Statistical analysis of pulses in systems with random modulation through an instability point

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A method for the analysis of pulse statistics in one-dimensional dynamical systems with random modulation of the control parameter is developed. The stationary properties of pulses are obtained via the solution of an integral equation for the probability density of the passage time by a reference level. The method is applied to a complex Landau-type model with random on-off modulation. In some cases analytical expressions are derived for the passage time distribution. This distribution is shown to be multimodal due to memory effects. An excellent agreement is obtained in all situations between theoretical results and numerical simulations. [S1050-2947(96)09709-0]

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I. INTRODUCTION

Switching processes in many physical systems can be described in the framework of externally driven dynamical systems. The system is forced to switch from one state to another by changing a parameter. Of special interest are those cases in which the system has two steady states, one stable and the other unstable, for which the change of the parameter only modifies the stability of these solutions. Pulses in these systems can be obtained by varying the control parameter between two values that correspond to two stable states *off* and *on*, usually associated to small and large values, respectively, for the variable of interest. When the control parameter is changed in such a way that the *off* state becomes unstable, the evolution is in a first stage governed by fluctuations. Then the analysis of this decay process is usually based on stochastic models of the Langevin type, in which fluctuations are modeled by a white noise. The decay of unstable states is one of the fundamental problems of nonequilibrium statistical mechanics, in which nonlinearities and fluctuations are crucial to have a correct description [1]. In most theoretical analysis the system is in the *off* steady state when the control parameter is either instantaneously changed driving the system to an unstable state [1,2], or swept across the unstable state with a finite velocity [3–5]. Another interesting situation corresponds to the case of periodic modulation through the instability point [6]. This problem is relevant for several physical systems: modulated convection [7], stochastic resonance [8], Q-switched lasers [9], and gain-switched modulated semiconductor lasers [10]. In this case the system is not in the *off* steady state when the control parameter is changed because the initial condition, at the beginning of a pulse, is determined by the final evolution of the previous pulse. Hence, some kind of consistency condition for the statistical properties associated with two consecutive periods must be used to solve this problem. This consistency condition has been indirectly used in different methods: generalization [11] of Suzuki's matching procedure [12] and the path-integral approach [13]. In [6] we have developed a method in which the consistency condition is explicitly worked out following the main idea of the quasideterministic theory [2]. This theory has been applied to gas lasers [6,14] and semiconductor lasers [15].

In this paper we develop a method to study the statistical properties of pulses in stochastic systems that are randomly modulated through an instability point. A random sequence of ''1'' (pulses) and ''0'' (associated with the *off* state) is produced by using two different evolutions for the control parameter a during one period. In *on* periods a is changed through the instability point to generate a pulse, while during *off* periods a is kept below the instability point. In our analysis we consider systems described by a one-dimensional Langevin equation, in which fluctuations are modeled by a white noise. The statistical properties of pulses for the ''intensity'' (modulus of the variable of interest) are studied. The probability density of the pulse characteristics, such as width and height, can be obtained from the passage time distribution by using the deterministic evolution. The passage time τ is defined as the delay between the beginning of the *on* period and the emission of the pulse, given by the time in which a reference value is reached by the intensity. After an initial transient the passage time distribution $P(\tau)$ becomes independent of the initial conditions. In this steady-state case the statistical properties of τ are the same for two consecutive pulses. An integral equation for $P(\tau)$ is derived from this consistency condition in a way similar to the periodic modulation case [6]. The passage time distribution is obtained by solving this equation and analytical expressions are given in some cases. Due to the random modulation of the control parameter, produced by the random sequence of *off* and *on* periods, new features appear with respect to the periodic modulation case (only *on* periods). One of the most interesting results is the multimodal character due to memory effects of the probability density of some of the pulse characteristics. The different peaks of the probability distribution are associated with different periodic sequences. The method is applied to a complex Landau dynamical system that can describe one-dimensional systems such as a randomly modulated class-A laser. In this system a random sequence of pulses and ''zeros'' can be generated by loss or gain switching. Analytical expressions are obtained in some cases for the switch-on (passage) time distribution that shows a bimodal character due to memory effects. This behavior has also been found in numerical simulations of semiconductor lasers modulated at high bit rates [16]. Memory effects have been experimentally corroborated in fiber lasers [17]. These lasers are two-dimensional dynamical systems and the population dynamics must also be considered. However, our results show that multimodal distributions due to memory effects are also obtained for one-dimensional dynamical systems. Moreover, analytical expressions are derived for the statistical distributions associated with the pulses described by these systems. These expressions are shown to be in good agreement with numerical simulations.

The paper is organized as follows. In Sec. II we describe the method in a one-dimensional general system, deriving the basic integral equation for the passage time distribution. The method is applied in Sec. III to a complex Landau-type dynamical system with random on-off modulation of the control parameter. Analytical results are obtained for this system in some cases. In Sec. IV theoretical results are compared with numerical simulations.

