Introducing nonlinear gauge transformations in a family of nonlinear Schrödinger equations

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In earlier work we proposed a family of nonlinear time-evolution equations for quantum mechanics associated with certain unitary group representations [Doebner and Goldin, Phys. Lett. A **162**, 397 (1992); J. Phys. A **27**, 1771 (1994)]. Such nonlinear Schrödinger equations are expected to describe irreversible and dissipative quantum systems. Here we introduce and justify physically the group of nonlinear gauge transformations necessary to interpret our equations. We determine the parameters that are actually gauge invariant and describe some of their properties. Our conclusions contradict, at least in part, the view that any nonlinearity in quantum mechanics leads to unphysical predictions. We also show how time-dependent nonlinear gauge transformations connect our equations to those proposed by Kostin [J. Chem. Phys. **57**, 3589 (1972)] and by Bialynicki-Birula and Mycielski [Ann. Phys. **100**, 62 (1976)]. We believe our approach to be a fundamental generalization of the usual notions about gauge transformations in quantum mechanics. [S1050-2947(96)06710-8]

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I. INTRODUCTION AND BACKGROUND

Infinite-dimensional local current algebras and groups play a fundamental role in quantum mechanics [1–7]. In trying to understand and interpret certain unitary representations of such groups, we obtained in our earlier work a family of nonlinear Schrödinger equations different from those commonly studied [8,9]. In the present article we consider a group of nonlinear transformations on the Hilbert space $\mathcal{H}$ that can linearize a subset of these equations. We observe that the significance of these transformations goes considerably beyond providing a technique for obtaining solutions. We argue for their interpretation as *nonlinear gauge transformations* or *gauge transformations of the third kind,* in that distinct nonlinear Schrödinger equations of our type, related by such transformations, are properly understood as forming equivalence classes within which the time evolutions describe the same physics. Our view is that this provides a fundamental generalization of the usual notions about (linear) gauge transformations. Of course, the theory is of interest due to the existence of gauge classes that are *inequivalent* to ordinary quantum mechanics.

One consequence of our perspective is that a whole family of nonlinear Schrödinger equations are in fact physically equivalent to the linear theory. This immediately contradicts the claim that *any* nonlinearity in the time evolution of pure states in quantum mechanics leads to some unphysical predictions [10–13] and demonstrates that a more sensitive analysis is necessary. Another implication is that the coefficients in our nonlinear Schrödinger equation are not susceptible to direct interpretation, as they are not invariant under nonlinear gauge transformations. Rather, gauge-invariant combinations must be found and their physical meaning determined. We obtain a set of gauge invariants and describe some of their properties. We also consider the case of time-dependent nonlinear gauge transformations and show how these connect our equations with those proposed by Kostin [14] and by Bialynicki-Birula and Mycielski [15]. Our analysis implies that nonlinear gauge transformations must be explicitly considered in interpreting the results of experiments investigating possible deviations from linearity in quantum mechanics [16–21].

We provide here the necessary background. Let $\psi(\mathbf{x},t)$ be a time-dependent quantum-mechanical wave function for a particle of mass m, with probability density $\rho(\mathbf{x},t)$ and probability flux density $\mathbf{j}(\mathbf{x},t)$ defined as usual from ψ by the formulas

$$\rho=\overline{\psi}\psi, \quad \mathbf{j}=\frac{\hbar}{2mi}[\overline{\psi}\nabla\psi-(\nabla\overline{\psi})\psi]. \tag{1.1}$$

Our previously derived equations may be written

$$i\hbar\frac{\partial\psi}{\partial t}=H_0\psi+iI[\psi]\psi+R[\psi]\psi, \tag{1.2}$$

where H_0 is the usual, linear Hamiltonian operator with potential energy $V(\mathbf{x},t)$, given (in the absence of any external magnetic field) by

$$H_0\psi=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi, \tag{1.3}$$

and $I[\psi],R[\psi]$ are real-valued nonlinear functionals. The latter are given by

$$R[\psi]=\hbar D'\sum_{j=1}^{5}c_jR_j[\psi], \quad I[\psi]=\frac{1}{2}\hbar DR_2[\psi], \tag{1.4}$$

where

$$R_1=\frac{\nabla\cdot\hat{\mathbf{j}}}{\rho},\quad R_2=\frac{\nabla^2\rho}{\rho},\quad R_3=\frac{\hat{\mathbf{j}}^2}{\rho^2},\quad R_4=\frac{\hat{\mathbf{j}}\cdot\nabla\rho}{\rho^2},$$

$$R_5=\frac{(\nabla\rho)^2}{\rho^2}, \tag{1.5}$$

with

$$\hat{\mathbf{j}}=\frac{m}{\hbar}\mathbf{j}=\frac{1}{2i}[\overline{\psi}\nabla\psi-(\nabla\overline{\psi})\psi]. \tag{1.6}$$

In (1.4), D and D' are real numbers with the dimensions of diffusion coefficients.

