Two simple expressions for the spontaneous emission factor β

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In the literature two different simplified expressions exist for the spontaneous emission factor β, being the fraction of spontaneous emission radiated into a specific optical mode. These expressions are valid in two mutually exclusive regimes. By treating the coupling between one discrete cavity mode and the outside optical continuum quasi one dimensionally in the spirit of Fano [Phys. Rev. **124**, 1866 (1961)], we extend the validity range of both expressions and show that they lead to essentially the same result. The relevance and limitation of this result are discussed. [S1050-2947(96)04210-2]

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I. INTRODUCTION

We consider an inverted gain medium inside an optical cavity of some kind. The spontaneous emission factor β is then defined as the fraction of spontaneous emission radiated into a specific optical mode, here denoted as the cavity or lasing mode. Lasers with a large β are interesting because for these lasers much of the spontaneous emission is radiated into the lasing mode and there is thus limited emission ''wasted'' in other modes. Such lasers will automatically have a low lasing threshold. The holy grail is a laser with $\beta=1$, where no radiation is lost, making it a ''thresholdless laser'' [1–3].

An accurate calculation of β is often lengthy and tedious. It involves a summation or integration of the atom-field coupling over all optical modes and thereby touches the heart of quantum electrodynamics. However, with a few simple assumptions the calculation can be greatly simplified. In this introduction we start by reminding the reader of the two simplified expressions for β that exist in the literature [1,4–8] and point out that these are valid in two mutually exclusive regimes.

Figure 1 sketches a typical system for which we will calculate β. The emitters in the active medium, which for convenience will be denoted by ''atoms,'' although they might equally well be molecules, electrons, excitons, or any other entities interacting with light, will radiate spontaneously in a multitude of directions and frequencies. It is now our task to determine the fraction β emitted into a single discrete mode, which occupies inside the cavity an optical volume V_{cav} and subtends a space angle $\Delta\Omega_{\mathrm{cav}}$. This mode has a finite spectral width due to outcoupling at the mirrors. The remaining optical modes are treated as a continuum.

A calculation of β is largely a geometric problem, which can be very complicated due to the three-dimensional structure of the cavity. The calculation simplifies considerably when a quasi-one-dimensional approach is taken, in which the optical continuum is subdivided into modes that interact with the discrete cavity mode, through the mirrors, and modes that do not. The former part of the continuum is associated with the ''longitudinal'' modes, as they have their wave vectors oriented more or less along the cavity axis, whereas the latter part may then be associated with the ''transverse'' modes. This separation is depicted in Fig. 2. The various arrows show the pump as energy supplier, additional loss of atomic coherence due to collisional dephasing (rate γ_{coll}), spontaneous emission to the transverse-optical continuum, coherent coupling to the discrete cavity mode (coupling constant g) and optical loss out of this mode (loss rate γ_{cav}). We restrict our attention to the weak-coupling regime, i.e., $g \ll \gamma_{\mathrm{cav}}$. In that case the coherent coupling and cavity loss combine to an effective exponential decay from the excited atomic state to the longitudinal-optical continuum outside the cavity [9,10]. The spontaneous emission factor β is then equal to the relative contribution of the latter decay to the total spontaneous emission rate γ_{rad}, which also includes (and for $\beta \ll 1$ is dominated by) decay to the transverse-optical continuum. For the atomic system the radiative decay (as T_1 process) and collision dephasing (as pure T_2 process) combine to what we will call the atomic decay rate $\gamma_{\mathrm{atom}}=\gamma_{\mathrm{rad}}+2\gamma_{\mathrm{coll}}$.

In the literature one finds basically two different approaches to calculate β. These approaches are valid in two mutually excluding regimes, being determined by the ratio between the cavity decay rate γ_{cav}, i.e., the spectral width of the cavity mode, and the atomic decay rate γ_{atom}, i.e., the width of the atomic spontaneous emission spectrum (see Fig. 3). When $\gamma_{\mathrm{cav}} \ll \gamma_{\mathrm{atom}}$ one often speaks about the ''good-cavity'' regime, whereas $\gamma_{\mathrm{cav}} \gg \gamma_{\mathrm{atom}}$ is then called the ''bad-cavity'' regime [11–13]. This nomenclature is somewhat misleading, as it does not refer to the quality of the cavity, as determined, e.g., by the mirror reflectivities, in an absolute sense, but only to the comparison between the field and atomic decay rates. In this paper the two regimes will therefore be denoted by characterizing the nature of the dominant decay: atom-dominated decay ($\gamma_{\mathrm{cav}} \ll \gamma_{\mathrm{atom}}$) and field-dominated decay ($\gamma_{\mathrm{cav}} \gg \gamma_{\mathrm{atom}}$).