II. THEORY FOR RANDOMLY MODULATED SYSTEMS

We consider systems that can be modeled by one-dimensional equations of the form

$$\dot{x}(t)=F(a(t),x)+\eta(t), \tag{2.1}$$

where $a(t)$ is a randomly modulated control parameter and $\eta(t)$ is a Gaussian white noise with zero mean value and intensity D,

$$\langle \eta(t)\eta(s)\rangle = D\delta(t-s). \tag{2.2}$$

We assume for the nonlinear function F the conditions

$$F(a(t),0)=0; \quad F(a(t),x)\sim a(t)x \quad \text{when } |x|<x_r, \tag{2.3}$$

where x_r is a reference value such that the noise can be neglected when $|x|>x_r$. Then the quasideterministic approach [2] can be applied by considering two kinds of evolution. When the variable of interest $|x|$ is small ($|x|<x_r$) the evolution is linear and noise effects are important. In the deterministic regime ($|x|>x_r$) noise can be neglected but nonlinear effects are important. It is also assumed that there are two attractors $x_{\rm off}=0$ and $|x_{\rm on}|\gg x_r$ for $|x|$ corresponding to values $-a_{\rm off}<0$ and $a_{\rm on}>0$ of the control parameter, respectively. Two different situations can be considered: one positive attractor $x_{\rm on}$ or two attractors with the same modulus. This situation appears, for instance, when the evolution is symmetric, $F(a(t),x)=-F(a(t),-x)$.

The control parameter $a(t)$ can have two different evolutions during one period T. During an *on* period a crosses the instability point and a pulse is generated. In this case the control parameter first increases reaching a maximum value $a_{\rm on}$. A decreasing evolution follows towards a minimum value $-a_{\rm off}$. In the *off* periods a is below the instability point and the system stays below the reference value, $|x|<x_r$ (this corresponds to a ''0''). A sequence of *on* and *off* periods with a probability p for an *on* period and $(1-p)$ for the *off* period is considered. When $p=1/2$ a fully random sequence of pulses and zeros is generated. A possible evolution for $|x(t)|$ and $a(t)$ is shown in Fig. 1, where a sequence ''1 0 1'' has been plotted.

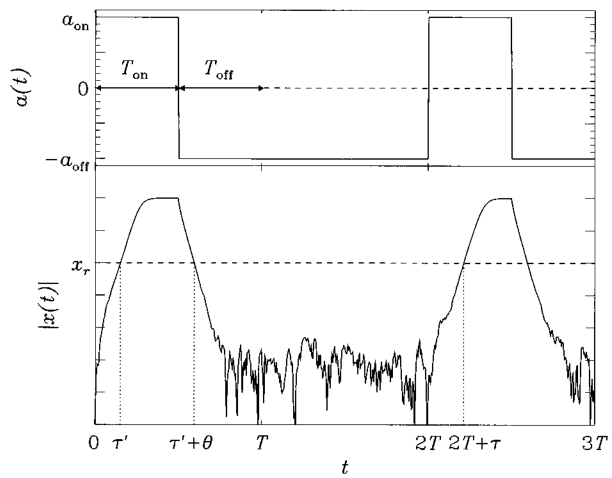

FIG. 1. Time evolution for the control parameter $a(t)$ (in linear scale) and the variable of interest $|x(t)|$ (in logarithmic scale) during an ''*on-off-on*'' sequence.

We assume that in the *on* periods all the pulses decay from their maximum values to reach the reference value x_r. This corresponds to a control parameter that takes large negative values during a large enough fraction of the period. The different evolution regions are shown in Fig. 1. The first region is defined from the beginning of the *on* period with $a(0)=a_{\rm on}$ until a time τ' such that $|x(\tau')|=x_r$. It is a region such that $|x(t)|<x_r$ dominated by fluctuations where the linear equation

$$\dot{x}(t)=a(t)x(t)+\eta(t) \tag{2.4}$$

holds [see (2.3)]. In the second region from τ' to $\tau'+\theta(\tau')$ such that $|x(t)|>x_r$ the noise is not relevant. Then the time spent in this deterministic region $\theta(\tau')$ can be calculated from the deterministic equation

$$\dot{x}(t)=F(a(t),x(t)) \tag{2.5}$$

as a function of τ'. In this region the control parameter crosses the instability point from above. The last region until the end of the period is similar to the first one, but now the control parameter is below the instability point. In the following *off* period a is always below the instability point. Then it can be considered as a prolongation of the last region of the *on* period. This region lasts until the reference value x_r is crossed at time $2T+\tau$ in the last pulse shown in Fig. 1.

Having defined the model and the different evolution regions we now generalize the method used in the periodic modulation case [6] to the case of random modulation. Our task is to obtain the passage time distribution $P(\tau)$. Since τ is associated to pulse emission, it is defined only in the *on* periods. The passage time is defined as the delay between the beginning of the *on* period and the time in which the reference value x_r is reached. In order to connect two consecutive *on* periods we consider the function $W_m^T(\tau/\tau')$, defined as the conditional probability to have a passage time τ in an *on* period when the passage time of the previous *on* period was

τ' and when there are m *off* periods between both periods. The m *off* periods between two consecutive pulses can be considered as a prolongation of the last noise-dominated region of an *on* period. Then we have

$$W_m^T(\tau/\tau')=W_0^{(m+1)T}(\tau/\tau'), \tag{2.6}$$

where $W_0^{(m+1)T}(\tau/\tau')$ is the conditional probability corresponding to a periodic case with a period $(m+1)T$ that is formed by the *on* period followed by m *off* periods. This conditional probability can be obtained from the expression derived for the periodic modulation case [6].

