schrödinger's equation describes a simple counting problem involving the sample paths on the lattice. Schrödinger's equation is obtained in the continuum limit with the result that this equation is confirmed to have a classical as well as a quantum context. [S1050-2947(96)08710-0]

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I. INTRODUCTION

Although the character of the solutions of the diffusion and free-particle Schrödinger equations are qualitatively very different, the two equations themselves bear a strong formal resemblance. Mathematically one may obtain Schrödinger's free-particle equation by letting the time coordinate in the diffusion equation be imaginary. This process is a formal analytic continuation (FAC) and it suggests that we may compare solutions of the two equations in spite of their qualitative difference.

The limitations of the FAC may be seen by examining the simplest case. Consider the diffusion equation in 1+1 dimensions

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

As shown by Einstein in 1905, Brownian motion provides this phenomonological equation with a microscopic model. That is, the $u(x,t)$ of Eq. (1) describes the ensemble average concentration of small particles undergoing Brownian motion on a scale much less than the scale of observation. Analytically continuing the diffusion equation to imaginary time we obtain the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = iD \frac{\partial^2 \psi}{\partial x^2}. \tag{2}$$

If we graft an interpretation that is appropriate for quantum mechanics onto this equation, then we have arrived at quantum mechanics in a formal way. The mathematical relation between the two equations is useful for *calculations* pertaining to Schrödinger's equation, but it is not particularly useful in the interpretation of the equation itself. Note that the FAC, which transplants the diffusion equation into the domain of quantum mechanics, does not transfer the microscopic model that goes with it. The random walks of Brownian particles that underlie the diffusion equation have a correspondence in the Feynman paths of Schrödinger's equation [1,2], but the mapping also takes the real, positive Boltzman weights of these paths and converts them into complex numbers, thus removing the transformed ''model'' from the domain of classical probability. The easily interpreted concentration or probability density $u(x,t)$ of Eq. (1) becomes a ''wave function'' under the FAC. This object, while exceedingly useful in quantum mechanics, has no known physical counterpart and is not observable in nature.

The comparison of classical and quantum equations that differ by a FAC may be extended to more interesting quantum systems by considering the path-integral formulation of quantum mechanics and its relation to the classical Wiener integral. The resulting connection between quantum field theory and statistical mechanics has been of considerable interest in both fields now for some years [3,4]. In addition, the formal similarity between the quantum and classical equations has encouraged efforts to effect an analytic continuation by invoking physical processes that mimic self-interference, thereby motivating an otherwise formal procedure. Some references include the works of Fényes [5], Nelson [6], and Nottale [7] who have proposed physical systems that combine diffusion and antidiffusion to produce interference and El Naschie and co-workers [8,9], who have used a variational principle to time symmetrize a diffusive process to accomplish the same effect. In the relativistic domain, the relationship between random walks and quantum propagators is discussed in Refs. [10–13].

In Fig. 1 we illustrate the different context in which Schrödinger's equation appears in this paper. Here *both* the diffusion and Schrödinger equations occur within the domain of classical statistical mechanics. The model that we use is the standard lattice random-walk model of Brownian motion [14,15]. Schrödinger's equation appears in the solution of this model without having to alter the dynamics in any way [16]; however, to show this one has to retain more informa-

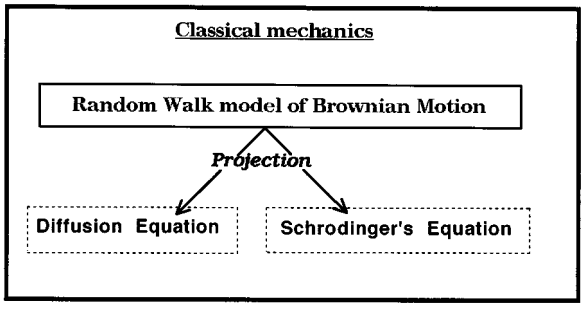

FIG. 1. In this paper, both the diffusion and Schrödinger equations occur within the domain of classical statistical mechanics, both equations occurring as projections from the same random-walk model. This provides a context for Schrödinger's equation that is independent of its context in quantum mechanics.

tion about the random walks than previously considered. This information is described by a vector in a four-dimensional space. Solutions of the diffusion equation appear directly in a first-order projection out of this space and solutions of Schrödinger's equation appear directly in a second-order projection. The fact that these two projections are orthogonal allows the two qualitatively different behaviors to coexist in the same physical system. No formal analytic continuation is required to produce the Schrödinger equation from this classical model and, figuratively speaking, both equations simply represent different ''views'' of a single object, the object being an ensemble of Brownian particles. In contrast to the usual FAC that links the two equations, this derivation produces a ''wave function'' that is an observable property of ensembles of real point particles.