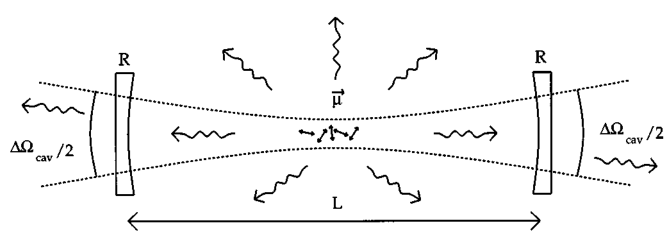

FIG. 1. Typical geometry for which β is calculated.

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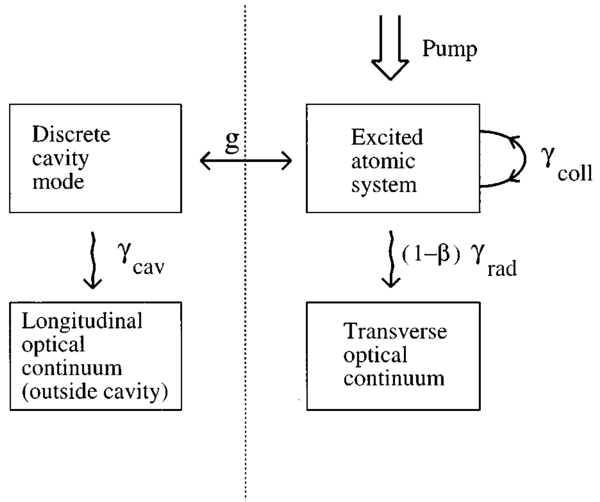

FIG. 2. Schematic diagram of the couplings between the atomic excitation, the discrete cavity mode, and the optical continuum.

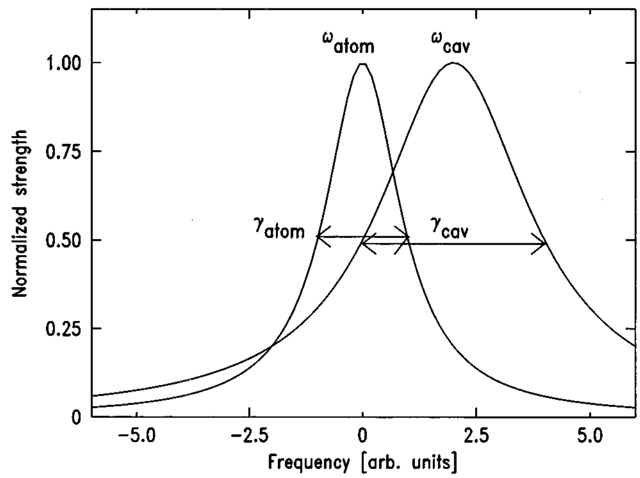

FIG. 3. Spectra of the spontaneous emission and the cavity resonance.

In the atom-dominated decay regime ($\gamma_{cav} \ll \gamma_{atom}$) the cavity mode has a relatively well-defined frequency. The standard approach is then to assume full lateral confinement, by setting the optical quantization volume V_{quant} equal to the volume V_{cav} taken up by the cavity mode, and simply counting the number of modes p with eigenfrequencies falling underneath the atomic spectral profile. At resonance this ''mode count'' gives [1,4]

$$\beta = \frac{1}{p} = \frac{1}{4\pi^2} \times \frac{\lambda^3}{V_{cav}} \times \frac{\omega}{\gamma_{atom}}, \tag{1}$$

where λ and ω are the optical wavelength and optical frequency. In Sec. II we will show that the introduction of rigorous lateral confinement is not really neccesary and that the above result has a much wider validity. By writing the result as a product of three factors β is shown to be basically the inverse of the modal volume, in units of λ^3, multiplied by the Q factor of the atomic transition.