In the stationary regime $P(\tau)$ is independent of the initial condition and the following chain equation holds:

$$P(\tau)=\sum_{m=0}^{\infty} p(1-p)^m \int_0^T W_0^{(m+1)T}(\tau/\tau')P(\tau')d\tau'. \tag{2.7}$$

This is the main equation that allows the calculation of the statistical properties of the passage time. Moreover, the statistics of the pulse characteristics can be obtained [6] from $P(\tau)$.

III. RANDOM ON-OFF MODULATION. MODEL AND ANALYTICAL RESULTS

In this section we apply our method to the following dynamical system:

$$\dot{E}=a(t)E-A|E|^2E+\phi(t), \tag{3.1}$$

where $a(t)$ is the randomly modulated control parameter and ϕ is a Gaussian white noise. When E is a real magnitude (3.1) corresponds to the Landau equation and it has been used to study the Rayleigh-Bénard convection with a control parameter modulated around the threshold [7]. When $E=E_1+iE_2$ is a complex magnitude (3.1) can describe, for instance, the electric field of a single-mode class-A laser [18] near threshold. In this case $a(t)$ corresponds to the pump parameter and $\phi=\phi_1+i\phi_2$ is the spontaneous emission noise with intensity D and correlations

$$\langle \phi_i(t)\phi_j(t')\rangle = D\delta_{i,j}\delta(t-t'). \tag{3.2}$$

In the following we apply the theory developed in the previous section to this dynamical system. The random modulation of $a(t)$ can be obtained in the case of a class-A laser by varying the quality factor of the cavity (Q-switching), for instance with an acousto-optic modulator [14], or by gain switching. In this one-dimensional dynamical system both processes lead to the same result. However, when the dynamics of the population inversion must be also considered (class-B lasers), the statistical properties of pulses for gain and loss switching are different [19]. The model with a complex variable can be treated with the method shown in the previous section by introducing the intensity $I(t)=E_1^2(t)+E_2^2(t)$ and taking into account that in the linear noise-dominated region both field components are decoupled. For small enough D a reference value for the intensity I_r can be introduced to separate this region, $I<I_r$, from the deterministic region, $I>I_r$.

Now we consider a modulation that involves only two values of $a(t)$. In an *on* period $a(t)$ takes the value $a_{\rm on}>0$ above threshold during the first $T_{\rm on}=\alpha T$ fraction of period T and the value $-a_{\rm off}<0$ below threshold during the second part $T_{\rm off}=(1-\alpha)T$ (see Fig. 1). Obviously, during an *off* period the parameter $a(t)$ has the value $-a_{\rm off}$ during all the period time. We assume in the following that a pulse is always emitted during $T_{\rm on}$. The passage time is then smaller than $T_{\rm on}$. It is also assumed that the reference level I_r is crossed in the pulse decay during $T_{\rm off}$. In a way similar to the periodic modulation case (see [6] for a detailed calculation) one obtains for the conditional probability $W_0^{\mathcal{T}}(\tau/\tau')$ the expression

$$W_0^{\mathcal{T}}(\tau/\tau')=\frac{a_{\rm on}I_r}{\sigma_Z^2}e^{2a_{\rm off}[\mathcal{T}-\theta(\tau')-\tau']}e^{-2a_{\rm on}\tau} \times\exp\left[-\frac{I_r(1+e^{2a_{\rm off}[\mathcal{T}-\theta(\tau')-\tau']-2a_{\rm on}\tau})}{2\sigma_Z^2}\right] \times\mathcal{J}_0\left[\frac{I_r}{\sigma_Z^2}e^{a_{\rm off}[\mathcal{T}-\theta(\tau')-\tau']-a_{\rm on}\tau}\right], \tag{3.3}$$

where τ and τ' are smaller than $T_{\rm on}$, $\mathcal{J}_0$ is the Bessel function of the first order [20], and σ_Z is given by

$$\sigma_Z^2=\frac{D}{2}\left[\frac{1}{a_{\rm off}}(1-e^{-2a_{\rm off}[\mathcal{T}-\theta(\tau')-\tau']}) +\frac{1}{a_{\rm on}}(1-e^{-2a_{\rm on}\tau})\right]e^{2a_{\rm off}[\mathcal{T}-\theta(\tau')-\tau']}. \tag{3.4}$$

The time spent in the nonlinear deterministic region, $\theta(\tau')$, can be calculated [6] from the deterministic equation for the intensity with the initial $I(\tau')=I_r$ and final $I(\tau'+\theta)=I_r$ conditions.

The stationary passage time probability $P(\tau)$ can be calculated from (3.3), (3.4), and the integral equation (2.7). The validity conditions of this method can be explicitly obtained in the following way [6]. The reference value I_r must be in a deterministic region, $I_r\gg D/\min\{a_{\rm on},a_{\rm off}\}$, such that saturation effects are negligible, $AI_r\ll\min\{a_{\rm on},a_{\rm off}\}$. Then the following condition is obtained:

$$D\ll\left[\frac{(\min\{a_{\rm on},a_{\rm off}\})^2}{A}\right]. \tag{3.5}$$

To understand the behavior of the system under random modulation conditions it is useful to introduce two dimensionless parameters:

$$r_{\rm on}=a_{\rm on}T_{\rm on},\quad r_{\rm off}=a_{\rm off}T_{\rm off}. \tag{3.6}$$

