BRIEF REPORTS

Brief Reports are accounts of completed research which do not warrant regular articles or the priority handling given to Rapid Communications; however, the same standards of scientific quality apply. (Addenda are included in Brief Reports.) A Brief Report may be no longer than four printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.

Absence of classical and quantum mixing

L. L. Salcedo
Departamento de Física Moderna, Universidad de Granada, E-18071 Granada, Spain
(Received 22 April 1996; revised manuscript received 14 June 1996)

It is shown, under mild assumptions, that classical degrees of freedom dynamically coupled to quantum ones do not inherit their quantum fluctuations. It is further shown that if the assumptions are strengthened by imposing the existence of a canonical structure, only purely classical or purely quantum dynamics is allowed. [S1050-2947(96)05210-9]

PACS number(s): 03.65.Bz, 03.50.−z, 03.65.Sq

I. INTRODUCTION

There is consensus among physicists that quantum mechanics is the correct description of nature, at least within the range of presently observable scales. Nevertheless, some systems are routinely described using a classical, and thus approximated, dynamics. This can be so either for simplicity or due to the lack of a consistent quantum theory. Einstein's theory of general relativity is an example of the latter. In an excellent speculative paper [1], Boucher and Traschen considered several physical systems that require a mixed description in terms of quantum and classical degrees of freedom, mutually interacting. A good example is provided by early universe physics, where fully quantum matter fields are coupled to classical gravitational fields. The traditional approach to this problem has been to couple the gravitational fields to the expectation values of the quantum energy-momentum tensor; see, e.g., Ref. [2]. This kind of approach has been criticized [1] on the grounds that the classical fields evolve deterministically, hence the quantum fluctuations in these fields, induced by their coupling to the quantum fields, are missed.

This criticism, as well as presumably the challenge it presents, has led to the search for a mathematically consistent description of semiquantized systems, i.e., mixed classical-quantum systems [3,1,4–6]. These systems are considered by themselves, that is, not as the limit of a fully quantum theory. The fact that the classical description is just an approximation is disregarded in this context since the purpose is to define a mathematical structure with some physical input. Let us remark that the use of approximated treatments mixing classical and quantum degrees of freedom of enormous success, particularly in molecular theory and optics, is not under debate here. Only the existence of mathematically exact and consistent semiquantization schemes is.

In this work it is shown that, in fact, there are severe obstructions to constructing such a description and, if it exists at all, it will not enjoy the elegant mathematical structures common to classical and quantum mechanics. A similar conclusion was reached by De Witt using uncertainty principle arguments [7]. Since presently there is no widely accepted definition of what is meant by a semiquantized system, and in order not to discard potentially interesting choices, we should rely on properties as general as possible, which must hold, in particular, for the purely classical and purely quantum cases.

II. ASSUMPTIONS

It is assumed (i) that the observables are (in an algebraic sense) constructed out of the coordinates q_i and the conjugate momenta p_i, $i=1,\ldots,N$, as well as the identity E. These generators satisfy some commutation relations to be specified. In classical mechanics we have the set of complex functions in phase space and E is the unity function. In quantum mechanics it is the algebra of operators in the Hilbert space of the system. Here the word ''observable'' is being used in a slightly wider sense than usual since it includes nonreal functions and non-Hermitian operators as well.

A second axiom (ii) refers to the time evolution of the observables (Heisenberg picture). We assume that the evolution is a bijection that preserves the algebraic structure, that is, if two observables $A(t_0)$ and $B(t_0)$ evolve to $A(t)$ and $B(t)$, respectively, and a,b are constant complex numbers, the observable $aA(t_0)+bB(t_0)$ evolves to $aA(t)+bB(t)$ and $A(t_0)B(t_0)$ evolves to $A(t)B(t)$. Certainly, this axiom holds both in classical and in quantum mechanics and it is hard to imagine an interesting formulation that would violate it. Note that we are referring only to the dynamic time-dependence of observables. On the other hand, it is not assumed that the system is conservative; there can be time dependent external fields that break invariance under time translations. Similarly, time-reversal invariance is not required.

Some relevant conclusions can be extracted from these two axioms. If a set of elements generates the algebra, this property is maintained through time evolution. Also, the observable E is time independent. Finally, commutation relations of the form $[A(t_0),B(t_0)]=cE$ are also preserved since they evolve to $[A(t),B(t)]=cE$. In particular, if two observables commute at any given time, they do so at any other time.