In the field-dominated decay regime ($\gamma_{cav} \gg \gamma_{atom}$) the cavity modes are spectrally broad and it is common to consider the problem from the atomic point of view. When the emitted light is followed on its path through the cavity, the interference from the various reflecting surfaces are found to lead to radiation enhancement for some emission angles and suppression for others [6–8]. As $\gamma_{cav} \gg \gamma_{atom}$ this modification will be more or less constant over the full spontaneous emission spectrum and for a very thin ($\ll \lambda$) active medium positioned in an antinode of the optical field one finds at resonance [5,6]

$$\beta = \frac{\Delta\Omega_{cav}}{8\pi}\left(\frac{4}{1-R}\right), \tag{2}$$

where $\Delta\Omega_{cav}$ is the solid angle subtended by the cavity mode and R is the intensity reflectivity of the cavity mirrors.

It is not at all obvious that the above equations (1) and (2) are limiting cases of a single more general form. In Ref. [1] a substitution of typical experimental parameters into similar equations showed their practical equivalence, but a reason for this was not given. Other papers report numerical calculations of β as a function of $\gamma_{atom}/\gamma_{cav}$ and presented the results in the form of figures, but analytic expressions have not been given [4,8]. In this paper we will give the analytic expression and demonstrate the intimate relation between the two above equations. Furthermore, we will discuss under which, rather general, conditions these equations are valid and when they break down. In order to focus on the essential point we use some simplifying assumptions and skip much detail. As a consequence, in practical cases the simplified expressions are correct only to within factors of order unity. We apologize beforehand for this inconvenience.

In Sec. II we will calculate β by considering the cavity mode as a single discrete but lossy mode. The derivation is rather informal and more directed towards insight and applicability than rigor. Reasonable assumptions to simplify the problem are pointed out. In Sec. III a more formal Fano-type approach is followed in which the cavity mode is interpreted as a resonance in the longitudinal mode continuum. In Sec. IV the resulting expressions are compared, and their applicability is discussed. Finally, conclusions are drawn and a summary is given.

II. THE ''ONE PHOTON PER MODE'' PICTURE

The pedestrian view of spontaneous emission is based on the notion that ''spontaneous emission is equivalent to stimulated emission produced by vacuum fluctuations, the latter corresponding to one photon per optical mode'' [14]. We consider an oscillating dipole at position $\vec{r}$ having an eigenfrequency ω. Combining the well-known dipolar atom-field interaction Hamiltonian $\vec{\mu} \cdot \vec{E}(\vec{r})$ with Fermi's golden rule for decay to a continuum of final states one finds that the spontaneous emission rate of this dipole into each specific mode i is proportional to $\langle|\vec{\mu} \cdot \vec{E}_i(\vec{r},\omega)|^2\rangle$, where $\langle|\vec{E}_i(\vec{r},\omega)|^2\rangle$ denotes the mean-square vacuum field per unit frequency. The generic equation for the now *position and frequency-dependent* spontaneous emission factor follows immediately:

$$\beta(\vec{r},\omega)=\frac{\langle|\vec{\mu}\cdot\vec{E}_{\rm cav}(\vec{r},\omega)|^2\rangle}{\Sigma_{\rm i}\langle|\vec{\mu}\cdot\vec{E}_{\rm i}(\vec{r},\omega)|^2\rangle}, \quad (3)$$

where in the numerator $|\vec{E}_{\rm cav}(\vec{r},\omega)|^2$ represents the vacuum field density in the cavity mode at the atomic position $\vec{r}$ and frequency ω and where the sum in the denominator runs over all modes.

The position and frequency dependence of $\beta(\vec{r},\omega)$ reflects the obvious notion that the amount of spontaneous emission coupled into the cavity mode depends both on the position and frequency of the emitting dipoles. The overall spontaneous emission factor β that appears for instance in the laser rate equations [1,8] is simply the average value of $\beta(\vec{r},\omega)$ integrated over the positions and spectra of all emitters.

The generic equation (3) immediately suggests some natural assumptions. For simplicity we assume that: (i) the radiating dipoles $\vec{\mu}$ are randomly oriented in space, (ii) the active medium is spread out over the nodes and antinodes of the field profile $\vec{E}_{\rm cav}(\vec{r})$, (iii) the volume $V_{\rm atom}$ of the active medium is relatively small ($V_{\rm atom}\ll V_{\rm cav}$) so that the coarse-grained field, i.e., the field averaged over regions of a few cubic wavelengths, is more or less constant over the active medium, and (iv) the total spontaneous emission rate, i.e., the denominator in Eq. (3) which sums over all available modes, is equal to its value in free space.