Equations (1.2)–(1.6) were first derived from consideration of the representations of an infinite-dimensional group and the corresponding Lie algebra of local currents. Certain unitary group representations led us to replace the usual continuity equation for ρ and $\mathbf{j}$ by a Fokker-Planck type of equation

$$\frac{\partial\rho}{\partial t}=-\nabla\cdot\mathbf{j}+D\nabla^2\rho \tag{1.7}$$

and to accompany this by appropriate conditions on the corresponding Schrödinger equation. The parameter D in (1.7) then characterizes the original group representation up to unitary equivalence, via the dimensionless quantity $\Gamma=Dm/\hbar$. The other diffusion coefficient D' in (1.4) permits the real coefficients c_j to be dimensionless parameters. Thus far the products $D'c_j$ take arbitrary values.

Some related nonlinear Schrödinger equations have been considered by others, though without the group-theoretical motivation. A nonlinear Schrödinger equation proposed by Guerra and Pusterla [22] falls within our class of equations, but always with $I[\psi]=0$. The interpretation of this equation and its relation with ours is clarified by the analysis in the present paper; see also papers by Smolin, Kaloyerou, and Vigier [23–25]. Equation (1.7) was applied by Schuch, Chung, and Hartmann to the quantum-mechanical probability density and current [26–28], but in place of the nonlinearity in (1.4) they introduced a logarithmic nonlinear term that is actually independent of (1.7). Equations of Stenflo, Yu, and Shukla governing surface waves at a boundary between (non-quantum-mechanical) plasmas [29] were generalized by Malomed and Stenflo to a family that intersects the above for special values of the coefficients in one space dimension [30]. Another special equation of this sort was considered but rejected by Kibble [31], and a related class of equations is discussed by Auberson and Sabatier [32,33]. Still more general nonlinear quantum time-evolution equations, motivated by those discussed here, were developed by Dodonov and Mizrahi [34–36]. Comments on the relationship with the rather different equations of Kostin and of Bialynicki-Birula and Mycielski are given by Goldin and Svetlichny [37]. This relationship turns out to be a deep one, as shown below. A discussion of nonlinearity in the Schrödinger equation and two-level atoms is given by Czachor [38].

To understand the physics of quantum mechanics with the above nonlinear time-development, we must determine the physical content of the set of coefficients in (1.4). We summarize some of the already-established properties of these equations, which will be useful in the later development. The total probability $\int\rho(\mathbf{x},t)d\mathbf{x}$ is conserved by the time evolution for all values of the coefficients. Furthermore, the equations are *strictly homogeneous*, i.e., if ψ is a solution, then $\alpha\psi$ is a solution for any complex number α. This means one can construct a hierarchy of N-particle equations, satisfying the separation property, i.e., for which initially uncorrelated, noninteracting subsystems remain uncorrelated [15,37]. With $V\equiv 0$, plane waves are always solutions [39] and we have Euclidean and time-translation invariance. When V is a stationary potential that in ordinary quantum mechanics would accommodate bound states, there exist corresponding stationary-state solutions to (1.2). These typically go over smoothly to the stationary solutions of the linear Schrödinger equation, as we let $D,D'\to 0$.

Through (1.7), the sign of D appears to introduce an *arrow of time* into the quantum mechanics in a fundamental way. This is one of the reasons for interest in our model. However, the discussion in Secs. II and III below modifies this direct interpretation importantly.

When one calculates the time-rate of change of the expectation values $\langle -i\hbar\nabla\rangle=\int\overline{\psi}(\mathbf{x},t)[-i\hbar\nabla\psi(\mathbf{x},t)]d\mathbf{x}$ and $\langle i\hbar\partial_t\rangle=\int\overline{\psi}(\mathbf{x},t)[i\hbar\partial_t\psi(\mathbf{x},t)]d\mathbf{x}$, one obtains extra terms that are *dissipative.* There is a subfamily of our equations, characterized by the conditions $D'c_1=D=-D'c_4$ and $c_2+2c_5=c_3=0$, for which the dissipative terms in $(d/dt)\langle -i\hbar\nabla\rangle$ are zero, i.e., for which Ehrenfest's theorem formally holds. Equations in this subfamily are linearizable by means of nonlinear transformations [33,40,41]. A larger subfamily of the equations are Galileian invariant; necessary and sufficient conditions for Galileian invariance of the equations are $c_1+c_4=c_3=0$, so that the family obeying Ehrenfest's theorem is Galileian invariant, as are other subfamilies. In addition to the above results, we mention that explicit time-dependent solutions to (1.2) have been obtained by several authors [42–45] and that symmetries of the equations have also been studied in some detail [46–49].

