Cavity modified quantum beats

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The coupling of an atom to its environment can be strongly dependent on frequency when that atom is placed in, for example, a cavity. We consider here the exact dynamics of a three-level atom for a resonant form of the atom-environment coupling of the type found in a cavity. The three-level atom forms a quantum-beat V system in the general model that we consider. Without the use of perturbation theory, we derive a set of three coupled differential equations that describe the system. Results are compared to quantum beats in free space and an interpretation is provided in terms of the coupling of the three-level system to a *pseudomode*. The pseudomode is defined by the differential equations involving its amplitude and possesses the properties of a finite-Q cavity mode. The signal from a detector is formulated in terms of a resonant coupling between the detector and the cavity modes. Limits for a broadband and a narrow-band detector are considered. [S1050-2947(96)05310-3]

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I. INTRODUCTION

The process of spontaneous emission is well known to be dependent not only on the properties of the excited atomic system, but also on the nature of the environment to which that system is optically coupled. That environment has not been of particular interest while spectroscopic measurements on atomic systems have been performed in free space, but environmental considerations play an important role when the atom is placed in a structure such as a cavity. The influence of a cavity on spontaneous emission was noted long ago by Purcell [1] and has been the subject of much study (and recently reviewed in [2]). Very often, for low-Q cavities, the theoretical treatment involves the atom coupling to a modified density of states, leading to enhanced, or inhibited, spontaneous emission, which can still be based on a Wigner-Weisskopf theory of decay [3]. Spontaneous emission inside cavities has been largely undesirable within the context of such cavity-atom systems as micromasers and in optical cavity QED. When high-Q cavities have been realized, they have allowed a perturbative treatment of spontaneous emission as, essentially, the leaking of a cavity mode. In this case, the strong coupling of the atom to the cavity leads to reversible dynamics where the radiated photon can return to the atom. However, sometimes the photon escapes from the cavity and then the environmental coupling is treated perturbatively. This paper addresses situations where neither kind of perturbative treatment is possible.

A nonperturbative treatment of the atom-cavity problem becomes desirable when the width of the cavity resonance is comparable to the width of the spontaneous emission of the atom placed in that cavity. The previous treatments of lossy cavity-atom problems cover, as mentioned, the two extreme scenarios. In the low-Q cavity the spontaneous-emission width is rather smaller than the cavity linewidth and decreases as the cavity Q is further decreased (in line with the Purcell formulation). In a high-Q cavity, as we shall see, the free-space spontaneous-emission linewidth is greater than the cavity linewidth.

The current interest in this kind of regime is stimulated by efforts to make microscopic lasing devices, that is, microcavity lasers, or *microlasers* [4–6]. These devices comprise an active medium inside a microscopic cavity. Current aims of the technology are to increase the coupling of the active region to an extent that there is no longer a threshold for laser action [5]. This will enable the production of high-efficiency, and low-power, devices. However, the necessity to obtain an output from the device [7] (that is, the built-in need for a ''cavity loss'') combined with the trend to high couplings can lead us directly into the nonperturbative regime that we consider in this paper. The experimental observation of Rabi oscillations in a semiconductor microcavity [8] (where energy is reversibly exchanged between the cavity field and a quantum two-level system) demonstrates the need for nonperturbative treatments.

The focus in this paper will be on the nonperturbative cavity coupling of a three-level V system in a microcavity. Because both of the transitions of the V system will be nearly resonant with the cavity there is a possibility for quantum beats of the radiated field in the cavity. The atom may be embedded in a dielectric cavity, as considered in Refs. [9,10], or it may be suspended in the cavity by means of a trapping mechanism. Another possible realization of the system is that there is no real three-level atom, but the existence of the energy levels is provided by a quantum-well structure in a semiconductor microcavity. Quantum beats from excitons have already been observed [11] and there has been theoretical consideration of quantum beats from coupled quantum wells [12], though in the absence of cavity effects such as those considered here.

There have been some theoretical explorations of the nonperturbative cavity regime. Frerichs *et al.* [9] have examined a three-level ladder system in which the upper pair of levels are coupled to the cavity. Rippin and Knight [10] have made calculations of the mode structures in distributed Bragg re-

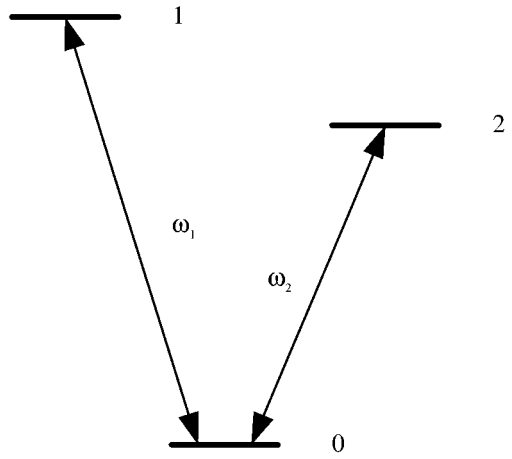

FIG. 1. Three-level system considered in this paper. The ground state 0 is coupled to the upper states 1 and 2 by transitions with frequencies ω_1 and ω_2. The frequency separation of the two upper levels is $\omega_{21}=\omega_2-\omega_1$.

flecting cylinders and examined the two-level system dynamics in the nonperturbative regime in such a cylinder. These papers both go beyond the usual assumptions of Markovian behavior, as used in the Wigner-Weisskopf description of spontaneous decay and other treatments of spontaneous emission in microcavities [13–15]. However, there is some conceptually related work on the spontaneous emission of two-level atoms into photonic band gaps [16,17]. There is much interest in the behavior of photons confined in semiconductor structures [18].

The three-level system that we consider here is illustrated in Fig. 1. It comprises two upper states 1 and 2 that are coupled by dipole transitions to the ground state 0. The three states of the system will be denoted as $|0\rangle$, $|1\rangle$, and $|2\rangle$ and, relative to the lower level 0, the energy of the states 1 and 2 will be ω_1 and ω_2 (throughout this paper we will take $\hbar=1$ so that frequencies are equivalent to energies). With these definitions it will be convenient to denote the frequency separation of the two upper levels as $\omega_{21}=\omega_2-\omega_1$, which plays a central role in the quantum beats. Indeed, in free space, ω_{21} is the frequency of the beats, which, in this paper, will be seen to be modified by the presence of a cavity.

In Sec. II of this paper we present the formulation of the problem starting from a fundamental Hamiltonian. The governing differential equations are derived and the pseudomode amplitude is identified. In Sec. III we describe features of the time evolution of the atomic state amplitudes. This includes the low-Q behavior and the values of the eigenvalues for different limiting cases. In Sec. IV we determine the energy of the cavity field and the signal from a photodetector. The photodetector is coupled to the cavity in a way that is similar to the coupling of the three-level system to the cavity. The relationship of the detector signal to the pseudomode amplitude and three-level system amplitudes is shown in the broadband detector and narrow-band detector limits. In Sec. V we explore the connections between the amplitude approach used in Secs. II and III and a master-equation approach based on a density matrix. Finally, some concluding remarks are presented in Sec. VI.