Assumption (i) implies that the vector nature drops out of the problem and assumptions (ii)–(iii) imply that the active dipoles interact on average with the same electric-field strength. Together they allow us to eliminate $\vec{\mu}$ out of Eq. (3), thus clearly showing the geometric nature of the problem. If desired, some of the restrictions can be easily removed. For instance β can be increased by preferential alignment of the dipoles $\vec{\mu}$ along $\vec{E}_{\rm cav}(\vec{r})$; this increase is at most a factor 3. Preferential positioning of the dipoles in the antinodes of the field yields another factor 2. Also, if the volume of the active medium is not small compared to the modal volume, some atoms may couple more efficiently with the cavity mode than others and the (spatially averaged) β will decrease as compared to β for a cavity in which a compact active medium is located at the maximum of the modal field.

Assumption (iv), which takes the total spontaneous emission rate $\gamma_{\rm rad}$ to be more or less equal to its free space value, is of course a crude assumption, as we know that the cavity leads to enhanced spontaneous emission in some modes and we are in fact trying to calculate this enhancement and its effect on β. However, the spontaneous emission will generally be enhanced only for relatively few modes whereas it remains unaffected or will even be suppressed for other modes. Therefore, the cavity mainly leads to a redistribution over phase space or a directionality of the emitted photons, but its effect on total emission rate $\gamma_{\rm rad}$ is generally small and assumption (iv) is approximately valid for cavities with relatively small β values (e.g., $\beta<0.1$). Experimentally, it has been found extremely difficult to change the total spontaneous emission rate in optical experiments. Record experiments, leading typically to variations of up to 30%, have been performed on large-angle concentric cavities [15], (sub)wavelength-size cavities with dye solutions [2,16], and semiconductor microcavities [16].

We return now to the calculation of β. In quantitative form the notion that ''spontaneous emission is similar to stimulated emission produced by vacuum fluctuations, the latter corresponding to one photon per optical mode,'' means that the mean-square fluctuating field of the cavity mode $\langle|\vec{E}_{\rm cav}(\vec{r})|^2\rangle$ satisfies the following requirement:

$$\int dV\epsilon_0\langle|\vec{E}_{\rm cav}(\vec{r})|^2\rangle=\hbar\omega. \quad (4)$$

The integral in Eq. (4) can be rewritten as a product by introducing the effective, i.e., intensity-averaged, modal volume as

$$V_{\rm cav}^{\rm eff}=\int dV\langle|\vec{E}_{\rm cav}(\vec{r})|^2\rangle/\langle|\vec{E}_{\rm cav,max}|^2\rangle, \quad (5)$$

where $\langle|\vec{E}_{\rm cav,max}|^2\rangle$ corresponds to the one-mode vacuum fluctuations in the spatial region of maximum coarse-grained intensity. Note that with the above definition $V_{\rm cav}^{\rm eff}$ is equal to the product of the cavity length times the *smallest* mode cross section, assuming that the compact gain medium is located at that position. When the mode cross section in the cavity is more or less constant we have $V_{\rm cav}^{\rm eff}\approx V_{\rm cav}$, independent of the location of the compact gain medium (in this case the gain medium may also fill the complete mode volume). In a lossy cavity the fluctuating field is distributed over a Lorentzian spectrum of width $\gamma_{\rm cav}$, resulting in

$$\langle|\vec{E}_{\rm cav,max}(\omega)|^2\rangle=\frac{\hbar\omega}{\epsilon_0 V_{\rm cav}^{\rm eff}}\left(\frac{2}{\pi\gamma_{\rm cav}}\right)\frac{(\gamma_{\rm cav}/2)^2}{(\omega-\omega_{\rm cav})^2+(\gamma_{\rm cav}/2)^2}. \quad (6)$$

When we apply the same ''one photon per mode'' argument to spontaneous emission in free space, we find that the fluctuating field of each free-space mode is $\langle|\vec{E}(\vec{r})|^2\rangle=\hbar\omega/(\epsilon_0 V_{\rm quant})$, where $V_{\rm quant}$ is the quantization volume. Multiplied by the mode density per unit volume per unit angular frequency of $\omega^2/(\pi^2c^3)V_{\rm quant}$ we find the standard result

$$\langle|\vec{E}(\vec{r},\omega)|^2\rangle=\frac{\hbar\omega}{\epsilon_0}\frac{\omega^2}{\pi^2c^3}. \quad (7)$$