In Sec. II we introduce the discrete random-walk model that produces both the diffusion and Schrödinger equations as part of its description. However in this section, we concentrate on the dynamics of the ensemble of walks without regard to the spatial distribution of the walks. In Sec. III we reintroduce the distribution of walks in space and consider a continuum limit.

II. BINARY RANDOM WALKS AND MARKOV CHAINS

In this paper we will consider only symmetric random walks in discrete time. There will be no internal or external mechanism to change the dynamics of the walks. Instead we shall simply retain more information about the dynamics than is usually considered and we shall see that this is enough to show that Schrödinger's equation occurs naturally in the description of such walks.

Consider a space-time lattice with respective spacings δ and ϵ. Particles hop on this lattice a distance $\pm\delta$ at each time step ϵ. The walks are symmetric and at each lattice site walks are equally likely to take either direction. We shall be interested in the statistics of the number of direction changes in trajectories on the lattice. In particular, between lattice sites, each particle will be in one of two direction states (right or left moving) and one of two ''spin'' states. We use the Ising spin variable $\sigma=\pm1$ to describe these two states. The direction state will change with every ''collision'' (direction change) and the spin will change with every two collisions (Fig. 2). Thus, a particle starting off in state one (right moving, $\sigma=+1$) changes to state two (left moving, $\sigma=+1$) at the first collision, state three (right moving, $\sigma=-1$) at the second collision, state four (left moving, $\sigma=-1$) at the third collision, and back to state one at the fourth collision. Note that states one and three both correspond to right-moving particles and states two and four correspond to left-moving particles. A particle that starts in state one (two) and ends in the same *direction* state three (four) has changed its spin from $+1$ to -1. Notice here that the Ising spin is *not* a new property that has been added to our particles. It is simply a convenient label that helps to classify particle trajectories. In Ord [16], the difference between two states with identical directions was called parity.

Let $p_\mu(m\delta,s\epsilon)\delta$ $(\mu=1,2,3,4)$ be the probability that a particle leaving the space-time point $(m\delta,s\epsilon)$ is in state μ $(m=0,\pm1,\ldots;\ s=0,1,\ldots)$. The difference equations for p_μ that we study in this paper are

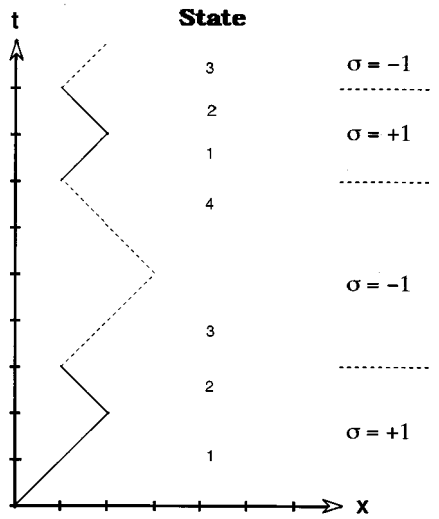

FIG. 2. A typical path on the space-time lattice. The particle changes state cyclically with each collision, the ''state'' being a means of counting the number of times a direction occurs modulo 2. The ''spin'' of the trajectory changes every two collisions.

$$
\begin{aligned}
p_1(m\delta,(s+1)\epsilon)&=\tfrac{1}{2}p_1((m-1)\delta,s\epsilon)+\tfrac{1}{2}p_4((m+1)\delta,s\epsilon),\\
p_2(m\delta,(s+1)\epsilon)&=\tfrac{1}{2}p_2((m+1)\delta,s\epsilon)+\tfrac{1}{2}p_1((m-1)\delta,s\epsilon),\\
p_3(m\delta,(s+1)\epsilon)&=\tfrac{1}{2}p_3((m-1)\delta,s\epsilon)+\tfrac{1}{2}p_2((m+1)\delta,s\epsilon),\\
p_4(m\delta,(s+1)\epsilon)&=\tfrac{1}{2}p_4((m+1)\delta,s\epsilon)+\tfrac{1}{2}p_3((m-1)\delta,s\epsilon).
\end{aligned}
\tag{3}
$$

Upon multiplying these equations by δ, the first equation implies that the probability that a particle leaves node $(m\delta,(s+1)\epsilon)$ in state one is equal to the sum of two probabilities. One is $\frac{1}{2}p_1((m-1)\delta,s\epsilon)\delta$ that is the probability that a particle leaves node $((m-1)\delta,s\epsilon)$ in state one and remains in this state when it leaves the node $(m\delta,(s+1)\epsilon)$. The other one is $\frac{1}{2}p_4((m+1)\delta,s\epsilon)\delta$ that is the probability that a particle leaves node $((m+1)\delta,s\epsilon)$ in state four and changes to state one when it leaves the node $(m\delta,(s+1)\epsilon)$. The remaining three equations have a similar interpretation, and the probability p_μ is uniquely determined once the initial conditions are specified. For $s\geqslant0$ we take

$$
\sum_{\mu=1}^{4}\sum_{m=-\infty}^{+\infty}p_\mu(m\delta,s\epsilon)\,\delta=1, \tag{4}
$$

which establishes the fact that the probability that a particle is somewhere on the lattice at a given time is one.