II. MATHEMATICAL DESCRIPTION

The three-level system is coupled to a bath of oscillators that may be the quantized modes of a cavity, microcavity, waveguide, or free space. For brevity we will refer to a cavity throughout the rest of this paper. The creation and annihilation operators for each oscillator are $a_k^\dagger$ and a_k, where the oscillator, which has frequency ω_k, is labeled here by the index k. This index can be understood as the wave vector of the mode (which need not satisfy the free-space dispersion relation $\omega_k=ck$), but it also stands for the directional and polarization labels. Then, within the rotating-wave approximation and with only dipole interactions, the Hamiltonian for the system can be written as

$$H=\sum_k \omega_k a_k^\dagger a_k+\omega_1|1\rangle\langle 1|+\omega_2|2\rangle\langle 2|$$
$$+\sum_k g_k^{(1)}(a_k^\dagger|0\rangle\langle 1|+a_k|1\rangle\langle 0|)$$
$$+\sum_k g_k^{(2)}(a_k^\dagger|0\rangle\langle 2|+a_k|2\rangle\langle 0|), \tag{1}$$

where $g_k^{(1)}$ and $g_k^{(2)}$ are the frequency-dependent couplings of the atomic transitions 2-0 and 1-0 to the mode denoted k when the atomic system is within the cavity. The sum over modes k is trivially converted to an integral by including the density of states ρ_k and taking account of any polarizations (or geometric factors depending on the orientation of the radiating dipole) so that

$$\sum_k \rightarrow \int d\omega_k \rho_k . \tag{2}$$

For example, this would mean that in free space, when we have $\rho_k \propto \omega_k^2$ we will find that $g_k^{(j)} \propto d_{j0}$ for $j=1,2$, where d_{j0} are the atomic dipole matrix elements. This case, which is not the focus of this article, leads to the well-known spontaneous-emission rate that is proportional to the cube of the transition frequency and the square of the dipole moment. In general, consideration of a frequency-dependent density of states can lead, for example, to modified Maxwell-Bloch equations [19] and to changes in the usual resonance fluorescence [20].

In a cavity we could expect to have a Lorentzian form for the coupling that reflects the phenomenon of resonance and the decay of the field. However, a cavity can also have a density of states that varies rapidly with frequency (and rather more so than in free space). Recently, Rippin and Knight have made detailed calculations of the spontaneous emission of a two-level atom in a cylindrical cavity containing distributed Bragg reflectors [10]. In that case they find that the Lorentzian approximation to the k dependence of the product of the square of the couplings and the density of states $\rho_k(g_k^{(j)})^2$ (where $j=1,2$) is an extremely good approximation for practical cases. In this paper we will take $g_k^{(j)}$ to be defined by

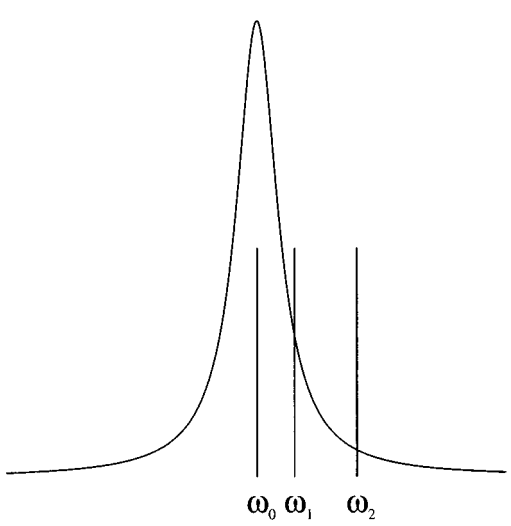

FIG. 2. Cavity resonance, Eq. (3), illustrated as a function of frequency. The center frequency ω_0 and the transition frequencies ω_1 and ω_2 are illustrated with vertical lines.

$$\rho_k(g_k^{(j)})^2=\frac{\Gamma_0 g_j^2/2\pi}{(\omega_k-\omega_0)^2+(\Gamma_0/2)^2}, \tag{3}$$

where $\Gamma_0/2$ is the frequency width of the resonance, ω_0 is the center frequency of the resonance, and g_j $(j=1,2)$ is a coupling constant. The weight of the resonance is g_j^2. Here we are assuming that these peak couplings are real, but it is very easy to generalize the results to the case of complex couplings. We may now also define the detunings of the energy levels from the center frequency of the cavity resonance as

$$\omega_{10}=\omega_1-\omega_0,$$

$$\omega_{20}=\omega_2-\omega_0, \tag{4}$$

as well as the mode detuning

$$\omega_{k0}=\omega_k-\omega_0. \tag{5}$$

These frequencies and the coupling Eq. (3) are illustrated in Fig. 2. Again, we emphasize that the coupling (3) contains contributions from *both* the density of states and consideration of the orientation of the atomic dipole with respect to the electric fields of the modes k at the position of the dipole.

For convenience, we may split the Hamiltonian Eq. (1) into two pieces comprising the interacting part and the noninteracting part so that

$$H=H_0+H_I,$$

$$H_0=\sum_k \omega_k a_k^\dagger a_k+\omega_1|1\rangle\langle 1|+\omega_2|2\rangle\langle 2|, \tag{6}$$

$$H_I=\sum_k g_k^{(1)}(a_k^\dagger|0\rangle\langle 1|+a_k|1\rangle\langle 0|)$$

$$+\sum_k g_k^{(2)}(a_k^\dagger|0\rangle\langle 2|+a_k|2\rangle\langle 0|).$$

Now it is clear that at zero temperature (an excellent approximation in the optical regime), if the atomic system is initially in a general state (a superposition of the levels 0, 1, and 2), we can only have the exchange of energy between one of the upper levels and the vacuum modes k. Because the total number of excitations in this model system cannot change we write down all the possible states to which the initial state can be coupled by the interaction. These states are

$$\psi_1=|1\rangle\otimes|000\cdots000\rangle,$$

$$\psi_2=|2\rangle\otimes|000\cdots000\rangle, \tag{7}$$

$$\psi_k=|0\rangle\otimes|000\cdots010\cdots000\rangle,$$

where the ket $|000\cdots000\rangle$ indicates the field state where all the radiation modes are in a vacuum state and the ket $|000\cdots010\cdots000\rangle$ indicates a state of the radiation field where all of the modes are in a vacuum state apart from mode k, which is in the first excited state. The unexcited state

$$\psi_0=|0\rangle\otimes|000\cdots000\rangle \tag{8}$$

is not coupled to anything else.