Another axiom (iii) is needed referring to the commutation relations since we have to specify in what sense our system is composed of a classical sector plus a quantum one. The classical dynamics is characterized by commuting coordinates and momenta that evolve according to Hamilton's equations. On the other hand, the quantum dynamics satisfies the canonical commutation relations and the Heisenberg evolution equation $dA/dt=-i[A,H(t)]$. For the semiquantum dynamics it is postulated that the classical commutation relations hold among the classical generators and similarly for the quantum sector. Furthermore, the generators of the classical sector commute with those of the quantum one. In other words, the commutation relations are

$$[q_i,q_j]=[p_i,p_j]=0\,, \quad [q_i,p_j]=i\lambda_i\delta_{ij}E, \quad i,j=1,\dots,N, \tag{1}$$

where λ_i is zero if i is the label of a classical degree of freedom and unity (or $\hbar$) if it labels a quantum one. These are the defining identities of the algebra of the semiquantized system.

This axiom can be justified as follows. Certainly, it is natural to demand Eqs. (1) if the semiquantized system consists of a classical sector and a quantum sector without any interaction among them. Since in both classical and quantum dynamics the commutation relations are unaffected by the choice of the interaction, one should expect that this is true as well in the semiquantized case and hence Eqs. (1) follow. For another argument, assume that the coupling among the two sectors can be switched on and off by playing with suitable time-dependent coupling constants. Now we can imagine starting with an uncoupled system, which satisfies the relations (1), and then switching on the interaction to end up with any given fully coupled system. Since the commutation relations are preserved by time evolution (even for nonconservative dynamics) Eqs. (1) will hold too in an arbitrary coupled semiquantized system. We think that these considerations make axiom (iii) inescapable.

Before extracting further conclusions, let us show that axiom (ii) is needed in order to exclude unacceptable behaviors of the semiquantized dynamics. Assume that the two sectors, classical and quantum, are coupled only during some time interval $t_1<t<t_2$. If axiom (ii) is dropped the commutation relations will no longer be preserved during that time interval. As a consequence, one finds that, except in particular cases, they will not hold either even when the two sectors are completely decoupled, i.e., before $t=t_1$ or after $t=t_2$. This is evident since one simply has to consider an arbitrary nontrivial coupled evolution and match it with free evolutions at $t<t_1$ and $t>t_2$ using the values of the generators at t_1 and t_2 as boundary conditions. This feature is certainly undesirable: physicists living before or after the coupling takes place, who can check that their systems are composed of a classical plus a quantum sector without relative coupling, will find a strange behavior (the classical variables do not commute and so on) just because there was or will be a coupling among sectors at a remote time in the past or in the future. Such a lack of locality in time is avoided if the time evolution preserves the algebraic structure of the observables. Further, if axiom (ii) is dropped it is problematic to define the dynamics. For instance, in order to define the Poisson bracket one may assume, as done, e.g., in Ref. [4], that the classical variables commute at zero time. If the commutation relations are not preserved by the evolution there will be a privileged time $t=0$: an observer in that world would find that the "classical" degrees of freedom commute (e.g., are fully classical) at some special time but not before or after. This privileged time is universal since it is independent of the Hamiltonian. Such behavior is nonexistent in classical or quantum dynamics.

Now, from the previous assumptions, a quite strong result can be derived, namely, the classical variables cannot inherit quantum fluctuations through their coupling to the quantum ones. To simplify the reasoning, let us consider a system with just two degrees of freedom—one of them (q,p) quantum and the other (x,k) classical in the sense of the commutation relations (1)—and let us denote the coordinates and momenta at $t=t_0$ by q, x, p, and k. At any time t, $\{E,q(t),p(t),x(t),k(t)\}$ is a set of generators. Since $x(t)$ commutes with all these generators, it commutes as well with all other observables and, in particular, with q and p and the same holds for $k(t)$. On the other hand, again using the commutation relations, every observable A is uniquely characterized by a set of coefficients C_{abcd}, with $a,b,c,d=0,1,2,\dots$, as $A=\Sigma_{abcd}C_{abcd}q^ap^bx^ck^dE$. We immediately see that any observable commuting with q cannot contain p and vice versa. Therefore, $x(t)$ must be of the form $\Sigma_{cd}C_{cd}(t)x^ck^dE$ and similarly $k(t)$. In other words, at all times x and k are commuting objects that evolve following well-defined trajectories, without fluctuations. On the other hand, $q(t)$ and $p(t)$ may depend on $x(t)$ and $k(t)$, which, in this regard, behave as external sources. This is the main result of this section.