We will be interested in the statistics of these binary random walks and, especially, in the difference in the number of particles reaching a given space-time point with opposite spin. However, before we perform this analysis, we examine the walks with regard to the distribution of particles in the four states before considering in Sec. III the additional complexity of the motion of particles in space. Let $q_i(s)$ be the probability that a particle is in the ith state $(i=1,2,3,4)$ at the sth step on the lattice. Then $q_i(s)$ is equal to $\Sigma_{m=-\infty}^{+\infty}p_i(m\delta,s\epsilon)\,\delta$. Multiplying Eq. (3) by δ and summing over m, the equations reduce to

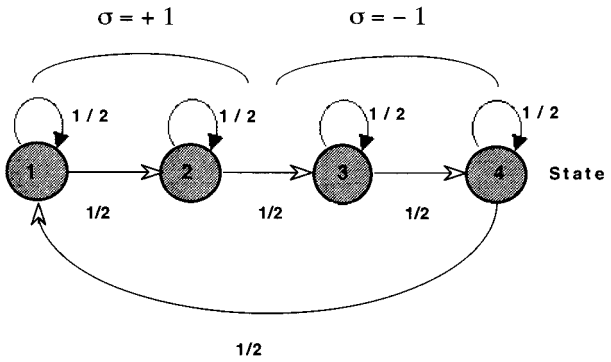

FIG. 3. The underlying Markov chain for the lattice walks. The probability of a state change at each time step is $\frac{1}{2}$.

$$[q_1(s+1),q_2(s+1),q_3(s+1),q_4(s+1)]^T = T(1) \times [q_1(s),q_2(s),q_3(s),q_4(s)]^T. \tag{5}$$

The transition matrix $T(1)$, which we now consider, defines a Markov chain.

It is convenient to define the more general Markov chain with four states where the transition matrix

$$T(\alpha)=\frac{\alpha}{2}\begin{bmatrix}1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix}. \tag{6}$$

With $\alpha=1$, T_{ij} is the probability of a transition from state j to state i in one step (Fig. 3). $[T^s]_{ij}$ is the probability of a transition from state j to state i in s steps. The behavior of $[T^s(1)]_{ij}$ with s is very simple and $[T^s]_{11}$ is illustrated in Fig. 4. Since in the limit of large s each state is equally likely, all elements of $T^s(1)$ approach $\frac{1}{4}$. The oscillation observed in Fig. 4 reflects the ''transient response'' of the chain. With $\alpha=2$ the matrix T counts paths in the Markov process. Thus $[T^s(2)]_{ij}$ is the number of distinct paths from j to i in s steps. The powers of the matrix T obey

$$\sum_{i=1}^{4}[T^s(2)]_{ij}=2^s \quad (j=1,2,3,4;\ \ s=1,2,\ldots) \tag{7}$$

expressing the fact that there are 2^s distinct s-step paths. Note that Eq. (7) may be reexpressed as an invariance property of T; namely,

$$\frac{1}{(2^s)}\sum_{i=1}^{4}[T^s(2)]_{ij}=\sum_{i=1}^{4}[T^s(1)]_{ij}=1 \quad (s=1,2,\ldots). \tag{8}$$

This is in accord with the idea that the elements of $T^s(1)$ may be regarded as probabilities.

Other features of the Markov process are easily understood by considering a change of variables from $q_1(s),q_2(s),q_3(s),q_4(s)$ to $u_1(s),u_2(s),\tilde{\xi}_1(s),\tilde{\xi}_2(s)$ where

$$u_1=q_1+q_2+q_3+q_4, \quad \tilde{\xi}_1=q_1-q_3,$$
$$u_2=(q_1+q_3)-(q_2+q_4), \quad \tilde{\xi}_2=q_2-q_4. \tag{9}$$

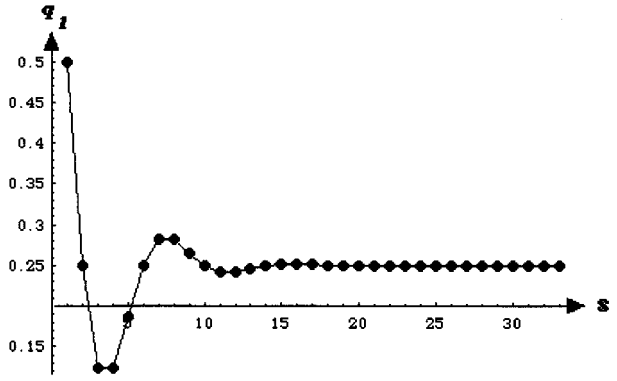

FIG. 4. The probability $q_1(s)$ that a particle starting in state one at time zero is in state one after n steps.