Now for the noninteracting part of the Hamiltonian we will trivially obtain

$$H_0\psi_1=\omega_1\psi_1,$$

$$H_0\psi_2=\omega_2\psi_2, \tag{9}$$

$$H_0\psi_k=\omega_k\psi_k,$$

while for the interacting part of the Hamiltonian

$$H_I\psi_1=\sum_k g_k^{(1)}\psi_k,$$

$$H_I\psi_2=\sum_k g_k^{(2)}\psi_k, \tag{10}$$

$$H_I\psi_k=g_k^{(1)}\psi_1+g_k^{(2)}\psi_2.$$

It is clear from these equations that we have a closed system of equations for the time evolution. We will now expand a general state vector of the system as

$$\Psi(t)=c_0\psi_0+c_1(t)\psi_1+c_2(t)\psi_2+\sum_k c_k(t)\psi_k \tag{11}$$

in terms of the states (7) and insert this into the Schrödinger equation $i(d/dt)\Psi=H\Psi$ to obtain the (infinite) set of coupled equations

$$i\frac{d}{dt}c_1=\omega_1 c_1+\sum_k g_k^{(1)}c_k,$$

$$i\frac{d}{dt}c_2=\omega_2 c_2+\sum_k g_k^{(2)}c_k, \tag{12}$$

$$i\frac{d}{dt}c_k=\omega_k c_k+g_k^{(1)}c_1+g_k^{(2)}c_2.$$

The coefficient c_0 is constant in time. It is convenient to move to an interaction representation by means of the time-dependent transformations

$$\begin{aligned}\tilde{c}_1(t)&=e^{i\omega_1 t}c_1(t),\\ \tilde{c}_2(t)&=e^{i\omega_2 t}c_2(t),\\ \tilde{c}_k(t)&=e^{i\omega_k t}c_k(t),\end{aligned}\tag{13}$$

so that we obtain the coupled equations

$$i\frac{d}{dt}\tilde{c}_1=\sum_k g_k^{(1)}e^{-i\Delta_k^1 t}\tilde{c}_k,\tag{14}$$

$$i\frac{d}{dt}\tilde{c}_2=\sum_k g_k^{(2)}e^{-i\Delta_k^2 t}\tilde{c}_k,\tag{15}$$

$$i\frac{d}{dt}\tilde{c}_k=g_k^{(1)}e^{i\Delta_k^1 t}\tilde{c}_1+g_k^{(2)}e^{i\Delta_k^2 t}\tilde{c}_2,\tag{16}$$

with the two detunings from the mode k defined by

$$\begin{aligned}\Delta_k^1&=\omega_k-\omega_1,\\ \Delta_k^2&=\omega_k-\omega_2.\end{aligned}\tag{17}$$

Now we can eliminate the coefficients $\tilde{c}_k$ by integrating Eq. (16) (in time) and substituting the resulting expression for $\tilde{c}_k$ into Eqs. (14) and (15). The integration of Eq. (16) yields

$$\tilde{c}_k(t)=-i\int_0^t dt'[g_k^{(1)}e^{i\Delta_k^1 t'}\tilde{c}_1(t')+g_k^{(2)}e^{i\Delta_k^2 t'}\tilde{c}_2(t')],\tag{18}$$

where the initial condition assumed is

$$\tilde{c}_k(0)=c_k(0)=0,\tag{19}$$

which simply means that there are no photons in the external bath (or cavity). We thus obtain the two coupled integro-differential equations

$$\begin{aligned}\frac{d}{dt}\tilde{c}_1(t)&=-\int_0^t dt'[G_{11}(t,t')\tilde{c}_1(t')+G_{12}(t,t')\tilde{c}_2(t')],\\ \frac{d}{dt}\tilde{c}_2(t)&=-\int_0^t dt'[G_{21}(t,t')\tilde{c}_1(t')+G_{22}(t,t')\tilde{c}_2(t')],\end{aligned}\tag{20}$$

where the functions $G_{ij}(t,t')$ are defined by

$$G_{ij}(t,t')=\sum_k g_k^{(i)}g_k^{(j)}\exp[i(\Delta_k^j t'-\Delta_k^i t)]\tag{21}$$

for $i,j=1,2$. In writing down Eq. (20) we have exchanged the order of summation over k and integration over time. This then allows us to write down the expressions (21) that can be evaluated analytically for a specific expression of the coupling such as given in Eq. (3). In that case we find

$$\begin{aligned}G_{ij}(t,t')&=\frac{g_i g_j}{2\pi}\int d\omega_k\frac{\Gamma_0\exp[i(\Delta_k^j t'-\Delta_k^i t)]}{(\omega_k-\omega_0)^2+(\Gamma_0/2)^2}\\ &=g_i g_j\exp[(i\omega_{i0}-\Gamma_0/2)t-(i\omega_{j0}-\Gamma_0/2)t'],\end{aligned}\tag{22}$$

where ω_{i0} and ω_{j0} are defined in Eq. (4). The ω integration in Eq. (22) can be performed as a contour integral closed in the lower-half complex ω plane [given that $t'\leqslant t$ is found in Eq. (20)].

At this point, the Lorentzian form of the couplings, Eq. (3), proves very useful because the t and t' parts of Eq. (22) factorize as a result. Thus, while Eqs. (20) can be expressed as two coupled second-order differential equations, whatever the form of G_{ij}, we can use the factorization to obtain three coupled first-order differential equations (a form that is more convenient for numerical evaluation)

$$\begin{aligned}i\frac{d}{dt}\tilde{c}_1&=g_1e^{i\omega_{10}t}\tilde{c}_f,\\ i\frac{d}{dt}\tilde{c}_2&=g_2e^{i\omega_{20}t}\tilde{c}_f,\\ i\frac{d}{dt}\tilde{c}_f&=-i\frac{\Gamma_0}{2}\tilde{c}_f+g_1e^{-i\omega_{10}t}\tilde{c}_1+g_2e^{-i\omega_{20}t}\tilde{c}_2.\end{aligned}\tag{23}$$

We have introduced the variable $\tilde{c}_f$,

$$\begin{aligned}\tilde{c}_f=-ie^{-\Gamma_0 t/2}\int_0^t dt'[&g_1e^{-(i\omega_{10}-\Gamma_0/2)t'}\tilde{c}_1(t')\\ &+g_2e^{-(i\omega_{20}-\Gamma_0/2)t'}\tilde{c}_2(t')],\end{aligned}\tag{24}$$

which plays the role of an amplitude for a single fictional *pseudomode*. The pseudomode is an abstract construction: the quantity $|c_0|^2+|c_1|^2+|c_2|^2+|c_f|^2$ is not conserved in time and in fact decays as $-\Gamma_0|c_f|^2$. The initial condition $\tilde{c}_f(0)=0$ is satisfied by Eq. (24). We emphasize that Eqs. (23) are exact, and the same equations could be obtained from a non-Hermitian Hamiltonian for the three-level system coupled to the pseudomode that has a complex frequency $\omega_0-i\Gamma_0/2$. That is,

$$\begin{aligned}H\equiv&(\omega_0-i\Gamma_0/2)a_f^\dagger a_f+\omega_1|1\rangle\langle 1|+\omega_2|2\rangle\langle 2|\\ &+g_1(a_f^\dagger|0\rangle\langle 1|+a_f|1\rangle\langle 0|)+g_2(a_f^\dagger|0\rangle\langle 2|+a_f|2\rangle\langle 0|)\end{aligned}\tag{25}$$

where the operators $a_f^\dagger$ and a_f are the creation and annihilation operators for the pseudomode f. It is clear that for very small Γ_0 the pseudo-mode f can be associated with the real cavity mode with frequency ω_0.