Division of Eq. (6) by Eq. (7) yields the spontaneous emission factor $\beta(\omega)$ for the case of monochromatic emitters

$$\beta(\omega)=\frac{1}{4\pi^2}\frac{(\lambda)^3}{V_{\rm cav}^{\rm eff}}\left(\frac{\omega}{\gamma_{\rm cav}}\right)\frac{(\gamma_{\rm cav}/2)^2}{(\omega-\omega_{\rm cav})^2+(\gamma_{\rm cav}/2)^2}. \quad (8)$$

Integration of $\beta(\omega)$ over the normalized free-space emission spectrum

$$L(\omega)=\frac{\gamma_{\rm atom}/(2\pi)}{(\omega-\omega_{\rm atom})^2+(\gamma_{\rm atom}/2)^2}, \quad (9)$$

yields

$$\beta=\int d\omega\ \beta(\omega)L(\omega)=\frac{1}{4\pi^2}\frac{\lambda^3}{V_{\text{cav}}^{\text{eff}}}\left(\frac{\omega}{\gamma_{\text{cav}}+\gamma_{\text{atom}}}\right)\times\frac{[(\gamma_{\text{cav}}+\gamma_{\text{atom}})/2]^2}{(\omega_{\text{cav}}-\omega_{\text{atom}})^2+[(\gamma_{\text{cav}}+\gamma_{\text{atom}})/2]^2},\tag{10}$$

where we have used that the convolution of two normalized Lorentzians with widths γ_{cav} and γ_{atom} give a single normalized Lorentzian with a width equal to the sum $\gamma_{\text{cav}}+\gamma_{\text{atom}}$. Note that to obtain this result there was no need for ''hard'' confinement of the optical field in the transverse direction. The derivation was based only on ''soft'' confinement of the cavity or lasing mode [as given by Eq. (5)] and on the assumptions (i)–(iv).

At resonance ($\omega_{\text{cav}}=\omega_{\text{atom}}$) the final result becomes

$$\beta=\frac{1}{4\pi^2}\frac{\lambda^3}{V_{\text{cav}}^{\text{eff}}}\frac{\omega}{\gamma_{\text{cav}}+\gamma_{\text{atom}}},\tag{11}$$

which is equal to Eq. (1) in the atom-dominated decay regime, but which contains a correction factor $\gamma_{\text{atom}}/(\gamma_{\text{cav}}+\gamma_{\text{atom}})$ otherwise.

III. THE FANO-TYPE APPROACH

In the preceding section β was calculated more or less *ad hoc* by considering the cavity mode as a single discrete mode with a finite bandwidth due to damping. A more rigorous route of calculation starts with the combined continuum of the modes of free space with the cavity embedded in it, i.e., the left-hand side of Fig. 2. In this continuum the cavity mode shows up as a resonant enhancement of the field strength inside the cavity around the mode frequency. These continuum mode functions may be viewed upon as stationary scattering states of the Maxwell field, which automatically account for the coupling between the discrete cavity mode and the optical continuum outside the cavity. In the limit $R\approx 1$, where R is the intensity reflectivity of the mirrors, the resonant enhancement factor of the optical intensity inside the cavity as compared to that outside, is given by [17,18]

$$A(\omega)=\left(\frac{2\,\omega_{\text{fsr}}}{\pi\gamma_{\text{cav}}}\right)\frac{(\gamma_{\text{cav}}/2)^2}{(\omega-\omega_{\text{cav}})^2+(\gamma_{\text{cav}}/2)^2},\tag{12}$$

where ω_{cav} is the cavity resonance frequency and $\omega_{\text{fsr}}=\pi c/L$ is the free spectral range. The interfering reflections are found to structure the originally ''white'' continuum into a Lorentzian-shaped spectrum. The enhancement and suppression is such that the integral of $A(\omega)$ over a full free spectral range remains unaffected; the cavity resonance only leads to a redistribution of the spectral intensity. For the symmetric cavity sketched in Fig. 1, the decay rate of the intracavity intensity is easily found to be

$$\gamma_{\text{cav}}=\frac{1-R}{\pi}\omega_{\text{fsr}}=\frac{\omega_{\text{fsr}}}{F},\tag{13}$$

where F is called the cavity finesse.