In words, $u_1(s)$ is the sum of all occupation probabilities, $u_2(s)$ is the difference of occupation probabilities by direction, and $\tilde{\xi}_1(s)$ and $\tilde{\xi}_2(s)$ are the differences of occupation of the two spin states for right- and left-moving particles, respectively. Rewriting Eq. (9) in matrix form, we have

$$\begin{bmatrix}u_1\\u_2\\\tilde{\xi}_1\\\tilde{\xi}_2\end{bmatrix}=R\begin{bmatrix}q_1\\q_2\\q_3\\q_4\end{bmatrix}, \quad R=\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}. \tag{10}$$

To examine the dynamics of $\tilde{\xi}_i$, we normalize it suitably by defining $\xi_i(s)=(\sqrt{2})^s\,\tilde{\xi}_i(s)$. Substituting Eq. (10) into Eq. (5), the resulting system of equations is

$$\begin{bmatrix}u_1(s+1)\\u_2(s+1)\end{bmatrix}=\begin{bmatrix}u_1(s)\\0\end{bmatrix}, \quad \begin{bmatrix}\xi_1(s+1)\\\xi_2(s+1)\end{bmatrix}=V\begin{bmatrix}\xi_1(s)\\\xi_2(s)\end{bmatrix}, \tag{11}$$

where

$$V=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}. \tag{12}$$

These equations can also be expressed in the form

$$u_1(s)=u_1(0)=1, \quad u_2(s+1)=0, \quad \Xi(s)=V^s\Xi(0), \tag{13}$$

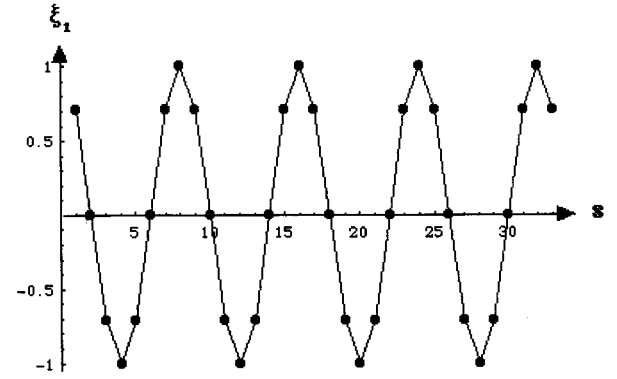

FIG. 5. $\xi_1(s)$ for particles starting in state one at time zero. Notice that ξ_1 is periodic with period eight.

where $\Xi(s)=[\xi_1(s),\xi_2(s)]^T$. Here we see that R block diagonalizes the matrix T. Equation (13) shows that whatever the dynamic behavior of $\tilde{\xi}_i$, the behavior is intrinsic to the four-state Markov chain described by T, and independent of the dominant variable u_1. The dynamics of ξ_i become clear when we notice that V is just a rotation matrix with a rotation angle of $\pi/4$, and hence, $V^8=I$ where I is the 2×2 identity matrix. Thus, we see that $V^s=V^{(s+8k)}$ $(k=0,\pm1,\ldots)$ so that ξ_i is periodic with period eight. That is, replacing s by $s+8k$ in Eq. (13), we have

$$\xi_i(s)=\xi_i(s+8k) \quad (i=1,2;\ \ k=0,1,2,\ldots). \tag{14}$$

In Fig. 5, the graph of $\xi_1(s)$ illustrates this periodicity. In contrast to the periodicity of $\xi_i(s)$, u_1 is itself an invariant in time. Finally, Eq. (13) shows that $\|\Xi\|^2$ is invariant in discrete time. Since $V^T=V^{-1}$,

$$\Xi^T(s)\Xi(s)=\Xi^T(0)(V^T)^s(V^s)\Xi(0)=\Xi^T(0)\Xi(0). \tag{15}$$

III. THE CONTINUUM LIMIT

In this section we consider Eqs. (3) of the full random walk, and we show that the solutions of these equations can be approximated in terms of solutions of the diffusion equation and Schrödinger's equation. Throughout this section we shall use the usual diffusive scaling for random walks in which for small δ,

$$\frac{\delta^2}{2\epsilon}=D+O(\delta) \ \text{ or } \ \epsilon=\frac{\delta^2}{2D}+O(\delta^3), \tag{16}$$

where D is a diffusion constant. To express Eq. (3) in matrix form, consider the shift operators $E_x^{\pm1}$ and E_t such that