Finally, a note on the form of the coupled differential equations is in order. We see that by changing our amplitudes $\tilde{c}_j$ for the original c_j we can also write the coupled equations (23) in the form

$$i\frac{d}{dt}c_1=\omega_1c_1+g_1c_f,$$

$$i\frac{d}{dt}c_2=\omega_2 c_2+g_2 c_f, \tag{26}$$

$$i\frac{d}{dt}c_f=(\omega_0-i\Gamma_0/2)c_f+g_1 c_1+g_2 c_2.$$

Further, if we define the new amplitudes $\tilde{c}_j(t)=e^{i\omega_0 t}c_j(t)$, which are rotating at the cavity frequency, we obtain the differential equations

$$i\frac{d}{dt}\tilde{c}_1=\omega_{10}\tilde{c}_1+g_1\tilde{c}_f,$$

$$i\frac{d}{dt}\tilde{c}_2=\omega_{20}\tilde{c}_2+g_2\tilde{c}_f, \tag{27}$$

$$i\frac{d}{dt}\tilde{c}_f=-i\frac{\Gamma_0}{2}\tilde{c}_f+g_1\tilde{c}_1+g_2\tilde{c}_2,$$

which are convenient for numerical integration.

III. FEATURES OF THE TIME EVOLUTION

A. Solution for large cavity width

If the cavity width Γ_0 becomes very large, we can approximate Eq. (24) by

$$\tilde{c}_f=\frac{-2i}{\Gamma_0}[g_1 e^{-i\omega_{10}t}\tilde{c}_1(t)+g_2 e^{-i\omega_{20}t}\tilde{c}_2(t)]. \tag{28}$$

This allows us to make an adiabatic elimination of the pseudomode. By substituting Eq. (28) into Eq. (26) we can obtain the approximate equations

$$\frac{d}{dt}c_1=-(i\omega_1+\gamma_1/2)c_1-(\overline{\gamma/2})c_2,$$

$$\frac{d}{dt}c_2=-(i\omega_2+\gamma_2/2)c_2-(\overline{\gamma/2})c_1, \tag{29}$$

where we have defined the decay rates

$$\gamma_1=4g_1^2/\Gamma_0,$$

$$\gamma_2=4g_2^2/\Gamma_0, \tag{30}$$

$$\overline{\gamma}=\sqrt{\gamma_1\gamma_2}.$$

These decay rates are, apart from 2π factors, the values of the couplings and density of states factors, Eq. (3), at the position of resonance. If we were to decouple one of the energy levels, for example, by setting $g_2=0$, we would find the amplitude c_1 decaying at a rate $\gamma_1/2$. This would mean a decay of population at the rate γ_1, which can be interpreted as a decay rate in free space. However, we note that when Γ_0 is finite, γ_j need not be the free space decay rate because, when defined as in Eqs. (30), the factor g_j necessarily changes when the environment changes.

The above equations (29) clearly result in decaying oscillations of the amplitudes c_j. It is straightforward to find the frequencies and damping rates of these decaying oscillations by a determination of the eigenvalues of Eq. (29). These are found to be

$$\lambda_{\pm}=-i\frac{\omega_1+\omega_2}{2}-\frac{\gamma_1+\gamma_2}{4}\pm\frac{1}{2}\sqrt{\left(\frac{\gamma_1-\gamma_2}{2}+i\omega_{12}\right)^2+\gamma_1\gamma_2} \tag{31}$$

which is essentially a result given in Refs. [21,22]. The properties of these eigenvalues and the associated eigenvectors show that it is even possible to obtain quantum beats in the case where a single atomic energy level is initially excited [23,22]. This is because a single excited state is not an exact eigenvector of Eq. (29). The physical explanation for this is that in a careful consideration of a three-level system we find that the total system dipole couples to the vacuum and not the dipole of any single transition. Clearly this is true, and this is why we can get quantum beats at all. But it also means that even if only a single upper level is excited, there is a possibility for oscillations from another transition to become ''mixed in'' [23]. We will see later in this paper that to neglect this possibility is equivalent to making a secular approximation on the master equation for the complete system.

B. The general eigenvalue problem

If we now return to the exact problem Eq. (26) [or Eq. (27)], we note that the eigenvalues are determined by the equation

$$\lambda^3+(i\omega_1+i\omega_2+i\omega_0+\Gamma_0/2)\lambda^2+(g_1^2+g_2^2-\omega_1\omega_2-\omega_0\omega_1$$
$$+i\omega_1\Gamma_0/2-\omega_0\omega_2+i\Gamma_0\omega_2/2)\lambda+[i\omega_1 g_2^2+i\omega_2 g_1^2$$
$$-\omega_1\omega_2(i\omega_0+\Gamma_0/2)]=0, \tag{32}$$

which in a basis rotating at the cavity center frequency may be expressed as

$$\mu^3+(i\omega_{10}+i\omega_{20}+\Gamma_0/2)\mu^2+(g_1^2+g_2^2-\omega_{10}\omega_{20}+i\omega_{10}\Gamma_0/2$$
$$+i\omega_{20}\Gamma_0/2)\mu+(i\omega_{10}g_2^2+i\omega_{20}g_1^2-\omega_{10}\omega_{20}\Gamma_0/2)=0, \tag{33}$$

where $\mu=\lambda-i\omega_0$.

C. Limits for a central cavity frequency

In this section we will consider the limiting behavior for some special cases of the three-level system. There are two important simplifications. First, we will assume that we have a symmetric arrangement of the system frequencies such that the cavity frequency lies exactly midway between the atomic frequencies ω_1 and ω_2, i.e., $\omega_{20}=-\omega_{10}$. We will also assume that the two cavity-atom coupling constants are equal so that we can let $g_f=g_1=g_2$. Under these special conditions the equation for the roots, Eq. (33), reduces to

$$\mu^3+\Gamma_0\mu^2/2+(2g_f^2+\omega_{10}^2)\mu+\omega_{10}^2\Gamma_0/2=0. \tag{34}$$

Of course, this cubic equation cannot be solved in a simple form, but by the use of approximate expansions in appropri-

ate variables we can obtain approximate expressions for the roots μ. We will denote these roots by μ_0, μ_+, and μ_-.

We present here the following approximate results

(a) *Small* Γ_0. This corresponds to the case of a very narrow cavity resonance, and an expansion in Γ_0 yields

$$\mu = \begin{cases} -\dfrac{\omega_{10}^2/2}{\omega_{10}^2+2g_f^2}\Gamma_0 - \dfrac{g_f^2\omega_{10}^4/4}{(\omega_{10}^2+2g_f^2)^4}\Gamma_0^3 \\ \pm i\sqrt{\omega_{10}^2+2g_f^2} - \dfrac{g_f^2/2}{\omega_{10}^2+2g_f^2}\Gamma_0 . \end{cases} \tag{35}$$

The eigenvalues show weak damping at a rate depending on the cavity width and ω_{10}, and oscillations at a modified frequency $\sqrt{\omega_{10}^2+2g_f^2}$. If we let g_f become small, such that $|\omega_{10}| \gg g_f \gg \Gamma_0$, we obtain a limit for large separations ω_{10} with a damped mode at $\Gamma_0/2$ and oscillatory modes at $\pm\omega_{10}$. Despite the presence of a rather definite frequency for the cavity resonance, this limit leads to oscillations at the usual beat frequency. If instead we let ω_{10} become small, such that $g_f \gg |\omega_{10}| \gg \Gamma_0$, we obtain a limit for strong coupling of the atomic system to a high-Q cavity mode. In this case the damping of the μ_0 mode is simply $\omega_{10}^2\Gamma_0/(4g_f^2)$ and remains strongly dependent on ω_{10}^2 and the dipole matrix elements of the atomic transitions. This regime may be realized in microcavities with strong-coupling constants.