Our aim is to obtain the distribution of the spontaneously emitted photons over the modes of the structured continuum, which also yields the factor β. For this purpose Fano's diagonalization procedure [19] is well suited, where we now have to account for the resonance structure of the atom-field coupling. This procedure gives the eigenstates for a system consisting of a single discrete state $|0\rangle$ coupled to a continuum. Since Fano's approach requires a pure-state description of the atom-field system, it cannot account for external perturbations such as collisions. Therefore we first restrict ourselves to the case of a free atom coupled to the radiation field, i.e., $\gamma_{\text{coll}}=0$. We denote the excited state of the atom without photons by $|0\rangle$, and the ground-state atom combined with a photon of frequency ω propagating in the direction Ω by $|\omega,\Omega\rangle$. The continuum states are normalized according to

$$\langle\omega,\Omega|\omega',\Omega'\rangle=\delta(\omega-\omega')\delta(\Omega-\Omega').\tag{14}$$

The system is described by a Hamiltonian with a coupling between the excited state $|0\rangle$ and the continuum equal to

$$\int\int d\omega\ d\Omega|0\rangle f(\omega,\Omega)\langle\omega,\Omega|+\{\text{Hermitian}-\text{conjugate}\}.\tag{15}$$

Here $f(\omega,\Omega)$ contains the transition dipole moment multiplied by the normalized-mode function inside the cavity at the position of the atom. Fano diagonalization gives formal but exact expressions for normalized eigenstates of the Hamiltonian including the atom-field interaction [19]. If we expand the initial state $|0\rangle$ in these eigenstates, the long-time evolution immediately gives the probability distribution over the frequency ω and direction Ω of the emitted photons in the form

$$p(\omega,\Omega)=\frac{|f(\omega,\Omega)|^2}{(\omega-\omega_{\text{atom}}-\Delta\omega)^2+(\gamma_{\text{atom}}/2)^2},\tag{16}$$

where $\Delta\omega$ is a small Lamb-type frequency shift, which will be neglected. The radiative transition rate γ_{atom} is related to the coupling strength by the relation:

$$\gamma_{\text{atom}}=2\pi\int d\Omega|f(\omega,\Omega)|^2.\tag{17}$$

These results (16) and (17) are justified when the right-hand side of Eq. (17) varies negligible with the frequency ω within a width of the order γ_{atom}. Then we may equate the frequency ω to ω_{atom} in (17) and one easily checks that the distribution (16) is normalized, as it should.

It is now an easy matter to extract an expression for β, if we once more use a quasi-one-dimensional approach and assume that only a well-defined fraction of modes $|\omega,\Omega\rangle$ is affected by the cavity, i.e., only the longitudinal modes for which Ω lies within the space angle $\Delta\Omega_{\text{cav}}$. For Ω outside this space angle $|f(\omega,\Omega)|^2$ is not affected by the cavity and must have the value $\gamma_0/(8\pi^2)$, with γ_0 the free-space decay rate [see Eq. (17)]. Within $\Delta\Omega_{\text{cav}}$, $|f(\omega,\Omega)|^2$ is enhanced by the factor $A(\omega)$ given in Eq. (12). Hence the atomic decay rate (17) is equal to

$$\gamma_{\text{atom}}=\left(1-\frac{\Delta\Omega_{\text{cav}}}{4\pi}\right)\gamma_0+\frac{\Delta\Omega_{\text{cav}}}{4\pi}\gamma_0A(\omega).\tag{18}$$

The spontaneous emission factor is found by integrating Eq. (16) over frequency and the solid angle $\Delta\Omega_{\mathrm{cav}}$ with as result

$$\beta=\Delta\Omega_{\mathrm{cav}}\frac{\gamma_0}{8\pi^2}\int d\omega\frac{A(\omega)}{(\omega-\omega_{\mathrm{atom}})^2+(\gamma_{\mathrm{atom}}/2)^2}. \quad (19)$$