In Sec. II we introduce a two-parameter group of nonlinear transformations, and justify its interpretation as a nonlinear gauge group. Then we rewrite our nonlinear Schrödinger equation in a more convenient general form and display the group action on the space of coefficients. In Sec. III we write a set of mutually independent, gauge-invariant quantities in terms of the original coefficients and discuss further the physical interpretation of our equations in terms of the gauge invariants. Explicitly time-dependent nonlinear gauge transformations are treated in Sec. IV. These require the widening of our class of equations to include terms of the type introduced by Kostin and by Bialynicki-Birula and Mycielski. We summarize our conclusions in Sec. V.

II. THE NONLINEAR GAUGE GROUP AND ITS ACTION

A. Rationale for nonlinear gauge transformations

Gauge transformations of both the first and second kind in quantum mechanics are implemented by *unitary* (linear) operators in $\mathcal{H}$; i.e., they preserve inner products between wave functions. Gauge transformations of the first kind correspond to the unitary group U(1), and leave the form of both position and momentum operators unchanged. Gauge transformations of the second kind correspond to the group of U(1)-valued functions of $\mathbf{x}$ and t. They leave the form of all functions of the position operator invariant, while the form of operators depending on the momentum operator is changed.

Now it has been remarked by various authors that *all* actual quantum-mechanical measurements consist of or are obtained from positional measurements performed at various times, for example, the Feynman–Hibbs statement (see [50], p. 96):

> Indeed all measurements of quantum-mechanical systems could be made to reduce eventually to position and time measurements. Because of this possibility a theory formulated in terms of position measurements is complete enough in principle to describe all phenomena.

Adopting this point of view, quantum theories (whether linear or nonlinear) for which corresponding time-dependent wave functions give the same probability density in space at all times, are ''in principle'' (as well as in practice) equivalent. Two different time evolution equations thus related will not predict different physical effects. This is also consistent with the analysis of Mielnik [51], who considers quantum logic with an eye to possible nonlinear time evolutions. Further discussion of related issues is provided by Lücke [52].

The preceding statement about the physical equivalence of different time-evolution equations does *not* simply mean that one theory has been obtained from the other through a mathematical change of coordinates in the Hilbert space. It embodies the *physical assumption* stated by Feynmann and Hibbs about the special nature of positional measurements.

To say that all measurements are fundamentally positional means that when we ''measure'' momentum, energy, angular momentum, etc., in quantum mechanics, we are really making *inferences* (i.e., performing calculations) from the outcomes of positional measurements at various different times. But the *process of inferring* momentum, energy, angular momentum, etc., from positional outcomes (i.e., the actual choice of calculation to perform) depends on what is *assumed* about the time evolution.

For instance, consider the idea that the linear operator $-i\hbar\boldsymbol{\nabla}$ describes the momentum of a particle. This assertion means that the *distribution* of the outcomes of a large number of repeated momentum measurements on identically prepared pure states ψ, outcomes that are *necessarily obtained by making positional measurements at different times on the identically prepared states,* can be predicted from the operator $-i\hbar\boldsymbol{\nabla}$ according to the usual mathematical rules of quantum mechanics. But the way that we infer momentum from the positional outcomes is based on the *assumption* that the time evolution is given by the usual linear Schrödinger equation, which has the property (for a noninteracting particle) that the Fourier transform of the wave function time evolves so as to conserve any particular value of momentum (i.e., so as to preserve the probability density in momentum space). If the time evolution is assumed to be given by a different equation, e.g., a nonlinear one, this property may no longer hold. The inferred ''momentum'' outcomes will then be different for the same positional measurements and the operator $-i\hbar\boldsymbol{\nabla}$ will no longer predict the momentum distribution.

Thus we consider, in general, transformations in $\mathcal{H}$ that leave the positional probability density invariant. Such transformations change the time-evolution equation, but do not change the physical content of a theory. If they are linear they must be implemented by unitary operators and we have gauge transformations of the first or second kind. A well-known additional possibility is that of antilinearity, so that the transformations are implemented by antiunitary operators. This enlarges the group by admitting the operation of complex conjugation of wave functions, which yields the time-reversed Schrödinger equation. In this paper we discuss a particular, two-parameter group of *nonlinear* transformations acting on Eqs. (1.2). In addition, we investigate the case where the two parameters depend explicitly on time.