The first parameter $r_{\rm on}$ is associated with the pulse rising and it corresponds to the ratio between the time spent by the system above threshold and the time needed to reach the *on* state. The second one, $r_{\rm off}$, gives the ratio between the time spent by the system below threshold and the pulse decay time. These parameters can be used to characterize all possible situations. The case $r_{\rm on}\ll 1$ is not considered because it corresponds to a situation such that the probability of occur-

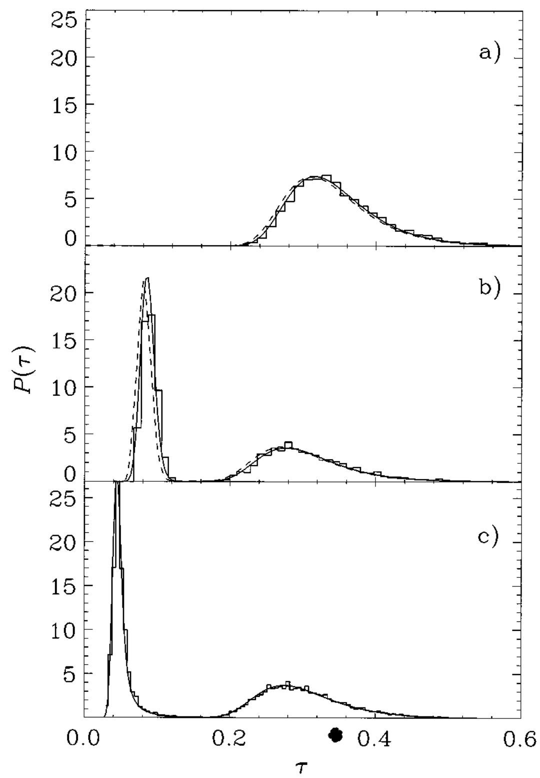

FIG. 2. Probability density function $P(\tau)$ for different values of the parameters (a) $T=2$, $a_{\rm off}=10$, $\alpha=0.6$; (b) $T=2$, $a_{\rm off}=3$, $\alpha=0.6$; (c) $T=1.2$, $a_{\rm off}=3$, $\alpha=0.45$. Other parameters are $a=10$, $A=1$, $D=10^{-3}$, $I_r=0.1$, and $p=0.5$. Histograms correspond to numerical simulations of Eq. (3.1), solid lines correspond to theory [Eq. (2.7)], and dashed lines to the analytical approximations given by Eq. (3.9) (a) and Eq. (3.16) (b).

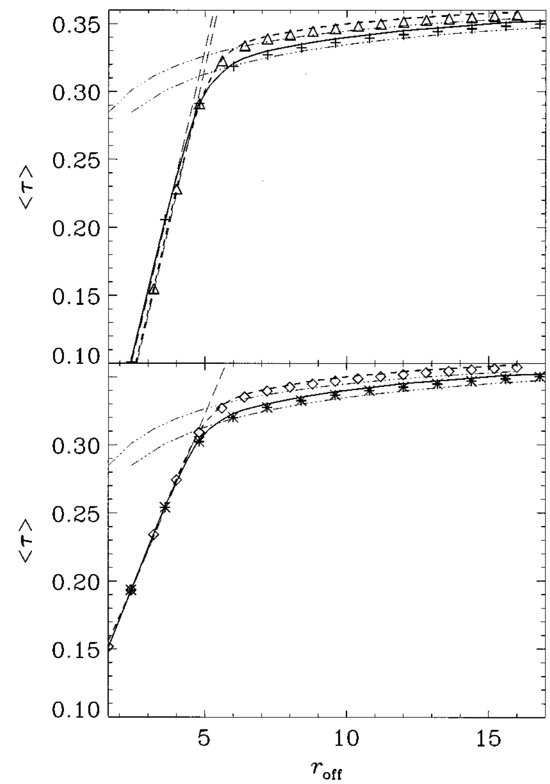

FIG. 3. Mean switch-on time $\langle\tau\rangle$ as a function of $r_{\rm off}$ for periodic, $p=1$ (upper plot) and random, $p=0.5$ (lower plot) modulation. Asterisks and crosses (diamonds and triangles), obtained from simulations, and solid (dashed) lines, obtained from theory [Eq. (2.7)] correspond to $\alpha=0.4$ ($\alpha=0.6$). Three-dot–dashed lines correspond to Eq. (3.10) and long-dashed lines correspond to Eq. (3.17). Parameter values are the same as in Fig. 2, with $T=2$ and $a_{\rm off}$ variable.

rence of a pulse in an *on* period is low. Note that in our analysis we have assumed that in an *on* period a pulse is always emitted. In the opposite situation, $r_{\rm on}\gg 1$, the system spends enough time above threshold to reach the *on* state, given by $I_{\rm on}=a_{\rm on}/A$. Then, all the pulses saturate to this value. In this case a complete analytical treatment is possible (see below). In the intermediate case $r_{\rm on}\sim 1$ different pulse heights are obtained for different values of τ. The probability density of the pulse maximum intensity, $P_H(I_m)$, can be easily obtained [6] from $P(\tau)$ by solving the deterministic equation for I with the initial condition $I(\tau)=I_r$. The pulse height is given by $I_m=I(\alpha T)$ as a function of τ. By using the inverse function [6]

$$\tau(I_m)=\alpha T-\frac{1}{2a_{\rm on}}\ln\frac{I_m(AI_r-a_{\rm on})}{I_r(AI_m-a_{\rm on})}, \tag{3.7}$$

we get the pulse height distribution

$$P_H(I_m)=P[\tau(I_m)]/[2I_m(AI_m-a_{\rm on})]. \tag{3.8}$$

With regard to pulse decay, two cases $r_{\rm off}\gg 1$ and $r_{\rm off}\sim 1$ are considered. In the first case the *off* state is reached at the end of an *on* period. Then the initial condition for a pulse always corresponds to this state. However, in the second case the behavior of the system during a pulse depends on the previous sequence of ''zeros'' and pulses. Note that in this case $r_{\rm off}$ cannot be very small, since we have assumed that the reference level I_r is always crossed during the pulse decay.