$$E_x^{\pm1}p_i(m\delta,s\epsilon)=p_i((m\pm1)\delta,s\epsilon) \tag{17}$$

and

$$E_t p_i(m\delta,s\epsilon)=p_i(m\delta,(s+1)\epsilon). \tag{18}$$

The difference equations (3) may then be written as

$$E_tP(m\delta,s\epsilon)=\frac{1}{2}\begin{bmatrix}E_x^{-1}&0&0&E_x\\E_x^{-1}&E_x&0&0\\0&E_x&E_x^{-1}&0\\0&0&E_x^{-1}&E_x\end{bmatrix}P(m\delta,s\epsilon), \tag{19}$$

where

$$P(m\delta,s\epsilon)=[p_1(m\delta,s\epsilon),p_2(m\delta,s\epsilon),p_3(m\delta,s\epsilon),p_4(m\delta,s\epsilon)]^T. \tag{20}$$

As in Sec. II, we now consider the change of variables

$$z_1=p_1+p_2+p_3+p_4, \quad \tilde{\phi}_1=p_1-p_3,$$
$$z_2=(p_1+p_3)-(p_2+p_4), \quad \tilde{\phi}_2=p_2-p_4. \tag{21}$$

Here, $z_1(m\delta,s\epsilon)\delta$ $(m=0,\pm1,\ldots;\ s=0,1,\ldots)$ is the probability that a particle is at $x=m\delta$ at time $t=s\epsilon$ in any direction or spin state. $z_2(m\delta,s\epsilon)\delta$ is the expected difference in the probabilities for the two direction states (right or left moving). $\tilde{\phi}_1\delta$ is the expected spin $(+1)p_1\delta+(-1)p_3\delta$ for right-moving particles; while, $\tilde{\phi}_2\delta$ is the expected spin $(+1)p_2\delta+(-1)p_4\delta$ for left-moving particles. The transformation (21) expresses P in terms of an orthogonal basis,

$$P=\frac{z_1}{4}[1,1,1,1]^T+\frac{z_2}{4}[1,-1,1,-1]^T+\frac{\tilde{\phi}_1}{2}[1,0,-1,0]^T+\frac{\tilde{\phi}_2}{2}[0,1,0,-1]^T, \tag{22}$$

where $[1,1,1,1]^T$ and $[1,-1,1,-1]^T$ are eigenvectors of $T(1)$. As in the case of the Markov chain, P is a sum of four orthogonal vectors such that the coefficient of each vector involves only one of z_1, z_2, $\tilde{\phi}_1$, and $\tilde{\phi}_2$.

The change of variables (21) gives

$$E_t\begin{bmatrix}z_1\\z_2\\\tilde{\phi}_1\\\tilde{\phi}_2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}E_x+E_x^{-1}&E_x^{-1}-E_x&0&0\\0&0&0&0\\0&0&E_x^{-1}&-E_x\\0&0&E_x^{-1}&E_x\end{bmatrix}\times\begin{bmatrix}z_1\\z_2\\\tilde{\phi}_1\\\tilde{\phi}_2\end{bmatrix}. \tag{23}$$

In Eq. (23), $z_2(m\delta,(s+1)\epsilon)=0$ [$z_2(m\delta,0)$ may not be 0] and we can analyze $\tilde{\phi}_i$ and z_1 separately. Hence, we start with the equations for $\tilde{\phi}_i$ and consider the simpler case for z_1 later. As in the case of the Markov chain, we normalize $\tilde{\phi}_i$ suitably by defining

$$\phi_i(m\delta,s\epsilon)=(\sqrt{2})^s\tilde{\phi}_i(m\delta,s\epsilon). \tag{24}$$

In matrix notation, the equations for ϕ_i in Eq. (23) may be expressed as

$$E_t\Phi(m\delta,s\epsilon)=\mathcal{E}_\delta\Phi(m\delta,s\epsilon), \tag{25}$$

where $\Phi(m\delta,s\epsilon)=[\phi_1(m\delta,s\epsilon),\phi_2(m\delta,s\epsilon)]^T$ and

$$\mathcal{E}_\delta=\frac{1}{\sqrt{2}}\begin{bmatrix}E_x^{-1}&-E_x\\E_x^{-1}&E_x\end{bmatrix}. \tag{26}$$

In the following, we consider the equations

$$E_t^k\Phi(m\delta,s\epsilon)=\mathcal{E}_\delta^k\Phi(m\delta,s\epsilon), \tag{27}$$

which are derived from Eq. (25) by applying the operator E_t to Eq. (25) k times and using Eq. (25) repeatedly.