In principle Eq. (18) allows for a large deviation of the atomic decay rate γ_{atom} from the free-space value, in which case γ_{atom} can even become frequency dependent. Likewise, β as given by Eq. (19) can have an appreciable value so that the restriction $\beta\ll 1$ does not necessarily apply. For convenience, however, we assume that the total decay rate is hardly affected, so that $\gamma_{\mathrm{atom}}\approx\gamma_0$. In this case the integral in Eq. (19) can be directly performed, and we obtain

$$\beta=\frac{\Delta\Omega_{\mathrm{cav}}}{4\pi}\frac{\omega_{\mathrm{fsr}}}{2}\frac{(\gamma_{\mathrm{cav}}+\gamma_{\mathrm{atom}})/(2\pi)}{(\omega_{\mathrm{cav}}-\omega_{\mathrm{atom}})^2+(\gamma_{\mathrm{cav}}+\gamma_{\mathrm{atom}})^2/4}, \quad (20)$$

where an extra factor 1/2 has been applied to single out one polarization of the cavity mode.

If we now once more specialize to the resonant case ($\omega_{\mathrm{atom}}\approx\omega_{\mathrm{cav}}$) we find

$$\beta=\frac{\Delta\Omega_{\mathrm{cav}}}{8\pi}\frac{2\omega_{\mathrm{fsr}}/\pi}{\gamma_{\mathrm{cav}}+\gamma_{\mathrm{atom}}}=\frac{\Delta\Omega_{\mathrm{cav}}}{8\pi}\left(\frac{2}{1-R}\right)\frac{\gamma_{\mathrm{cav}}}{\gamma_{\mathrm{cav}}+\gamma_{\mathrm{atom}}}, \quad (21)$$

which in the cavity-dominated decay regime ($\gamma_{\mathrm{cav}}\gg\gamma_{\mathrm{atom}}$) is a factor 2 smaller than Eq. (2), because here we assumed the atoms to be evenly distributed over the nodes and antinodes of the field instead of concentrated in the antinodes, but which contains an extra correction factor $\gamma_{\mathrm{cav}}/(\gamma_{\mathrm{cav}}+\gamma_{\mathrm{atom}})$ otherwise.

The effect of collisions, or any other type of dephasing, on the distribution of emitted photons over the modes can be easily incorporated when the duration of collision τ_{c} may be ignored. This is the case in a spectral region of the order of $1/\tau_{\mathrm{c}}$ around resonance, where the impact theory of line broadening holds. Then the effect of collisions is a simple additional damping of the transition dipole at a rate γ_{coll} and the autocorrelation function of the dipole is simply multiplied by the exponential $\exp(-\gamma_{\mathrm{coll}}t)$. This implies that the distribution Eq. (16) must be convoluted by the normalized Lorentzian with width $2\gamma_{\mathrm{coll}}$, which again gives Eqs. (20) and (21), with $\gamma_{\mathrm{atom}}=\gamma_{\mathrm{rad}}+2\gamma_{\mathrm{coll}}$, but now for the general case $\gamma_{\mathrm{coll}}\neq 0$.

IV. DISCUSSION AND CONCLUSION

After having extended the two simplified expression of β found in the literature into the more general Eqs. (11) and (21) it is easy to show that these are in fact equivalent: it is sufficient to rewrite the effective modal volume $V_{\mathrm{cav}}^{\mathrm{eff}}$ as the product of cavity length L times minimum mode area A and to use the diffraction result

$$\Delta\Omega_{\mathrm{cav}}\approx\frac{2\lambda^2}{A}. \quad (22)$$

Uncertainties to within a factor of order unity not only stem from the exact nature of the transverse-optical boundary conditions, but also from the specific definition of $\Delta\Omega_{\mathrm{cav}}$ and A, which can be, e.g., either in terms of full widths at half maximum or in terms of integrals over the mode profile. Whether Eq. (11) or Eq. (21) is more convenient depends in practice on the geometry of the cavity.

It follows directly from our analysis that for a given cavity β is largest when the spontaneous emission spectrum is narrow, while for a given emission spectrum β is largest in a narrow-band cavity. Separately, these statements may also be found in the literature [6,4,8]. However, stressing only one of the two statements, in a phrase like [8] ''the atomic-gain linewidth must be narrower than the cold-cavity linewidth to realize the optimum β'' is somewhat misleading, since it is now obvious that to obtain a large β *both* the cavity decay rate γ_{cav} *and* the atomic decay rate γ_{atom} have to be small.