A. Repetitive switching ($r_{\rm off}\gg 1$)

For large values of $r_{\rm off}$ the system is below threshold at the end of an *on* period during a large enough time $T_{\rm off}$ to reach the *off* state. In this steady state associated with a control parameter $-a_{\rm off}$ the intensity is mainly due to the noise. In this case the system is in the *off* state at the end of all the periods. Then the initial condition at the beginning of a pulse always corresponds to this stationary state (repetitive switching), and the conditional probability $W_0^{(m+1)T}$ is independent of the number m of ''zeros'' between two pulses. Therefore the passage time distribution is independent of the kind of modulation, periodic or random. As shown in [6] in these conditions the argument of the Bessel function in (3.3) is small. Then, using [20] $\mathcal{J}_0\sim 1$, the conditional probability (3.3) becomes separable and one recovers from (2.7) and (3.3) the well-known expression of $P(\tau)$:

$$P_r(\tau)=N\exp\left[-2a_{\rm on}\tau-\frac{a_{\rm off}a_{\rm on}I_r}{D(a_{\rm on}+a_{\rm off})}e^{-2a_{\rm on}\tau}\right]. \tag{3.9}$$

The mean passage time and variance are readily obtained as

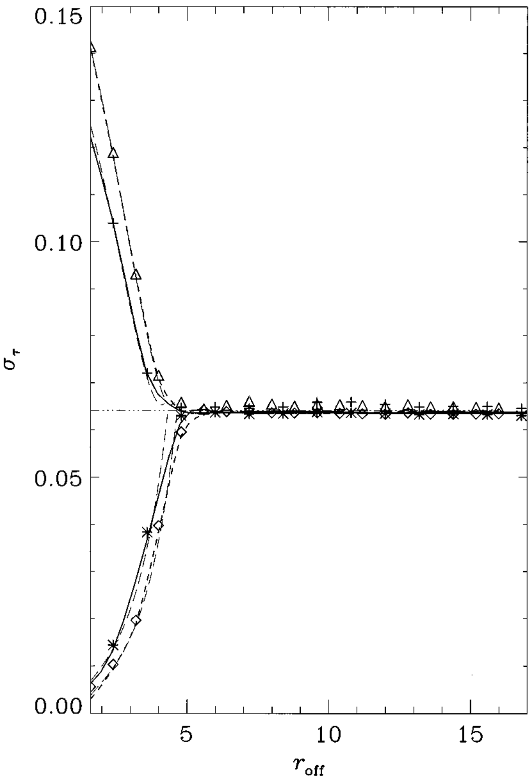

FIG. 4. Variance of the passage time σ_τ as a function of $r_{\rm off}$ for the same parameter values as in Fig. 3. Long-dashed lines correspond to Eq. (3.18).

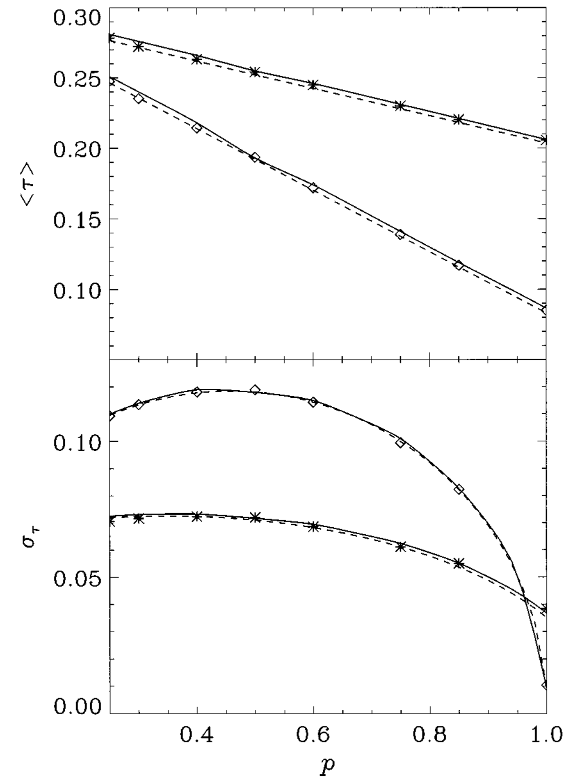

FIG. 5. Mean value $\langle \tau \rangle$ and variance σ_τ of the passage time as a function of p for two values of α as obtained from simulation (symbols) and from theory (lines). Asterisks correspond to $\alpha = 0.4$ and diamonds to $\alpha = 0.6$. Solid lines correspond to Eq. (2.7) and dashed lines to Eqs. (3.17) and (3.18)

$$\tau_r \sim \frac{1}{2a_{\rm on}}\left(\ln\left[\frac{I_r a_{\rm on} a_{\rm off}}{D(a_{\rm on}+a_{\rm off})} \right] - \psi(1) \right), \qquad \sigma_r^2 \sim \frac{\psi'(1)}{4a_{\rm on}^2}, \tag{3.10}$$

ψ and ψ' being the digamma function and its derivative, respectively [20].