Now Eq. (25) is an exact difference equation describing the expected spin over our ensemble of symmetric random walks. It may be solved for all finite positive values of ϵ and δ. However, if we want to approximate these solutions for small δ by the solutions of a partial differential equation, we first have to determine if the *vector* $\Phi(m\delta,s\epsilon)$ is rotated through large angles at each time step. If we assume that $p_i(m\delta\pm\delta,s\epsilon)=p_i(m\delta,s\epsilon)+O(\delta)$ for small δ, then

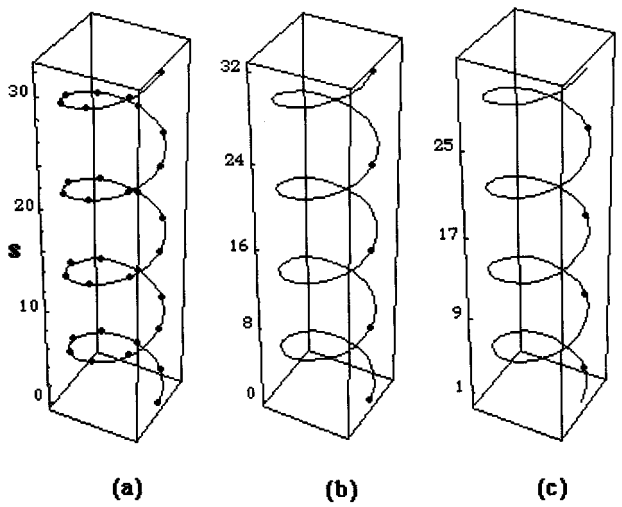

FIG. 6. The vector $\Phi=[\phi_1,\phi_2]^T$ versus discrete time is shown in (a). Note that the vector rotates through an angle $\pi/4$ in a single time step. The vectors Γ [see (b)] and $\Gamma^{[1]}$ [see (c)], which take eight steps, avoid the discontinuity between steps.

$$\mathcal{E}_\delta\Phi(m\delta,s\epsilon)=V\Phi(m\delta,s\epsilon)+O(\delta), \tag{28}$$

where V is the matrix (12). Thus $\mathcal{E}_\delta$, as an operator, rotates Φ through an angle of approximately $\pi/4$ (see Fig. 5 and the last paragraph of Sec. II). However, notice that $\mathcal{E}_\delta^8\Phi(m\delta,s\epsilon)=\Phi(m\delta,s\epsilon)+O(\delta)$ and hence

$$\begin{aligned}\Phi(m\delta,(s+8)\epsilon)&=E_t^8\Phi(m\delta,s\epsilon)=\mathcal{E}_\delta^8\Phi(m\delta,(s+8)\epsilon)\\&=\Phi(m\delta,s\epsilon)+O(\delta).\end{aligned} \tag{29}$$

This shows that the vector $\Phi(m\delta,s\epsilon)$ *is* changed by a small amount every eight steps of the discrete process, even though it is rotated by about $\pi/4$ at each single step.

We will approximate the function $\Phi(m\delta,(k+8l)\epsilon)$ $(l=0,1,\ldots)$ by eight different functions corresponding to $k=0,1,\ldots,7$. For clarity, we define the eight functions $\Gamma^{[k]}$ $(k=0,1,\ldots,7)$ as Φ where

$$\Gamma^{[k]}(m\delta,s\epsilon)=\Phi(m\delta,s\epsilon),\quad s=k+8l\quad(l=0,1,\ldots). \tag{30}$$

The domain of $\Gamma^{[k]}$ involves the nodes on the lattice at times where $s=k$, $s=k+8$, etc. Thus, from the details in the last paragraph, $\Gamma^{[k]}$ changes by $O(\delta)$ when we move by eight steps on the lattice. For simplicity, we define $\Gamma^{[0]}=\Gamma$ and relate $\Gamma^{[k]}$ to Γ by

$$\Gamma^{[k]}=E_t^k\Gamma^{[0]}=\mathcal{E}_\delta^k\Gamma^{[0]}=\mathcal{E}_\delta^k\Gamma. \tag{31}$$

The relationship of Γ and $\Gamma^{[1]}$ to Φ is represented schematically in Fig. 6.

We are interested in approximating the solution of the equation

$$E_t^8\Gamma(m\delta,s\epsilon)=\mathcal{E}_\delta^8\Gamma(m\delta,s\epsilon),\quad s=0\mod 8 \tag{32}$$

for small δ by a solution of a partial differential equation. We set up the limit process as follows. Suppose we are interested in (x,t) in a neighborhood of a point (X,T) and we choose a nearby point $(M\delta,S\epsilon)$ on the lattice as follows. Given δ and ϵ, we select (m,s) as (M,S) such that

$$M\delta\leqslant X<(M+1)\delta,\quad S\epsilon\leqslant T<(S+8)\epsilon,\quad S=0\mod 8. \tag{33}$$

At this point, we extend the domain of definition of Γ to all (x,t) and assume that this may be done in such a way that $\Gamma(x,t)=[\gamma_1(x,t),\gamma_2(x,t)]^T$ is a continuously differentiable function. We now proceed to determine the equation of this function.