B. A two-parameter group of nonlinear gauge transformations

To motivate the group of transformations introduced here, we consider linearizing transformations for the subfamily of our equations obeying Ehrenfest's theorem [40,41]. This subfamily is characterized by the conditions

$$D'c_1 = D = -D'c_4, \quad D'c_2 + 2D'c_5 = D'c_3 = 0\,. \tag{2.1}$$

Consider the nonlinear transformation given by

$$\psi \mapsto \psi' = N(\psi) = |\psi| \exp[i(\gamma \ln|\psi| + \Lambda \arg\psi)], \tag{2.2}$$

where γ and Λ are real numbers $(\Lambda \neq 0)$. If ψ solves an equation in the Ehrenfest class (1.2)–(1.5) under the constraints (2.1), then ψ' solves the linear Schrödinger equation

$$i\frac{\hbar}{\Lambda_E}\frac{\partial \psi'}{\partial t} = -\frac{\hbar^2}{2\Lambda_E^2 m}\nabla^2\psi' + V(\mathbf{x},t)\psi', \tag{2.3}$$

when

$$\gamma = \gamma_E = -\frac{2mD}{\hbar}\left(1 - \frac{4m}{\hbar}D'c_2 - \frac{4m^2D^2}{\hbar^2}\right)^{-1/2},$$

$$\Lambda = \Lambda_E = \left(1 - \frac{4m}{\hbar}D'c_2 - \frac{4m^2D^2}{\hbar^2}\right)^{-1/2}, \tag{2.4}$$

on condition that

$$\frac{4m}{\hbar}D'c_2 < 1 - \frac{4m^2D^2}{\hbar^2}. \tag{2.5}$$

We remark that the transformation N is not actually well defined by (2.2) as a mapping having a domain whose elements are wave functions (since the argument of ψ is only defined *modulo* integer multiples of 2π). This is not a serious difficulty; given ψ obeying one of our nonlinear Schrödinger equations, it is sufficient for our purposes that ψ' exists obeying the transformed equation. An appropriate selection of ψ' can always be made, and we thus have a concrete example of a nonlinear gauge transformation.

Now two transformations N having different arbitrary values of γ and Λ, with $\Lambda \neq 0$, can be performed successively to yield a third, i.e.,

$$N_{(\gamma_1,\Lambda_1)}\circ N_{(\gamma_2,\Lambda_2)}=N_{(\gamma_1+\Lambda_1\gamma_2,\Lambda_1\Lambda_2)}, \tag{2.6}$$

so that we have the group law of the affine group in one dimension. Let us call this group $\mathcal{N}$ (for ''nonlinear''). Any transformation $N_{(\gamma,\Lambda)}\in\mathcal{N}$ is *local* in the sense that it depends only on the value of ψ at a point, not on its derivatives or its values at other points. Transformations in $\mathcal{N}$ also respect *rays* in the Hilbert space; i.e., for any complex number α, the vector $(\alpha\psi)'$ belongs to the same ray as does ψ'. Furthermore, the transformations $N_{(\gamma,\Lambda)}$ leave the probability density ρ invariant for all $\mathbf{x}$ and t. In accordance with our earlier discussion, the quantum theory in which wave functions ψ obey a particular nonlinear time-evolution equation is physically equivalent to the corresponding theory, transformed by $N_{(\gamma,\Lambda)}$, in which wave functions ψ' obey a transformed equation. Distinct nonlinear Schrödinger equations related by elements of $\mathcal{N}$ should then be regarded as belonging to equivalence classes predicting exactly the same physics. In particular, this means that the parameters D and $D'c_j$ in (1.4) are not themselves susceptible to direct physical interpretation, as the coefficients that depend immediately on them are not invariant under $\mathcal{N}$.

C. Action of the nonlinear gauge group on nonlinear Schrödinger equations

To determine just how N transforms a general nonlinear Schrödinger equation in the category under consideration, we now rewrite the entire right-hand side of (1.2), including H_0, as a nonlinear function of the density ρ and the current $\hat{\mathbf{j}}$, multiplied by ψ. From (1.5) we have

$$\frac{\nabla^2\psi}{\psi}=iR_1[\psi]+\frac{1}{2}R_2[\psi]-R_3[\psi]-\frac{1}{4}R_5[\psi], \tag{2.7}$$

giving us a general form for the nonlinear Schrödinger equation that includes the linear case

$$i\frac{\partial\psi}{\partial t}=i\left(\sum_{j=1}^{2}\nu_jR_j[\psi]\right)\psi+\left(\sum_{j=1}^{5}\mu_jR_j[\psi]\right)\psi+U(\mathbf{x},t)\psi, \tag{2.8}$$

where the ν_j $(j=1,2)$ and μ_j $(j=1,\ldots,5)$ are real coefficients. The relationship of (2.8) to (1.2)–(1.5) is given by

$$\nu_1=-\frac{\hbar}{2m},\quad \nu_2=\frac{1}{2}D,$$

$$\mu_1=D'c_1,\quad \mu_2=-\frac{\hbar}{4m}+D'c_2,\quad \mu_3=\frac{\hbar}{2m}+D'c_3,$$

$$\mu_4=D'c_4,\quad \mu_5=\frac{\hbar}{8m}+D'c_5,$$

$$U(\mathbf{x},t)=\frac{1}{\hbar}V(\mathbf{x},t). \tag{2.9}$$

From here on we shall think of ν_1 not as having the fixed value $-\hbar/2m$ given by (2.9), but simply as one of our real parameters subject to variation like the others. This widens the class of equations and is essential to understanding the physical meaning of the set of coefficients.