B. Memory and pattern effects ($r_{\rm off} \sim 1$)

When $r_{\rm off}$ is not large enough the *off* state is not reached during the pulse decay. Then the initial value for the intensity at the beginning of a pulse depends on whether the previous bit was a ''zero'' or a ''one'' (a pulse). In this case we say that the system has memory or that pattern effects are important. The system has a memory of length n (or n-period memory) if the behavior of the system during a pulse depends on the previous n periods. This memory length can be estimated as the number of *off* periods required after one pulse to reach the *off* state. In this case the system stays below threshold with $a = -a_{\rm off}$ during a time $T_{\rm mem} = T_{\rm off} + nT$ such that $a_{\rm off} T_{\rm mem} \gg 1$. Therefore, after n ''zeros'' the initial condition for a pulse will correspond to the *off* state. As a consequence there are only $n+1$ different terms in the integral equation (2.7), since the conditional probability $W_0^{(m+1)T}$ corresponds to that of the repetitive switching case for $m \geq n$.

Due to the memory, a multimodal distribution for the passage time can be obtained. The statistical properties can be understood as a superposition of the statistical properties associated with the last ''one'' of 2^n different periodic sequences. Each of these sequences, composed of $n+1$ bits, is fixed and is obtained by taking one of the 2^n possible distributions of n bits ''zero'' and ''one'' and a ''one'' as the last bit. The passage time distribution $P(\tau)$ for the random modulation case with $p = 1/2$ can be obtained in the following way [16]: When a sequence is periodically repeated the passage time distribution associated with the last ''one'' of the sequence is obtained. $P(\tau)$ is given by the superposition of these 2^n passage time distributions associated with the different periodic sequences. The maximum number of peaks of $P(\tau)$ is 2^n in the n-memory case. However, it should be noted that one is not likely to observe 2^n peaks since many of the different sequences could give close peaks which are not distinguishable. This will happen when the separation between the peaks is smaller than the width of the passage time distribution associated to the sequences. Multimodal distributions due to memory effects have also been observed in numerical simulations of semiconductor lasers modulated at high speed [16]. Two sequences are especially relevant: the periodic sequence without ''zeros'' between pulses and the one associated with the repetitive switching. This last sequence corresponds in the n-memory case to n ''zeros'' between two consecutive pulses. The intensity at the beginning of a pulse decreases when the number of previous ''zeros'' increases. As a consequence, the largest and smallest values for the passage time will correspond to the repetitive and periodic sequences, respectively. Therefore, a multimodal distribution will be observed when the overlapping between the probability distributions associated with these two sequences is small [16].

It is interesting to consider the limiting situation such that at the end of a pulse the intensity is below the reference level I_r, but large enough for the fluctuations to be negligible. This case corresponds to small values of $r_{\rm off}$. Then in the periodic modulation case the intensity at the beginning of a pulse is also large, and the switch on is mainly deterministic (that is, due to the stimulated emitted photons in the case of a laser). As shown in [6] in this deterministic limit when the system is above threshold on the average, i.e., $r_{\rm off}<r_{\rm on}$, a deterministic steady state is reached for the periodic modulation case, $P_{\rm pm}(\tau)=\delta(\tau-\tau_D)$, where

$$\tau_D=\frac{1}{2a_{\rm on}}\left[\ln\left(\frac{AI_r(a_{\rm on}+a_{\rm off})}{a_{\rm on}a_{\rm off}}\right)+2a_{\rm off}T(1-\alpha)\right.$$
$$\left.-\ln(1-e^{-2a_{\rm on}\alpha T+2a_{\rm off}(1-\alpha)T})\right]. \qquad (3.11)$$

Since $r_{\rm off}<r_{\rm on}$ the last term in τ_D can usually be neglected. Then, the switch-on time in the periodic modulation case is approximately a linear function of $r_{\rm off}$ with a slope given by $(a_{\rm on})^{-1}$ [6]. In this deterministic limit an analytical condition for the multimodal character of the distribution $P(\tau)$ can be given. Using the criterion based on the overlapping between the distributions associated with the periodic and repetitive sequences the condition $(\tau_r-\tau_D)>\sigma_r$ [see Eqs. (3.9) and (3.10)] is obtained.

C. Pulse saturation $r_{\rm on}\gg 1$

In this limiting case the pulses always saturate to reach the *on* state: $I_{\rm on}=a_{\rm on}/A$. The intensity in an *on* period at time $T_{\rm on}$ is then given by $I_{\rm on}$. Therefore, the passage time τ is independent of the passage time of the previous pulse τ'. After changing the pump parameter from above to below threshold the system spends a decay time

$$\tau_1=\frac{1}{2a_{\rm off}}\ln\frac{a_{\rm on}(a_{\rm off}+AI_r)}{AI_r(a_{\rm off}+a_{\rm on})} \qquad (3.12)$$

to cross the reference level I_r. Then $\tau+\theta(\tau)$ is a constant given by $T_{\rm on}+\tau_1$ (see Fig. 1). Substituting $\tau'+\theta(\tau')$ by this constant in (3.4) and (3.3), we obtain a conditional probability $W_{\rm sat}^{(m+1)T}(\tau)$ that only depends on τ. The integral equation (2.7) can be trivially solved in this case obtaining for $P(\tau)$

$$P(\tau)=\sum_{m=0}^{\infty}p(1-p)^m W_{\rm sat}^{(m+1)T}(\tau). \qquad (3.13)$$

According to the previous discussion, when the memory length is n all the terms in (3.13) with $m\geqslant n$ correspond to the repetitive switching case given by (3.9). Therefore, only $(n+1)$ different terms must be considered in (3.13).