In the case of an inhomogeneously broadened atomic spectrum the various homogeneous components will generally interact differently with the cavity and lead to different β's. By setting γ_{atom} in the various equations equal to the homogeneous linewidth one finds β for a specific class of atoms, e.g., for the ''resonant'' atoms that couple most strongly with the cavity mode, whereas a spectrally averaged value is found when the inhomogeneous linewidth is inserted.

In assumption (iv) of Sec. II the total spontaneous emission rate, summed over all available modes, was set equal to its free-space value. This is only reasonable for $\beta\ll 1$, and can be *ad hoc* generalized by taking the ratio $\beta/(1+\beta)$, with β as defined in Eq. (11) or Eq. (21), as a better expression for the spontaneous emission factor. A simple justification for this repair is that the latter expression shows the proper asymptotic behavior $\beta\rightarrow 1$ for small modal volumes and large $\Delta\Omega_{\mathrm{cav}}$ and R, whereas Eq. (11) and Eq. (21) do not. The repair implies that we separate the sum in the denominator of Eq. (3) into the cavity mode and the other modes and assume that the spontaneous emission summed over these other modes, which generally take up most of the phase space, remains unaffected by the cavity.

Up to now we have avoided a discussion about the existence of a refractive index different from 1. Inclusion of such a refractive index is straightforward and we have in fact written all equations for β given above in such a form that they remain valid when λ is taken to be the actual wavelength inside the gain medium for a uniformly filled cavity and a suitable weighted average of the wavelength in the medium and the vacuum wavelength for a partially filled cavity. Inclusion of dispersion, in the form of a frequency-dependent refractive index, is more difficult, for instance, because the amount of dispersion will generally depend on the degree of atomic excitation.

Three types of cavities play a rather prominent role in the literature: the concentric and confocal cavity, both of which are popular in atomic beam experiments [15,20], and the planar Fabry-Perot cavity, which is often used for miniature dye lasers or for semiconductor vertical-cavity surface-emitting lasers [1,2,7]. For a concentric cavity the finite mirror dimensions limit the opening angle $\Delta\Omega_{\mathrm{cav}}$ of the cavity mode and β can be directly evaluated from Eq. (21). For such a cavity Eq. (11) is of course also applicable, but it now becomes important to use Eq. (5) as a proper definition for the effective modal volume. For a confocal cavity the situation is less favorable as the opening angle of the fundamental TEM_{00} mode is generally much smaller, being basically determined by the cavity or mirror focal length. However, in Ref. [21] Morin, Yu, and Mossberg discuss how in this case the frequency degeneracy of transverse and longitudinal modes allows one to construct linear combinations in the form of ''hour-glass modes'' that still have a large opening angle. To obtain a large β the gain medium should then be well localized in the waist of one specific hour-glass mode as the different hour-glass modes are spatially incoherent.

For a planar cavity the situation is not so clear, because ideally there is no optical confinement in the in-plane direction, and it is thus difficult to define a single discrete cavity or lasing mode. The obvious choice is to use the interference condition for the intracavity light as a means to define a modal opening angle $\Delta\Omega_{\rm cav}$ and related coherence area [8]. Substitution of the relevant expressions into either Eq. (11) or Eq. (21) shows that in a planar cavity β is of the order of $\lambda/(2L)$ times $[1+\gamma_{\rm cav}/(\gamma_{\rm cav}+\gamma_{\rm atom})]^{-1}$ and will in practical cavities never approach unity, basically because an increase of the cavity finesse automatically leads to an equal increase of modal volume $V_{\rm cav}^{\rm eff}$ and decrease of opening angle $\Delta\Omega_{\rm cav}$ [5,8,16].

In conclusion we have rederived, in two different ways, the two simplified expressions for β that exist in the literature and extended them to be valid for any ratio $\gamma_{\rm atom}/\gamma_{\rm cav}$ of the atomic over the cavity decay rate. The resulting expressions (11) and (21) are valid under rather general conditions [see assumptions (i)–(iv)]. In most cases they provide a reasonable first estimate of β. They show that to obtain a large β both the modal volume and the volume of the active medium should be small, in combination with low field and atomic-decay rates. These criteria are linked to a large opening angle for the cavity mode [Eq.(22)], and a large field enhancement or equivalently a high mirror reflectivity [Eqs. (12) and (13)].

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