With $\psi'=N_{(\gamma,\Lambda)}(\psi)$, we have the transformation of ρ and $\hat{\mathbf{j}}$ given by

$$\rho'=\overline{\psi'}\psi'=\rho,$$

$$\hat{\mathbf{j}}'=\frac{1}{2i}[\overline{\psi'}\nabla\psi'-(\nabla\overline{\psi'})\psi']=\Lambda\hat{\mathbf{j}}+\frac{\gamma}{2}\nabla\rho. \tag{2.10}$$

Defining $R_j'[\psi]=R_j[N_{(\gamma,\Lambda)}(\psi)]=R_j[\psi']$, we have

$$R_1'=\Lambda R_1+\tfrac{1}{2}\gamma R_2,\quad R_2'=R_2,$$

$$R_3'=\Lambda^2R_3+\Lambda\gamma R_4+\tfrac{1}{4}\gamma^2R_5,$$

$$R_4'=\Lambda R_4+\tfrac{1}{2}\gamma R_5,\quad R_5'=R_5, \tag{2.11}$$

and writing the inverse transformation $R_j^{\sim}[\psi]=R_j[N^{-1}(\psi)]$, we have

$$R_1^{\sim}=\Lambda^{-1}(R_1-\tfrac{1}{2}\gamma R_2),\quad R_2^{\sim}=R_2,$$

$$R_3^{\sim}=\Lambda^{-2}(R_3-\gamma R_4+\tfrac{1}{4}\gamma^2R_5),$$

$$R_4^{\sim}=\Lambda^{-1}(R_4-\tfrac{1}{2}\gamma R_5),\quad R_5^{\sim}=R_5. \tag{2.12}$$

Now if ψ obeys (2.8), then ψ' satisfies

$$i\frac{1}{\psi'}\frac{\partial\psi'}{\partial t}=i\sum_{j=1}^{2}\nu_jR_j^{\sim}[\psi']-\gamma\sum_{j=1}^{2}\nu_jR_j^{\sim}[\psi']+\Lambda\sum_{j=1}^{5}\mu_jR_j^{\sim}[\psi']+\Lambda U(\mathbf{x},t), \tag{2.13}$$

where $R_j^{\sim}[\psi']=R_j[\psi]$. Using (2.12) to substitute for the $R_j^{\sim}[\psi']$ in (2.13), we find that ψ' obeys an equation of the type (2.8), but with new coefficients and a scaled potential term. Denoting these transformed values with primes, we finally have [53]

$$\nu_1'=\frac{\nu_1}{\Lambda},\quad \nu_2'=-\frac{\gamma}{2\Lambda}\nu_1+\nu_2,$$

$$\mu_1'=-\frac{\gamma}{\Lambda}\nu_1+\mu_1,\quad \mu_2'=\frac{\gamma^2}{2\Lambda}\nu_1-\gamma\nu_2-\frac{\gamma}{2}\mu_1+\Lambda\mu_2,$$

$$\mu_3'=\frac{\mu_3}{\Lambda},\quad \mu_4'=-\frac{\gamma}{\Lambda}\mu_3+\mu_4,$$

$$\mu_5'=\frac{\gamma^2}{4\Lambda}\mu_3-\frac{\gamma}{2}\mu_4+\Lambda\mu_5,\quad U'=\Lambda U. \tag{2.14}$$

Thus the parametrized family of equations given by (2.8) is invariant, as a family, under the affine group of nonlinear transformations.

In accordance with our discussion, members of this family of nonlinear Schrödinger equations, when they are related by one of the transformations $N_{(\gamma,\Lambda)}$, describe the same physics; there is indeed no measurement or sequence of measurements that can distinguish them. Therefore we regard this group $\mathcal{N}$ as a nonlinear generalization of the linearly acting U(1) gauge group in the Hilbert space. It seems reasonable to call such nonlinear transformations ''gauge transformations of the third kind.'' In particular, we have a class of nonlinear Schrödinger equations gauge equivalent to the linear Schrödinger equation.

III. GAUGE-INVARIANT PARAMETERS AND PHYSICAL INTERPRETATION

Next we look again at the physical interpretation of (2.8). Since the two-dimensional gauge group acts on the seven-dimensional space of coefficients, we expect (in general) five independent, gauge invariant quantities labeling the classes of equations in the family. These gauge invariants are nonlinear combinations of the original coefficients. It is the gauge invariants, rather than the original coefficients, that must in principle be the measurable quantities characterizing the physics described by the nonlinear Schrödinger equations.