An interesting situation occurs when $r_{\rm off}$ is small (deterministic periodic modulation) and the memory length is 1, that is, $Ta_{\rm off}\gg 1$. In this case we have $W_{\rm sat}^{T}(\tau)\sim\delta(\tau-\tau_D)$, whereas $W_{\rm sat}^{(m+1)T}(\tau)$ for $m>0$ corresponds to the repetitive switching case (3.9). A better approximation for $W_{\rm sat}^{T}$ including the effect of fluctuations can be obtained in the following way. As shown in [6] the deterministic limit corresponds to a large value of the argument of the Bessel function in (3.3). Then, using [20] $\mathcal{J}_0(x)\sim\exp(x)/\sqrt{2\pi x}$ and replacing $\tau'+\theta(\tau')$ by $T_{\rm on}+\tau_1$ in (3.3) we get

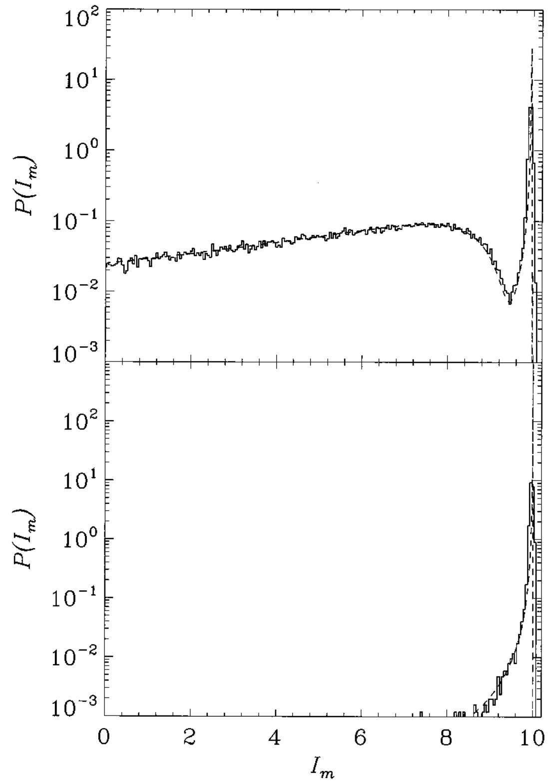

FIG. 6. Probability density function $P_H(I_m)$ as a function of height for $T=1.2$ (upper plot) and $T=2$ (lower plot). Histograms correspond to simulation and dashed lines to theory [Eq. (3.8)]. Other parameters are the same as in Fig. 2(c).

$$W_{\rm sat}^{T}(\tau)\sim P_D(\tau)=\frac{\exp[-a_{\rm on}(\tau-\tau_D)]}{\sqrt{2\pi\sigma_D^2}}$$
$$\times\exp\left[-\frac{(1-e^{-a_{\rm on}(\tau-\tau_D)})^2}{2a_{\rm on}^2\sigma_D^2}\right], \qquad (3.14)$$

where

$$\sigma_D^2=\frac{D(a_{\rm on}+a_{\rm off})}{2I_r a_{\rm on}^3 a_{\rm off}}\left[\frac{AI_r(a_{\rm on}+a_{\rm off})}{a_{\rm on}a_{\rm off}}e^{2r_{\rm off}}-1\right]. \qquad (3.15)$$

The mean and variance of the passage time associated with the deterministic periodic modulation are given by τ_D and σ_D^2, respectively.

By using (3.9) and (3.14) in (3.13) the following distribution is obtained for the random modulation case:

$$P(\tau)\sim pP_D(\tau)+(1-p)P_r(\tau). \qquad (3.16)$$

This distribution is bimodal when the separation between the peaks at τ_D and τ_r is greater than the width of $P_r(\tau)$ [see (3.10) and (3.11)]. The mean value of τ, $p\tau_D+(1-p)\tau_r$, is given by

$$\langle\tau\rangle=\frac{1}{2a_{\rm on}}\left[2pr_{\rm off}+p\ln AI_r-(1-p)\left(\psi(1)+\ln\frac{D}{I_r}\right)+(1-2p)\ln\frac{a_{\rm on}a_{\rm off}}{a_{\rm on}+a_{\rm off}}\right]. \quad (3.17)$$

When $p=1/2$ the last term in (3.17) is zero and the switch–on time is a linear function of $r_{\rm off}$ with a slope given by $(2a_{\rm on})^{-1}$. This slope is one-half of the one for τ_D in (3.11). The variance of τ is given from (3.16) by

$$\sigma_\tau^2=(1-p)\sigma_r^2+p\sigma_D^2+p(1-p)(\tau_r-\tau_D)^2. \quad (3.18)$$

This variance has a maximum value when the probability p of pulses is

$$p_{\rm max}=\frac{1}{2}-\frac{(\sigma_r^2-\sigma_D^2)}{(\tau_r-\tau_D)^2}. \quad (3.19)$$

In the bimodal case when the peaks are clearly separated $p_{\rm max}\sim 1/2$; that is the result for a random variable that can take only two values.