We note that this limit has been obtained by varying Γ_0 and keeping fixed g_f. Now in a practical case we would try to change Γ_0 by changing, for example, the cavity geometry. However, this is likely to change not just Γ_0, but g_f as well. In this way g_f effectively becomes a function of Γ_0, and in that case it is possible, though not inevitable, that the physical behavior in the limits of large and small Γ_0 (as achieved by physically changing the cavity) could differ from the results presented here. An example is presented below.

(b) *Small* g_f. This case corresponds to the weak-coupling limit

$$\mu = \begin{cases} -\Gamma_0/2 + \dfrac{\Gamma_0}{\omega_{10}^2+(\Gamma_0/2)^2}g_f^2 \\ \pm i\omega_{10} - \dfrac{1}{\Gamma_0/2 \pm i\omega_{10}}g_f^2 . \end{cases} \tag{36}$$

It is not surprising to see that if we neglect the higher-order terms in Eq. (36) above, we simply obtain the free evolution of the cavity, the μ_0 mode, damped at $\Gamma_0/2$, and the free evolution of the atomic transitions at $\pm\omega_{10}$ (relative to ω_0). The limit for $|\omega_{10}| \gg \Gamma_0 \gg g_f$ is shared with the $|\omega_{10}| \gg g_f \gg \Gamma_0$ limit found above for small Γ_0. This means that for $|\omega_{10}| \gg (g_f, \Gamma_0)$

$$\mu = \begin{cases} -\Gamma_0/2 + \dfrac{g_f^2\Gamma_0}{\omega_{10}^2} \\ \pm i\omega_{10}[1+(g_f/\omega_{10})^2] - \dfrac{g_f^2\Gamma_0}{2\omega_{10}^2}. \end{cases} \tag{37}$$

(c) *Large-*Γ_0 *limits*. The weak-coupling limit, Eqs. (36), also allows us to approach a broadband cavity limit where $\Gamma_0 \gg |\omega_{10}| \gg g_f$ and

$$\mu = \begin{cases} -\Gamma_0/2 + \dfrac{4g_f^2}{\Gamma_0} \\ \pm i\omega_{10} - \dfrac{2g_f^2}{\Gamma_0}. \end{cases} \tag{38}$$

Here the amplitudes of the atomic states decay at $2g_f^2/\Gamma_0 \equiv \gamma_{1,2}/2$ and oscillate at $\pm\omega_{10}$. There is a possibility of quantum beats in this case. The fact that the decay rate of the atomic states increases as the cavity Q increases ($Q=\omega_0/\Gamma_0$) is also reflected in the Purcell formula where, with γ regarded as a function of Q, $\gamma(Q) \propto Q\gamma(Q=0)$.

As mentioned above, the large-Γ_0 limit can be approached in several ways. We can choose to fix the peak value of the density of states, Eq. (3), while we change the resonance width. In that case we rewrite Eq. (34) in the form

$$\mu^3 + \Gamma_0\mu^2/2 + (\gamma\Gamma_0/2 + \omega_{10}^2)\mu + \omega_{10}^2\Gamma_0/2 = 0, \tag{39}$$

with $\gamma = 4g_f^2/\Gamma_0$. Then we will obtain the limit for $\Gamma_0 \gg (\gamma, \omega_{10})$, which on using the expression $\gamma = 4g_f^2/\Gamma_0$ becomes

$$\mu = \begin{cases} -\Gamma_0/2 + \gamma + 2\gamma^2/\Gamma_0 \\ -\gamma/2 \pm \sqrt{(\gamma/2)^2 - \omega_{10}^2} \\ -\dfrac{\gamma^2}{\Gamma_0}\left[1 \mp \dfrac{1}{\gamma}\dfrac{\gamma^2/2 - \omega_{10}^2}{\sqrt{(\gamma/2)^2-\omega_{10}^2}}\right] \end{cases} \tag{40}$$

In this equation we clearly see that we cannot expect to have beats in a broad width (low-Q) cavity unless $|\omega_{10}| > \gamma/2$. That is, the upper energy levels must be well separated.

(d) *Small* ω_{10}. This is the case where the two atomic energy levels are extremely close together. However, as we see below, the levels are then *too* close for quantum beats at the atomic energy separation:

$$\mu = \begin{cases} -\dfrac{\Gamma_0}{4g_f^2}\omega_{10}^2 + \dfrac{\Gamma_0}{8g_f^4}\left(1 - \dfrac{\Gamma_0^2}{8g_f^2}\right)\omega_{10}^4 \\ -\dfrac{\Gamma_0}{4} \pm \sqrt{\left(\dfrac{\Gamma_0}{4}\right)^2 - 2g_f^2} - O(\omega_{10}^2). \end{cases} \tag{41}$$

There can be beats caused by a Rabi splitting when $g_f \gg \Gamma_0/(4\sqrt{2})$.

(e) *A nontrivial exact solution*. Given appropriate parameters, we can always find simple solutions of Eq. (34) for given values of parameters. One such solution occurs if $g_f = 2|\omega_{10}| = \Gamma_0/(6\sqrt{3})$ (with $\omega_{20} = -\omega_{10}$). The determinant Eq. (34) then factorizes exactly and we have the three coincident roots

$$\mu = -\Gamma_0/6. \tag{42}$$

In this case we can solve the coupled differential equations exactly and we obtain

$$\tilde{c}_1(t)-c_1(0)=\tilde{c}_2(t)-c_1(0)$$
$$=-\frac{2g_f}{6^{3/2}\Gamma_0}(27+e^{-\Gamma_0 t/6+i\Gamma_0 t/(6\sqrt{3})}\{3[(\Gamma_0 t/2)^2$$
$$-(3/2)\Gamma_0 t-9]+i(\sqrt{3}/2)\Gamma_0 t(\Gamma_0 t/2+3)\}),$$

$$\tilde{c}_f(t)=-i\left(\frac{2}{3}\right)^{3/2}(\Gamma_0 t/2)(1-\Gamma_0 t/12)e^{-\Gamma_0 t/6}. \quad (43)$$

D. Large-Γ_0 limit for fixed $\gamma_1\neq\gamma_2$ and arbitrary ω_{10},ω_{20}

For future reference we include the approximate eigenvalues for large Γ_0 with fixed $\gamma_1\neq\gamma_2$. In this case we find from Eq. (33) that

$$\mu=\begin{cases}-\frac{\Gamma_0}{2}+\frac{\gamma_1+\gamma_2}{2}+\frac{(\gamma_1+\gamma_2)^2+2i(\gamma_1\omega_{10}+\gamma_2\omega_{20})}{2\Gamma_0}\\ -i\frac{\omega_{10}+\omega_{20}}{2}-\frac{\gamma_1+\gamma_2}{4}\\ \pm\frac{1}{2}\sqrt{\left(\frac{\gamma_1-\gamma_2}{2}+i\omega_{12}\right)^2+\gamma_1\gamma_2}.\end{cases} \quad (44)$$

The $\mu_\pm$ eigenvalues are consistent with the values in Eq. (31).