Functionally independent gauge invariants can be obtained by straightforward calculations. The following are one such set τ_j $(j=1,\ldots,5)$, as may be verified by direct substitution using (2.14):

$$\tau_1=\nu_2-\frac{1}{2}\mu_1,\quad \tau_2=\nu_1\mu_2-\nu_2\mu_1,\quad \tau_3=\frac{\mu_3}{\nu_1},$$

$$\tau_4=\mu_4-\mu_1\frac{\mu_3}{\nu_1},$$

$$\tau_5=\nu_1\mu_5-\nu_2\mu_4+\nu_2^2\frac{\mu_3}{\nu_1}. \tag{3.1}$$

In addition, let $\hat{U}=-\nu_1 U$ to obtain a potential that is invariant under the group $\mathcal{N}$. Note that neither the diffusion coefficient D nor its dimensionless counterpart $\Gamma=Dm/\hbar$, whose introduction led originally to the above development, is actually physically measurable, in that the coefficient ν_2 is not invariant under gauge transformations. Note also that the coefficient ν_1 is not a gauge invariant, so that the ratio $\hbar/m$ also requires more careful interpretation. These conclusions force us to change the perspective we took in our earlier papers about the physical interpretation of our equation.

We now begin the process of understanding the quantities τ_j in terms of their physical effects. First consider the usual linear Schrödinger equation (with $D=D'=0$). We have, from (2.9), the values $\nu_1=-\hbar/2m$, $\nu_2=0$, $\mu_1=0$, $\mu_2=-\hbar/4m$, $\mu_3=\hbar/2m$, $\mu_4=0$, $\mu_5=\hbar/8m$, and $U=(1/\hbar)V$. The corresponding gauge invariants are

$$\tau_1=0,\quad \tau_2=\frac{\hbar^2}{8m^2},\quad \tau_3=-1,\quad \tau_4=0,\quad \tau_5=-\frac{\hbar^2}{16m^2},$$

$$\hat{U}=\frac{1}{2m}V. \tag{3.2}$$

We see that $\hbar/m$ may be found from either of the gauge invariants τ_2 or τ_5 (or a linear combination of them). We *must* relate the physically observed value of $\hbar/m$ to gauge-invariant quantities. Though ν_1 is changed by a nonlinear gauge transformation, τ_2 and τ_5 are not; thus the ratio $\hbar/m$ of physical constants is observable. But its observed value within the class of linearizable Schrödinger equations is not, in general, equal to the original value entering the coefficients. We can write either

$$\left[\frac{\hbar}{m}\right]_{\rm obs}=(8\tau_2)^{1/2} \tag{3.3}$$

or, alternatively,

$$\left[\frac{\hbar}{m}\right]_{\rm obs}=(-16\tau_5)^{1/2}, \tag{3.4}$$

where the subscript means ''observed.'' This is one important reason to distinguish the *gauge-dependent parameter* ν_1 from the observable values of physical constants. The statement that $\nu_1=-\hbar/2m$ is just a (partial) choice of gauge, which is natural to call a *Schrödinger gauge*. For the usual, linear Schrödinger equation, either (3.3) or (3.4) gives the same value of $[\hbar/m]_{\rm obs}$. Since τ_2 and τ_5 here are varying independently, the effective observables $(8\tau_2)^{1/2}$ and $(-16\tau_5)^{1/2}$ can have different values. We therefore anticipate *two* in general distinguishable, observable physical constants, defined by (3.3) and (3.4), respectively, having the same limiting values $[\hbar/m]_{\rm obs}$ in the special case of linearizable Schrödinger equations (see below). Note also that if $V(\mathbf{x},t)\equiv 0$, m and $\hbar$ cannot be obtained individually (as is, of course, directly evident from the Schrödinger equation).

Consider now the ''Ehrenfest family'' of equations, which are linearizable via $\mathcal{N}$ using (2.4) and (2.5). The conditions (2.1) characterizing them narrow the values of the gauge-invariant parameters as

$$\tau_1=0,\quad \tau_2=\frac{\hbar^2}{8m^2}-\kappa,\quad \tau_3=-1,\quad \tau_4=0,$$

$$\tau_5=-\frac{1}{2}\left(\frac{\hbar^2}{8m^2}-\kappa\right), \tag{3.5}$$

where the parameter κ is given by

$$\kappa=\frac{\hbar}{2m}D'c_2+\frac{1}{2}D^2. \tag{3.6}$$

The case $\kappa=0$ is the linear Schrödinger equation. As long as $\kappa<\hbar^2/8m^2$ so that τ_2 is positive, the equation is linearizable by means of a nonlinear gauge transformation. The condition $\tau_2>0$ is just (2.5). Equations in the Ehrenfest family are a one-parameter subset of gauge-inequivalent theories, but they differ from (3.2) and from each other only in the physically observable, *effective* value of $[\hbar/m]_{\rm obs}$ described by means of either (3.3) or (3.4). This result also motivates the possible replacement [41] of the gauge invariant parameter τ_5 with the parameter $\iota_5=(1/2)\tau_2+\tau_5$, so that the Ehrenfest family can be characterized by $\tau_1=0$, $\tau_2>0$, $\tau_3=-1$, $\tau_4=0$, and $\iota_5=0$. Letting $\iota_5\neq 0$ moves one outside the Ehrenfest family, but as we shall shortly see, maintains Galileian and time-reversal invariance.