IV. COMPARISON BETWEEN THEORY AND NUMERICAL SIMULATIONS

In this section we compare theoretical and simulation results for the pulse statistics of the dynamical system (3.1). The stationary passage time probability density is obtained by solving the integral equation (2.7) with a standard numerical algorithm. In some cases (see Sec. III C) an analytical solution can be obtained. In the numerical simulations we have considered a transient of 100 periods to ensure that the system has reached the steady-state conditions. Very good agreement between theory and simulations is observed in Fig. 2 for different types of distributions $P(\tau)$. Figures 2(a) and 2(b) correspond to the case of pulse saturation, $r_{\rm on}\gg 1$, and different values of $r_{\rm off}$. In Fig. 2(a) this parameter is large and $P(\tau)$ is given by the repetitive switching distribution (3.9). When $r_{\rm off}$ is decreased a bimodal distribution is obtained in Fig. 2(b) due to memory effects. This situation corresponds to the last case discussed in the previous section with a memory length of 1 period. The two peaks are associated with the periodic modulation (nearly deterministic) and the repetitive switching. In this case the analytical result (3.16) for $P(\tau)$ is in good agreement with the numerical simulations. In Fig. 2(c) the period T decreases and then both $r_{\rm on}$ and $r_{\rm off}$ are also reduced. As a consequence different pulse heights are obtained (no pulse saturation). However, the passage time distribution is also bimodal, because the memory length is similar to that of the case in Fig. 2(b).

In Figs. 3 and 4 we show the mean value and variance of the passage time versus $r_{\rm off}$ for the pulse saturation case. Analytical results have been obtained from (3.13). The values of $a_{\rm on}$ and T are kept fixed and the value of $a_{\rm off}$ is changed. Two different values of α are considered that fulfill the condition $r_{\rm on}\gg 1$. In this way the value of $r_{\rm off}$ can be changed by varying $a_{\rm off}$ or $T_{\rm off}=(1-\alpha)T$. Two different types of modulation, periodic $(p=1)$ and random $(p=1/2)$, are compared in these plots. Very good agreement between theory and simulations is obtained. For large values of $r_{\rm off}$ the repetitive switching regime is reached and the results [see (3.10)] are independent of the type of modulation. When $r_{\rm off}$ decreases $\langle\tau\rangle$ and σ_τ are given by (3.17) and (3.18), respectively. This situation corresponds to a deterministic periodic modulation and 1 period memory. A linear behavior is observed for the mean passage time versus $r_{\rm off}$ with a relation of one half between the slopes of the random and periodic modulation cases, as predicted by the theory. In the random modulation case $\langle\tau\rangle$ is only a function of $r_{\rm off}$ and the same result is obtained for the two values of α. However, a slight difference is observed in the periodic modulation case due to the last logarithmic term in (3.17). With regard to the variance, when $r_{\rm off}$ decreases a very different behavior is observed in Fig. 4 for the periodic and random modulation cases. In the first case ($p=1$, only pulses) intensity at the beginning of a pulse increases when $r_{\rm off}$ decreases. Then the fluctuations decrease. However, in the random modulation case a bimodal distribution appears, and σ_τ is mainly due to the separation between the peaks of $P(\tau)$. Then the linear behavior of σ_τ with $r_{\rm off}$ is due to the linear increase of τ_D with this parameter.

The variation of the statistical parameters $\langle\tau\rangle$ and σ_τ with the probability p of occurrence of pulses is shown in Fig. 5. We have considered the pulse saturation case for two different values of α. Both values correspond to the situation described in Sec. III C by (3.16). Then the analytical results can be obtained from (3.17) and (3.18). Very good agreement is found between theory and simulations. The first value $\alpha=0.6$ corresponds for $p=1/2$ to the bimodal distribution shown in Fig. 2(b). In this case, according to (3.19), the maximum value of the variance is obtained when $p\sim 1/2$. In the second case ($\alpha=0.4$) the peaks of the distribution are not clearly separated, since we have $(\tau_r-\tau_D)\sim(\sigma_r+\sigma_D)$. Then the maximum of σ_τ is obtained for a smaller value of $p\sim 0.3$.

Finally we have analysed the statistics of pulse heights I_m. The corresponding density $P_H(I_m)$ is obtained from the passage time distribution by using (3.8). In Fig. 6(a) the pulse height distribution for the case of Fig. 2(c) is shown. In this case $r_{\rm on}$ is not large enough for all the pulses to saturate and a bimodal distribution for the pulse heights is obtained. When there is a ''zero'' before a pulse the passage time is large, and then the *on* state $I_{\rm on}$ is not reached during $T_{\rm on}$. These passage times correspond to the broad peak in Fig. 2(c). The narrow peaks in Figs. 2(c) and 6(a) are associated with pulses that follow another pulse. When the period is increased all the pulses have enough time during $T_{\rm on}$ to saturate and a distribution peak around $I_{\rm on}$ is obtained [see Fig. 6(b)]. In this case the parameters are similar to those of Fig. 2(b).

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