IV. THE RADIATED FIELD

The dynamics of the system are determined by Eqs. (23) [or by Eqs. (26)], and these dynamics can exhibit damped oscillations at up to three different frequencies. To make sense of the results it is necessary to determine a feature of the system that can be directly measured. The natural choice is the fluorescent signal, that is, the time-dependent intensity of the light radiated by the atom. To determine this there are two important considerations: first, the quantum mechanics of the radiation field, and second, complications in determining the fluorescence arising from the geometry of the cavity and specific orientation of the detecting system. The first consideration is dealt with by choosing the proper observable. We will avoid the second problem by choosing a rather general form for the coupling of the detector to the cavity-atom system.

A. Field energy

As a precursor to determining a radiated fluorescent field, it is instructive to determine the energy of the radiation field U_f as a function of time. One could naively propose that this corresponds to the total energy lost by the atom and that the rate of change of U_f could be the total intensity of radiated light. This quantity will be denoted I_f. We can formally write down the field energy as

$$U_f=\sum_k \omega_k|c_k|^2=\sum_k \omega_k|\tilde{c}_k|^2 \quad (45)$$

so that the derivative is given by

$$I_f=\frac{d}{dt}U_f=\sum_k \omega_k\tilde{c}_k\frac{d}{dt}\tilde{c}_k^*+\text{c.c.} \quad (46)$$

Then, by substituting Eq. (18) for c_k and the conjugate of Eq. (16) for the derivative of c_k^* we can show that

$$I_f=-\int_0^t dt'[G'_{11}(t,t')\tilde{c}_1^*(t)\tilde{c}_1(t')+G'_{12}(t,t')\tilde{c}_1^*(t)\tilde{c}_2(t')$$
$$+G'_{21}(t,t')\tilde{c}_2^*(t)\tilde{c}_1(t')+G'_{22}(t,t')\tilde{c}_2^*(t)\tilde{c}_2(t')]+\text{c.c.}, \quad (47)$$

which bears a close resemblance to Eqs. (20). However, this time the integration kernels are given by

$$G'_{ij}(t,t')=\sum_k g_k^{(i)}g_k^{(j)}\omega_k\exp[i(\Delta_k^j t'-\Delta_k^i t)]$$
$$\to(\omega_0-i\Gamma_0/2)G_{ij}(t,t'). \quad (48)$$

In the last line of Eq. (48) the sum has been converted to an integral, which has again been determined by a contour integration. If we now use the expressions (22) for the $G_{ij}(t,t')$ and the exact equation (24) we obtain the rate of change of the field energy in the form

$$I_f=i\omega_0(g_1c_1^*c_f-g_1c_1c_f^*+g_2c_2^*c_f-g_2c_2c_f^*)$$
$$+\frac{\Gamma_0}{2}(g_1c_1^*c_f+g_1c_1c_f^*+g_2c_2^*c_f+g_2c_2c_f^*). \quad (49)$$

The typical ratio of cavity width to frequency would suggest that the Γ_0 term can be ignored. Indeed, in the limit of large Γ_0, if we were to use the approximation (28), the Γ_0 term would vanish from Eq. (49). Neglecting Γ_0, we may write

$$I_f\to 2\omega_0\,\text{Im}[(g_1c_1+g_2c_2)c_f^*]=-\omega_0\frac{d}{dt}(|c_1|^2+|c_2|^2) \quad (50)$$

with some help from Eqs. (23) and (28). This laboriously derived though straightforward result shows that the cavity field picks up energy from population loss of the three-level system, as would be expected. It also shows that, in this limit, the energy feeds into the pseudomode. Note, however, that the 1-0 and 2-0 transitions contribute energy at the cavity resonance frequency rather than at the separate transition frequencies. This seems fair for a high-Q cavity, but not for a low-Q cavity or in the free-space limit when the separate photon energies ω_1 and ω_2 would be expected to appear. This problem results from the neglect of the Γ_0 term from Eq. (49).

In fact, neglecting the Γ_0 term in Eq. (50) amounts to a neglect of the interaction energy of the atom and cavity (enhanced by the factor Γ_0). The interaction energy is formally $U_I=\langle H_I\rangle$ with H_I given by Eq. (6). By inserting the state vector (11) we obtain

$$U_I(t)=c_1^*(t)\sum_k g_k^{(1)}c_k(t)+c_2^*(t)\sum_k g_k^{(2)}c_k(t)+\text{c.c.} \quad (51)$$

and if we then make use of the sums in Eq. (12) we finally obtain

$$U_I(t)=c_f(g_1c_1^*+g_2c_2^*)+c_f^*(g_1c_1+g_2c_2), \tag{52}$$

which is the Γ_0 term in Eq. (49). Thus we actually have

$$I_f=-\omega_0\frac{d}{dt}(|c_1|^2+|c_2|^2)+\frac{\Gamma_0}{2}U_I. \tag{53}$$

The rate of change of the field energy can also be cast in the form

$$I_f=i\omega_0(V_1+V_2)+\frac{\Gamma_0}{2}U_I, \tag{54}$$

where we introduce the quantities $V_j(j=1,2)$ such that

$$V_j=g_f^{(j)}(c_j^*c_f-c_jc_f^*). \tag{55}$$

Now we observe that for the rate of change of the interaction energy

$$\frac{d}{dt}U_I(t)=i\omega_{10}V_1(t)+i\omega_{20}V_2(t)-\frac{\Gamma_0}{2}U_I(t), \tag{56}$$

so that we can *exactly* write

$$I_f(t)=-\frac{d}{dt}[\omega_1|c_1(t)|^2+\omega_2|c_2(t)|^2+U_I(t)]. \tag{57}$$

This form shows the contributions of $|c_1(t)|^2$ and $|c_2(t)|^2$ at the appropriate energies. However, the equation is nothing more than a statement of the conservation of energy. We note that at $t=0$ any initial coherence does not contribute to I_f, so that the initial I_f is determined by the populations alone. This is because the initial rate of change of U_I in Eq. (57) is zero. This rate of change is zero because U_I and V_1, V_2 are zero in Eq. (56). However, as the system evolves in time, the U_I term introduces coherence into I_f through Eq. (57).