Next consider the conditions on our family of equations that (in the absence of interactions) establish Galileian invariance: $c_1+c_4=c_3=0$. The coefficients, from (2.9), take the values $\nu_1=-\hbar/2m$, $\nu_2=D/2$, $\mu_1=D'c_1$, $\mu_2=-\hbar/4m+D'c_2$, $\mu_3=\hbar/2m$, $\mu_4=-D'c_1$, and $\mu_5=\hbar/8m+D'c_5$, where $\hbar/m$, D, $D'c_1$, $D'c_2$, and $D'c_5$ are all freely chosen. We obtain the following values for the gauge invariants:

$$\tau_1=\frac{1}{2}(D-D'c_1),\quad \tau_2=\frac{\hbar^2}{8m^2}-\frac{\hbar}{2m}D'c_2-\frac{1}{2}DD'c_1,$$

$$\tau_3=-1,\quad \tau_4=0,$$

$$\tau_5=-\frac{\hbar^2}{16m^2}-\frac{\hbar}{2m}D'c_5+\frac{1}{2}DD'c_1-\frac{1}{4}D^2. \tag{3.7}$$

Here the condition $c_3=0$ fixes $\tau_3=-1$, the condition $c_1+c_4=0$ fixes $\tau_4=0$, while τ_1, τ_2, and τ_5 (or ι_5) vary freely and independently. The gauge invariants τ_3 and τ_4 therefore describe two different possible sources of deviation from Galileian invariance, insofar as the former differs from -1 and/or the latter from 0.

Consider finally the effect of time reversal on the nonlinear Schrödinger equation. Sending $t\to -t$ is equivalent to setting new coefficients $\nu_j^{\mathsf{T}}=-\nu_j$ $(j=1,2)$ and $\mu_j^{\mathsf{T}}=-\mu_j$ $(j=1,\dots,5)$ and setting $U^{\mathsf{T}}=-U$, where the superscript T denotes time reversal. Of course, $N_{(\gamma,\Lambda)}(\psi)=\bar{\psi}$ when $\gamma=0$ and $\Lambda=-1$. From (2.14), we see that if ψ solves Eq. (2.8), $\bar{\psi}$ solves the time-reversed equation just as long as $\nu_2=\mu_1=\mu_4=0$. But in terms of the τ's, we have that $\tau_j^{\mathsf{T}}=-\tau_j$ for $j=1$ or $j=4$, while $\tau_j^{\mathsf{T}}=+\tau_j$ for $j=2$, 3, or 5 and $\hat{U}^{\mathsf{T}}=+\hat{U}$. Nonzero values for *either* of the two parameters τ_1 or τ_4 introduce an ''arrow of time'' into quantum mechanics through nonlinear time evolutions.

We see that the condition of Galileian invariance still permits some irreversibility. Galileian invariance sets $\tau_4=0$, but one can still have $\tau_1\neq 0$ in a Galileian invariant theory. Since ν_2 is proportional to D, we see that $D\neq 0$ contributes to a possibly nonzero value for τ_1. But $\mu_1=D'c_1$ also contributes to τ_1. Thus the condition that $D\neq 0$ is, by itself, neither necessary nor sufficient to break the time-reversal invariance. This is a direct consequence of the fact that both the magnitude and the sign of ν_2, can change under nonlinear gauge transformations.

The nonlinear Schrödinger equations proposed from basic principles by Guerra and Pusterla [22] are obtained from the linear Schrödinger equation by nonlinear gauge transformations in our class with $\gamma=0$. They pose the question as to whether observable consequences are associated with this class of equations. The above discussion indicates there cannot be.

The particular dissipative equation studied by Ushveridze [43,44] corresponds to the values $D=0$ and $D'c_1\neq 0$ (with the remaining $c_j=0$). This equation breaks both time-reversal and Galileian invariance, in that $\tau_1\neq 0$ and $\tau_4=-2\tau_1\neq 0$, while $\tau_2>0$, $\tau_3=-1$, and $\iota_5=0$. Likewise the dissipative equation we considered in [8] breaks both invariances, with $\tau_1=-\tau_4\neq 0$, and $\tau_2>0$, $\tau_3=-1$, and $\iota_5=0$.

As noted in [9], the ''simplest'' way to introduce a nonlinearity associated with $D\neq 0$ is to choose all the $D'c_j=0$. Then we have the values

$$\tau_1=\frac{1}{2}D,\quad \tau_2=\frac{\hbar^2}{8m^2},\quad \tau_3=-1,\quad \tau_4=0,$$

$$\tau_5=-\frac{\hbar^2}{16m^2}-\frac{1}{4}D^2. \tag{3.8}$$

We may regard this as a two-parameter class of gauge-inequivalent, Galileian-invariant but non-time-reversal invariant theories, with $\tau_1\neq 0$, $\tau_2>0$, $\tau_3=-1$, $\tau_4=0$, and $\iota_5=-\tau_1^2$.