We start with $E_x\gamma_i(M\delta,S\epsilon)=\gamma_i(M\delta+\delta,S\epsilon)$ in Eq. (32) and expand $\gamma_i(M\delta+\delta,S\epsilon)$ in a power series in δ to obtain

$$E_x=1+L+\tfrac{1}{2}L^2+O(\delta^3), \tag{34}$$

where $L=\delta(\partial/\partial x)$. Henceforth, $L\gamma_i(M\delta,S\epsilon)$ denotes $L\gamma_i(x,t)$ evaluated at $(M\delta,S\epsilon)$. Then

$$\begin{aligned}\mathcal{E}_\delta&=\frac{1}{\sqrt{2}}\begin{bmatrix}1-L+\dfrac{L^2}{2} & -1-L-\dfrac{L^2}{2}\\ 1-L+\dfrac{L^2}{2} & 1+L+\dfrac{L^2}{2}\end{bmatrix}+O(\delta^3)=V+BL\\&\quad+\frac{1}{2}VL^2+O(\delta^3),\end{aligned} \tag{35}$$

where V is given by Eq. (12) and

$$B=-\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}.$$

Thus,

$$\begin{aligned}\mathcal{E}_\delta^2&=V^2+(VB+BV)L+(V^2+B^2)L^2+O(\delta^3)=C+\sqrt{2}BL\\&\quad+\sqrt{2}VL^2+O(\delta^3),\end{aligned} \tag{36}$$

where

$$C=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

The coefficients follow readily from the definition of A and B. Similarly,

$$\mathcal{E}_\delta^4=-I+2CL^2+O(\delta^3), \tag{37}$$

where I is the 2×2 identity. Finally,

$$\mathcal{E}_\delta^8=I-4CL^2+O(\delta^3). \tag{38}$$

Thus, the right-hand side of Eq. (32) becomes

$$\mathcal{E}_\delta^8\Gamma(M\delta,S\epsilon)=\left(I-4\delta^2\,C\,\frac{\partial^2}{\partial x^2}+O(\delta^3)\right)\Gamma(M\delta,S\epsilon). \tag{39}$$

Upon expanding the left-hand side of Eq. (32), we have

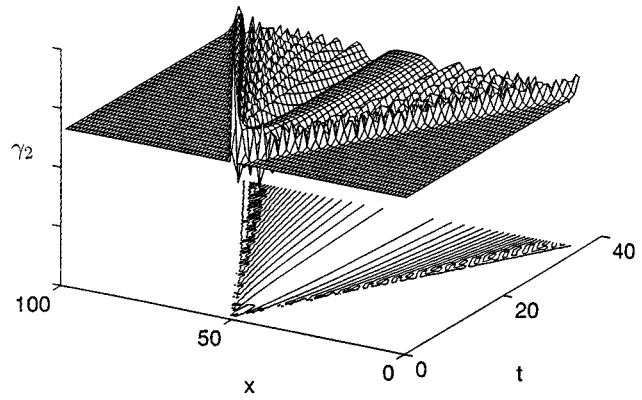

FIG. 7. γ_2 on the lattice along with the projection of the contours. The light cone in the figure is a result of the finite hopping speed on the lattice.

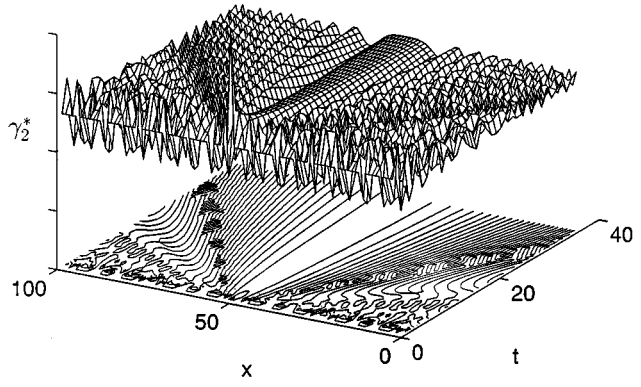

FIG. 8. The real part of the Feynman propagator (γ_2^* for a point source) evaluated on the lattice, as well as the projection of the contours. The ''effective'' light cone is an artifact of evaluating the exact propagator on a lattice with lattice spacing larger than the wavelength of the propagator.

$$E_t^8\Gamma(M\delta,S\epsilon)=\Gamma(M\delta,S\epsilon+8\epsilon)=\left(1+8\epsilon\frac{\partial}{\partial t}+O(\epsilon^2)\right)\Gamma(M\delta,S\epsilon). \tag{40}$$