If we now consider the low-Q cavity we can deduce from Eq. (56) that

$$U_I\rightarrow\frac{i\omega_{10}V_1+i\omega_{20}V_2}{\Gamma_0/2}, \tag{58}$$

and on utilizing Eqs. (53) and (23) we have

$$\begin{aligned}I_f(t)&\sim i\omega_1V_1(t)+i\omega_2V_2(t)\\&=(\omega_1\sqrt{\gamma_1}c_1^*+\omega_2\sqrt{\gamma_2}c_2^*)(\sqrt{\gamma_1}c_1+\sqrt{\gamma_2}c_2)\\&\quad\times(\sqrt{\gamma_1}c_1^*+\sqrt{\gamma_2}c_2^*)(\omega_1\sqrt{\gamma_1}c_1+\omega_2\sqrt{\gamma_2}c_2),\end{aligned} \tag{59}$$

where we have used Eq. (28) in the last line. In this limit the cavity frequency plays no role and the rate of change of the field energy contains interference terms of the kind expected in quantum beats.

B. A photodetection signal

The rate of change of field energy is not a suitable quantity for the signal from a photodetector. For example, in a high-Q cavity, there can be a reversible exchange of energy between the atom and the cavity, which means that the quantity I_f can be negative as energy returns to the atomic system. The signal from a photodetector, which is weakly coupled to the cavity field modes, will be proportional to $I_d(t)=\langle E_d^-E_d^+\rangle$, where $E_d^\pm$ are the positive and negative frequency components of the electric field operator at the surface of the detector. The electric field operator E_d^+ can be expanded in terms of the mode functions u_k and photon annihilation operators as

$$E_d^+=\sum_k u_ka_k, \tag{60}$$

so that if we utilize the expansion of the state vector Eq. (11) we can obtain the expression

$$I_d(t)=\left|\sum_k c_k(t)u_k\right|^2\equiv|V_d|^2, \tag{61}$$

which will also serve as a definition of the detector field V_d. The mode functions depend on spatial position and thus the detected signal will also be sensitive to the position and orientation of the detector system. This is not such a crucial issue in the case of an atom radiating in free space, but clearly the location of the detecting element is important when the atom is inside a cavity. For example, if the detector is near an antinode of dominating modes of a high-Q cavity critical parts of the fluorescence will be absent. We can avoid a discussion of specific cavity geometries and detecting arrangements by choosing a general form of the detector coupling that is similar to Eq. (3), but with modified center frequency and width. An appropriate limit can then be taken for a broadband detector or for a spectrally resolving detector. We proceed then by replacing the mode functions by generalized couplings $g_k^{(d)}$ of the detector to the cavity field. Then by using Eq. (18) we find that

$$\begin{aligned}V_d=-i\sum_k g_k^{(d)}e^{-i\omega_kt}\int_0^t dt'[&g_k^{(1)}e^{i\Delta_k^1t'}\tilde{c}_1(t')\\&+g_k^{(2)}e^{i\Delta_k^2t'}\tilde{c}_2(t')].\end{aligned} \tag{62}$$

We should emphasize that in this equation the sum over modes k is now at the site of the detector and so depends on the density of states at the position of the detector. We proceed to generalize Eq. (3) with

$$\rho_kg_k^{(j)}g_k^{(d)}=\frac{\Gamma_dg_dg_j/(2\pi)}{(\omega_k-\omega_d)^2+(\Gamma_d/2)^2}, \tag{63}$$

where the width $\Gamma_d/2$ gives a detection width in the specific cavity-detector arrangement. This should not be confused with any free-space frequency response of the detector. Indeed, if the detector is assumed to be broadband then Γ_d and ω_d will depend on the cavity parameters and relative location of the detector alone. By converting the sum over modes k in

Eq. (62) to an integral and using Eq. (63) together with the same argument used in Eqs. (21) and (22), we will find

$$V_d = -ig_d e^{-(i\omega_d+\Gamma_d/2)t}\int_0^t dt'[g_1 e^{-(i\omega_1-i\omega_d-\Gamma_d/2)t'}\tilde{c}_1(t') + g_2 e^{-(i\omega_2-i\omega_d-\Gamma_d/2)t'}\tilde{c}_2(t')]. \quad (64)$$

At this stage it is interesting to examine Eq. (64) for several limiting types of detector arrangement. First, we consider the broadband detector coupling where $\Gamma_d \to \infty$. In this limit we obtain

$$V_d(t) \to -i\frac{2g_d}{\Gamma_d}[g_1 c_1(t) + g_2 c_2(t)], \quad (65)$$

so that on using Eqs. (28), (61), and (49)

$$I_d(t) = |V_d(t)|^2 \to \left(\frac{g_d\Gamma_0}{\Gamma_d}\right)^2 |c_f(t)|^2 = \frac{g_d^2\Gamma_0}{\omega_0\Gamma_d^2} I_f(t), \quad (66)$$

in the low-Q cavity limit. Then the detector signal is proportional to the ''population'' of the pseudomode, which in turn is proportional to the rate of change of the energy of the cavity field.

Second, in the narrow-band detector limit $\Gamma_d \to 0$ ensures that the detector becomes very frequency selective. For large time $t \to \infty$ the detector signal is no more than the square of a Fourier component of the upper state amplitudes

$$I_d(\infty) \to g_d^2 \left| \int_0^\infty dt' e^{i\omega_d t'}[g_1 c_1(t') + g_2 c_2(t')] \right|^2. \quad (67)$$

V. MASTER-EQUATION APPROACHES

A. An equivalent cavity-atom master equation

The conventional approach to the study of the interaction between a cavity and a multi-level system almost always involves a description in terms of a master equation. The master equation is usually derived using time-dependent perturbation theory and the Markov approximation and is valid in the low-Q regime, where the dissipation applies directly to atomic operators, and in the high-Q regime, where the dissipation applies to the cavity field, which in turn is coupled to the atomic system.

In this paper the exact equations require no approximations from time-dependent perturbation theory or specific correlation times of a ''heat bath.'' But the question remains as to the relationship between the approach presented here, which utilizes an atomic state vector extended to include a pseudomode and the conventional master-equation approach where the description is in terms of a density matrix. More complex problems involving three-level systems can entail the use of the state vector Monte Carlo methods, which solve a dissipative problem using *stochastic* state vectors [24]. However, that method only applies for problems with a master equation specified in the Lindblad form [25]. In this section we seek to find the master equation decribing the atomic system and determine if it has Lindblad form.