IV. CONSEQUENCES OF TIME DEPENDENCE IN THE NONLINEAR GAUGE TRANSFORMATIONS

In this section we consider the consequences of letting γ and Λ depend explicitly on t. Writing $\gamma=\gamma(t)$ and $\Lambda=\Lambda(t)$, the transformation rules for the coefficients ν_j and μ_j in Eq. (2.8) are unchanged, but the coefficients are now also time dependent. In addition, the class of nonlinear Schrödinger equations is automatically extended to include two additional terms on the right-hand side of Eq. (2.8):

$$(\alpha_1\ln\rho)\psi+(\alpha_2\arg\psi)\psi, \tag{4.1}$$

where α_1 and α_2 are also real, time-dependent coefficients. The first term in (4.1) is just the nonlinear term proposed by Bialynicki-Birula and Mycielski, while the second term was proposed by Kostin. Although neither of these terms is strictly homogeneous, they are both consistent with the separation property for N-particle hierarchies of quantum-mechanical time evolutions [15,37]. This justifies our interpretation of (2.2) as a gauge transformation in the time-dependent case.

Again it cannot be the coefficients α_j that have direct physical meaning, as these change under gauge transformation. In fact, a straightforward calculation gives

$$\alpha_1'=\Lambda\alpha_1-\frac{\gamma}{2}\alpha_2+\frac{1}{2}\left(\gamma\frac{\dot{\Lambda}}{\Lambda}-\dot{\gamma}\right),$$

$$\alpha_2'=\alpha_2-\frac{\dot{\Lambda}}{\Lambda}, \tag{4.2}$$

where $\dot{\gamma}$ and $\dot{\Lambda}$ are time derivatives. It is apparent from (4.2) how the time dependence of γ and Λ yields non-zero values for α_1 and α_2, even if one begins with $\alpha_1=\alpha_2=0$. Equations (4.2) together with (2.14) describe the transformation rules for all the time-dependent coefficients.

The next step is to introduce gauge-invariant quantities associated with the α_j. A fairly simple choice is

$$\beta_1=\nu_1\alpha_1-\nu_2\alpha_2+\nu_2\frac{\dot{\nu}_1}{\nu_1}-\dot{\nu}_2,\quad \beta_2=\alpha_2-\frac{\dot{\nu}_1}{\nu_1}. \tag{4.3}$$

These are the quantities that, like the τ_j, are susceptible to physical interpretation. We note that β_2 changes sign under time reversal, which is consistent with Kostin's original interpretation of his equation as describing an external, velocity-dependent frictional force. On the other hand, β_1 is time-reversal invariant.

V. CONCLUSIONS AND FURTHER DISCUSSION

The results of this paper are all consistent with our original intuition that the general, nonlinear Schrödinger equation we proposed describes possible intrinsic, dissipative processes in quantum mechanics in a fundamental way. A group of nonlinear transformations is to be interpreted as a gauge group for the theory, with the gauge-invariant parameters $\tau_1, \ldots, \tau_5$ (or ι_5) characterizing the physical nature of the dissipation. Nonzero values for two of the τ parameters τ_1 and τ_4 break the time-reversal invariance. The parameters $\tau_3 \neq -1$ or $\tau_4 \neq 0$ break the Galileian invariance. For time-reversal invariant, Galileian-invariant theories, $\iota_5 \neq 0$ characterizes the deviation from linearizability. Allowing the gauge transformations to be time-dependent requires the addition of Bialynicki-Birula-Mycielski and Kostin terms, with corresponding additional gauge-invariant parameters β_1 and β_2. A nonzero value for β_2 also breaks time-reversal invariance.

The fact that a certain class of nonlinear time evolutions is physically equivalent to ordinary quantum mechanics also means that the usual arguments about nonlinearity in quantum mechanics having unphysical consequences do not apply, at least not with the generality that they are usually stated. The most compelling of these arguments is based on the idea that in systems with long-range correlations of the Einstein-Podolsky-Rosen type, nonlinearity in the time evolution would necessarily permit instantaneous (i.e., faster than light) communication. This is not exactly paradoxical in a nonrelativistic theory, but it does pose a problem of compatibility with special relativity. It has been noted that if one wishes to avoid this problem entirely, it is possible to maintain (1.7) by replacing (1.2)–(1.5) with a stochastic alternative [54]. But we have shown here, explicitly for the Ehrenfest class, that some nonlinear theories of the type considered do *not* permit arbitrarily fast communication when the more general notion of gauge invariance we have proposed is taken into account. We believe this notion of nonlinear gauge invariance to have other profound consequences for our understanding of quantum mechanics; this is a subject of our ongoing research.

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