Equating Eqs. (39) and (40), we have

$$\frac{\partial}{\partial t}\Gamma(M\delta,S\epsilon)=-\frac{\delta^2}{2\epsilon}C\frac{\partial^2}{\partial x^2}\Gamma(M\delta,S\epsilon)+O(\delta). \tag{41}$$

In Eq. (41), we have used the fact that diffusive scaling requires that δ and ϵ are related by Eq. (16). Hence, for small δ, $\Gamma(M\delta,S\epsilon)=\Gamma^*(M\delta,S\epsilon)+O(\delta)$, where $\Gamma^*(x,t)=[\gamma_1^*(x,t),\gamma_2^*(x,t)]^T$ satisfies

$$\frac{\partial}{\partial t}\gamma_1^*(x,t)=D\frac{\partial^2}{\partial x^2}\gamma_2^*(x,t),$$

$$\frac{\partial}{\partial t}\gamma_2^*(x,t)=-D\frac{\partial^2}{\partial x^2}\gamma_1^*(x,t). \tag{42}$$

Note that the γ_i, which solves the original difference equations on the lattice, is approximated by continuously differentiable functions γ_i^* that are solutions of the partial differential equations. Finally, the remaining functions $\Gamma^{[k]}$ are expressed as

$$\Gamma^{[k]}(M\delta,S\epsilon)=V^k\Gamma^*(M\delta,S\epsilon)+O(\delta). \tag{43}$$

To see the significance of $\gamma_i^*(x,t)$ in Eq. (42), write

$$\psi_+(x,t)=\gamma_2^*(x,t)+i\gamma_1^*(x,t),$$

$$\psi_-(x,t)=\gamma_2^*(x,t)-i\gamma_1^*(x,t). \tag{44}$$

Substituting Eq. (44) into Eq. (42) gives

$$-i\frac{\partial}{\partial t}\psi_+(x,t)=D\frac{\partial^2}{\partial x^2}\psi_+(x,t),$$

$$i\frac{\partial}{\partial t}\psi_-(x,t)=D\frac{\partial^2}{\partial x^2}\psi_-(x,t). \tag{45}$$

Both of these equations are of the Schrödinger form, and it is interesting that the derivation produces both the Schrödinger equation and its complex conjugate simultaneously. The importance of the complex conjugate equation has been stressed by El Naschie [8].

Finally, as stated earlier, the analysis of the equation for z_1 in Eq. (23) is much simpler than the equations for ϕ_i. In this case it is unnecessary to define the limit process as in (33). Instead, we define S as $S\epsilon\leqslant T<(S+1)\epsilon$ and define M as in (33). Assuming there exists a continuously differentiable function $z_1(x,t)$, the expansion of E_x is the same as Eq. (34). From Eq. (23),

$$\left(1+\epsilon\frac{\partial}{\partial t}+O(\epsilon^2)\right)z_1(M\delta,S\epsilon)=\left(1+\delta^2\frac{1}{2}\frac{\partial^2}{\partial x^2}+O(\delta^3)\right)z_1(M\delta,S\epsilon). \tag{46}$$

Thus, we require $\epsilon=O(\delta^2)$ so that we have

$$\frac{\partial}{\partial t}z_1(M\delta,S\epsilon)=\frac{\delta^2}{2\epsilon}\frac{\partial^2}{\partial x^2}z_1(M\delta,S\epsilon)+O(\delta). \tag{47}$$

For small δ, $z_1(M\delta,S\epsilon)=z_1^*(M\delta,S\epsilon)+O(\delta)$, where $z_1^*(x,t)$ is a solution of the diffusion equation

$$\frac{\partial}{\partial t}z_1^*(x,t)=D\frac{\partial^2}{\partial x^2}z_1^*(x,t). \tag{48}$$

IV. CONCLUSIONS

The pair of equations (42) which are equivalent to the two Schrödinger equations (45) were obtained from the binary random-walk model without a formal analytic continuation. The γ_2 and γ_1 that correspond to the real and imaginary part of ψ_+ are observable as expectations on the lattice. In this classical context, the ''wave functions'' that satisfy Schrödinger's equation have unambiguous interpretations in terms of ensembles of random walks. In Fig. 7, γ_2 is calculated for a point source on a lattice. The light cone is a result of the finite speed of the particles on the lattice, and the lack of

smoothness in the wave function near the light cone is due to the relatively few paths nearby. As the lattice is refined, the cone flattens (approaching the half plane as $\epsilon \to 0$) and the region of agreement with the exact wave function (Fig. 8) increases. For comparison, in Fig. 8 the real part of the solution of Schrödinger's equation is plotted on the lattice. The noisy values near the ''light cone'' are a result of the fact that the wavelength of the solution is smaller than the lattice spacing there.

In future work, we will show how this result may be generalized to include higher dimensions and smooth bounded potentials.

ACKNOWLEDGMENT

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