We have already seen, in Eq. (25), that the state vector, including the pseudomode, obeys a (non-Hermitian) Hamiltonian type of evolution. We will now assert that the equations (23) [or Eqs. (26) and (27)] are exactly equivalent to the Lindblad master equation

$$\frac{d}{dt}\rho = -i[H_s,\rho] - \frac{\Gamma_0}{2}(a_f^\dagger a_f \rho - 2a_f \rho a_f^\dagger + \rho a_f^\dagger a_f), \quad (68)$$

where the Hermitian system Hamiltonian is

$$H_s = \omega_0 a_f^\dagger a_f + \omega_1 |1\rangle\langle 1| + \omega_2 |2\rangle\langle 2| + g_1(a_f^\dagger |0\rangle\langle 1| + a_f |1\rangle\langle 0|) + g_2(a_f^\dagger |0\rangle\langle 2| + a_f |2\rangle\langle 0|). \quad (69)$$

The definition of ρ is crucial:

$$\rho = \begin{pmatrix} \Pi_{00} & c_0 c_1^* & c_0 c_2^* & c_0 c_f^* & 0 & 0 \\ c_1 c_0^* & |c_1|^2 & c_1 c_2^* & c_1 c_f^* & 0 & 0 \\ c_2 c_0^* & c_2 c_1^* & |c_2|^2 & c_2 c_f^* & 0 & 0 \\ c_f c_0^* & c_f c_1^* & c_f c_2^* & c_f c_f^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (70)$$

The basis used is identified by a pair of states, where the first state refers to the excitation of the (fictional) pseudomode (which may be zero photons or one photon) and the second state refers to the atomic state, that is, we use the basis

$$(|0_f\rangle|0\rangle, |0_f\rangle|1\rangle, |0_f\rangle|2\rangle, |1_f\rangle|0\rangle, |1_f\rangle|1\rangle, |1_f\rangle|2\rangle), \quad (71)$$

where the subscript f is used to distinguish the pseudomode basis. The zero elements in ρ do not exist because of the initial conditions and because the master equation does not couple them to nonzero elements. The amplitude c_0 is fixed at its initial value and the amplitudes c_1, c_2, and c_f obey the differential equation (26). We have a new variable Π_{00}, the probability of finding the atomic system in the ground state and no photons in the pseudomode. The initial value of Π_{00} is clearly $|c_0|^2$ and it satisfies

$$\Pi_{00}(t) = |c_0|^2 + \Gamma_0 \int_0^t |c_f(t')|^2 dt'. \quad (72)$$

The master equation (68) contains a ''sandwich'' term $2a_f \rho a_f^\dagger$ and the density matrix ρ, Eq. (70), clearly does not remain in the form of a pure ''state'' during the whole of the time evolution. [The term ''state'' is placed in quotes, because the pseudomode, in this paper, is a mathematical construct derived from Eqs. (23).] We recall that the original normalization of the extended system $|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_f|^2$ is not conserved in time. However, the trace of the density matrix (70) is preserved during the time evolution of the system because of the presence of the sandwich term. In this system the sandwich term feeds the $\Pi_{00} = \langle 0|\langle 0_f|\rho|0_f\rangle|0\rangle$ element in the master equation, but has no other effect. If we were to trace out the fictional pseudomode from the master equation (68), we would obtain an equation of motion for the reduced atomic density matrix. The sandwich term contributes to a diagonal element. This

means that if we wish to know the equation of motion governing the probability of being in the ground state, we require

$$\frac{d}{dt}\Pi_0=\frac{d}{dt}(\langle 0|\langle 0_f|\rho|0_f\rangle|0\rangle+\langle 0|\langle 1_f|\rho|1_f\rangle|0\rangle)$$
$$=\frac{d}{dt}|c_f|^2+\Gamma_0|c_f|^2, \tag{73}$$

where the right-hand side has been evaluated from Eqs. (70) and (68). Now we see that this result corresponds exactly to that found from a fundamental approach. In the original model the ground-state population is the same as the total population of the field [see Eq. (11)] and the original population of the ground state (which is constant in time). Thus we may write

$$\frac{d}{dt}\Pi_0=\frac{d}{dt}\sum_k c_k^* c_k=-\frac{d}{dt}[|c_1(t)|^2+|c_2(t)|^2] \tag{74}$$

on using Eqs. (12), first for the derivative of c_k and then for the sum over k. Then if we use Eqs. (23) we can obtain exactly Eq. (73).

B. Low-Q atomic master equation

In order to derive the low-Q atomic master equation we focus on the reduced atomic density matrix obtained by tracing over the pseudomode in Eq. (70):

$$\rho_A=\begin{pmatrix} \Pi_{00}+|c_f|^2 & c_0c_1^* & c_0c_2^* \\ c_1c_0^* & |c_1|^2 & c_1c_2^* \\ c_2c_0^* & c_2c_1^* & |c_2|^2 \end{pmatrix}. \tag{75}$$

It is now useful to define a collective dipole operator D such that

$$D=\sqrt{\gamma_1}|0\rangle\langle 1|+\sqrt{\gamma_2}|0\rangle\langle 2| \tag{76}$$

(where it is straightforward to generalize to the case where g_1, g_2 are complex). Then, given that in the low-Q limit, $|c_f|^2=|\sqrt{\gamma_1}c_1+\sqrt{\gamma_2}c_2|^2/\Gamma_0$, Eq. (28), we find, by using Eqs. (29) and (30), the low-Q master equation

$$\frac{d}{dt}\rho_A=-i[H_A,\rho_A]-\frac{1}{2}D^\dagger D\rho_A-\frac{1}{2}\rho_A D^\dagger D+D\rho_A D^\dagger, \tag{77}$$

where the effective atomic Hamiltonian is simply

$$H_A=\omega_1|1\rangle\langle 1|+\omega_2|2\rangle\langle 2|. \tag{78}$$

Of course, the cavity frequency ω_0 no longer plays a direct role in the atomic dynamics in this limit. We can, however, determine the excitation of the pseudomode, which is

$$|c_f|^2=\frac{\mathrm{Tr}(D^\dagger D\rho_A)}{\Gamma_0}. \tag{79}$$

As shown in Eq. (66), this is proportional to the signal of a broadband detector.

In the case of conventional quantum beats, where $|\omega_{12}|\gg(\gamma_1,\gamma_2)$, we are also able to make the secular approximation. Then the density-matrix coherences and populations evolve independently and we obtain the familiar master equation

$$\frac{d}{dt}\rho_A=-i[H_A,\rho_A]-\frac{1}{2}D_1^\dagger D_1\rho_A-\frac{1}{2}\rho_A D_1^\dagger D_1+D_1\rho_A D_1^\dagger$$
$$-\frac{1}{2}D_2^\dagger D_2\rho_A-\frac{1}{2}\rho_A D_2^\dagger D_2+D_2\rho_A D_2^\dagger, \tag{80}$$

where we have defined separate dipole operators for the 1-0 and 2-0 transitions:

$$D_1=\sqrt{\gamma_1}|0\rangle\langle 1|,$$
$$D_2=\sqrt{\gamma_2}|0\rangle\langle 2|. \tag{81}$$

Making the secular approximation removes the coupling between the coherences and populations. It is this coupling that makes it possible to have oscillations even if only one of the two excited states is excited [23,22].

VI. CONCLUDING REMARKS

The master equation (68) has the obvious interpretation that a three-level atomic system is coupled to a lossy mode: the pseudomode. Yet the theoretical development in this paper did not require the explicit use of a heatbath or perturbation theory, even if these concepts are implicit in the creation of a Lorentzian distribution in Eq. (3).

The pseudomode did not exist at the outset of this paper: we merely had a Lorentzian distribution of a density of states (and an atom-field coupling). However, it is clear that by the time we reach the master equation (68) the pseudomode has acquired reality; the complexity of a distribution of many modes has simplified to a single entity.

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