One realization of the above picture is the traditional approach to semiquantization, namely, the quantum degrees of freedom move in the presence of the classical background, whereas the classical degrees of freedom are coupled to the expectation values of the quantum variables. Such an approach is thus mathematically consistent. On the other hand, consider a naive approach in which both sectors are coupled directly. To avoid ambiguities from operator ordering we take the case of two coupled harmonic oscillators

$$\frac{dq(t)}{dt}=\frac{p(t)}{m}, \quad \frac{dp(t)}{dt}=-m\omega^2q(t)-g(t)x(t), \tag{2a}$$

$$\frac{dx(t)}{dt}=\frac{k(t)}{M}, \quad \frac{dk(t)}{dt}=-M\Omega^2x(t)-g(t)q(t). \tag{2b}$$

Assuming the commutation relations (1) at $t=0$, a simple computation shows that $[p(t),k(t)]=(\lambda_1-\lambda_2)ig(0)tE+O(t^2)$, which only vanishes if either $g=0$, and thus the two subsystems are decoupled, or $\lambda_1 = \lambda_2$, i.e., the purely classical case if they vanish or the purely quantum case if they do not. Similarly, $[q(t),k(t)]$ breaks down at $O(t^2)$. That is, Eqs. (2) do not preserve the algebraic structure in general. As it is easily shown, this does not happens if $q(t)$ is replaced by its expectation value in Eq. (2b), i.e., the traditional approach.

III. CANONICAL STRUCTURE

The canonical structure of both classical and quantum mechanics (Poisson bracket and commutator, respectively) has been invoked in the literature [1,4] as a guiding principle to define semiquantized theories. From this point of view, it is of interest to consider whether there exist canonical structures interpolating between the quantum and the classical limits. Let us then study which constraints are found if, in addition to previous assumptions (i)–(iii), a canonical structure is present. For convenience, the relations (1) are rewritten in the form

$$[\phi^\alpha,\phi^\beta] = \eta^{\alpha\beta}E, \quad \alpha,\beta = 1,\dots,2N, \tag{3}$$

where the single symbol ϕ^α has been introduced to denote both q_i and p_i, and $\eta^{\alpha\beta}$ is an antisymmetric tensor.

The canonical structure is introduced by three postulates. First, there exists (iv) a Lie bracket (,) (i.e., enjoying bilinearity, antisymmetry, and Jacobi's identity) that generates the (infinitesimal) canonical transformations by $\delta_A B = (A,B)$, A,B being arbitrary observables. In particular, time evolution is a canonical transformation

$$\frac{dA(t)}{dt} = (A(t),H(t)), \tag{4}$$

where the Hamiltonian of the system $H(t)$ is an observable. Second, it is assumed that (v) all canonical transformations (not only time evolution) preserve the algebraic structure. This is equivalent to saying that δ_A is a derivation, i.e., it satisfies the product (Leibniz) rule: $\delta_A(BC) = (\delta_A B)C + B(\delta_A C)$. Third, the following canonical relations are assumed (vi):

$$(\phi^\alpha,\phi^\beta) = \epsilon^{\alpha\beta}E, \quad \alpha,\beta = 1,\dots,2N, \tag{5}$$

where $\epsilon^{\alpha\beta}$ is δ_{ij} for (q_i,p_j) and vanishes for (q_i,q_j) or (p_i,p_j). Such relations are common to classical and quantum mechanics, of course with different meanings for the bracket in each case. The canonical structure encapsulates the information that the dynamics derives from an action functional. Axioms (i)–(iii) allowed for much more general dynamics, for instance, a classical sector evolving by itself with a quantum sector coupled to it. Such a violation of the action-reaction principle is removed by the Lie bracket structure.

Before proceeding, let us point out the importance of Jacobi's identity since it seems to have been overlooked so far in this context [1,3,4]. The identity can be written as $\delta_A(B,C) = (\delta_A B,C) + (B,\delta_A C)$ and thus expresses that the bracket itself is invariant under canonical transformations. In particular, the relationship $(A,B)(t) = (A(t),B(t))$ will be consistent with the equations of motion. Such an identity is required if the canonical relations (5) are to be preserved by the canonical transformations and hence to prevent the same kind of inconsistencies as noted for the commutation relations in Sec. II (e.g., lack of locality in time and the existence of universal privileged times in the dynamics). After some algebra, it can be shown, in fact, that (q,p) differs from its canonical value at $O(t^2)$ with the bracket in Ref. [4] and at $O(t^3)$ with that of Refs. [1,3], for generic Hamiltonians. The antisymmetry property of the bracket also seems to be an obvious requirement to guarantee that the energy is conserved and that the evolution preserves Hermiticity [8,9].

It is important to note that the bracket (A,B) of any two observables can be completely worked out using only bilinearity, antisymmetry, the product rule, and the canonical relations (5), hence the bracket is, in fact, completely determined. In particular, Jacobi's identity follows as a by-product [10]. On the other hand, the same is true for the commutator using Eq. (3). We have to check whether both sets of equations are compatible. Noting that $\delta_A E$ vanishes, we find the following chain of equalities for arbitrary $\alpha,\beta,\mu,\nu = 1,\dots,2N$:

$$\begin{aligned}0 &= (\phi^\alpha\phi^\beta,\eta^{\mu\nu}E) = (\phi^\alpha\phi^\beta,[\phi^\mu,\phi^\nu])\\ &= \epsilon^{\beta\mu}\eta^{\alpha\nu} + \epsilon^{\alpha\mu}\eta^{\beta\nu} + \epsilon^{\beta\nu}\eta^{\mu\alpha} + \epsilon^{\alpha\nu}\eta^{\mu\beta}.\end{aligned} \tag{6}$$

The last equality follows from repeatedly applying the product rule. Contracting this equation with $\epsilon^{\mu\beta}$, it is found that consistency is achieved only if $\eta^{\alpha\beta} = i\lambda\epsilon^{\alpha\beta}$ for some λ. In fact, from Eqs. (1), all the λ_i are equal to λ. In other words, there can be just one sector. Furthermore, $[A,B] = i\lambda(A,B)$, for arbitrary A,B. There are only two possibilities: first, that λ is nonvanishing. In this case, we end up with the usual purely quantum dynamics. Second, if λ vanishes, all variables are commuting. Moreover, since the bracket is completely determined, it coincides with the Poisson bracket. That is, the dynamics is purely classical. This is the main conclusion of this section. Note that this result is consistent with that found regarding Eq. (2), namely, the canonical evolution generated by an arbitrary quadratic Hamiltonian $\phi^\alpha\phi^\beta$ does not preserve the semiquantized commutation relations (1).

IV. CONCLUSION

We conclude that assumptions (i)–(iii) prevent the classical sector from inheriting quantum fluctuations and, further, assumptions (i)–(vi) actually discard any nontrivial semiquantized theory. Note that further details on how to actually extract physical information from the observables (e.g., expectations values in the quantum case) are not required to reach the previous results. They are not completely conclusive, however. The seemingly harmless assumption of a Heisenberg picture formulation is, in fact, relevant to the conclusion. The Schrödinger picture formulation of Ref. [1] is obviously free from universal privileged times and is thus inequivalent to (an antisymmetric version of) Ref. [4]. That both pictures are no longer equivalent in the semiquantized context can also be seen from uncertainty principle considerations since the commutation relations are trivially preserved in one picture but not in the other. The semiquantization proposed in Ref. [1], as well as that in Ref. [5], based directly on time-ordered vacuum expectation values must also be discarded, but this requires further physical arguments, namely, physical positivity of the expectation values [11]. It is entirely possible that there is no nontrivial (or at least elegant) semiquantization scheme since, after all, such a concept is not presently known to be physically required.

ACKNOWLEDGMENTS

The author thanks J. Caro and C. García-Recio for useful criticism and for providing the proof in Ref. [10] and also J. Kruytzer for calling my attention to Refs. [6,8,9]. This work was partially supported by the Spanish DGICYT under Contract No. PB92-0927 and the Junta de Andalucía.

[1] W. Boucher and J. Traschen, Phys. Rev. D **37**, 3522 (1988).
[2] R.H. Brandenberger, Rev. Mod. Phys. **57**, 1 (1985).
[3] I.V. Aleksandrov, Z. Naturforsch. Teil A **36**, 902 (1981).
[4] A. Anderson, Phys. Rev. Lett. **74**, 621 (1995).
[5] L.L. Salcedo (unpublished).
[6] L. Diósi (unpublished).
[7] B.S. De Witt, J. Math. Phys. **3**, 619 (1962).
[8] L. Diósi, Phys. Rev. Lett. **76**, 4088 (1996).
[9] A. Anderson, Phys. Rev. Lett. **76**, 4090 (1996).
[10] J. Caro (private communication).
[11] L.L. Salcedo (unpublished).