Fock states in a Kerr medium with parametric pumping

W. Leoński*
Nonlinear Optics Division, Institute of Physics, Adam Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland

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We discuss a model involving the interaction of a single-mode field with both a Kerr medium and a parametric nonlinearity of the kth order. The system is externally driven by the parametric pump field. We show that applying an appropriately chosen field frequency, and for weak nonlinear process, k-photon Fock state of the electromagnetic field can be achieved. [S1050-2947(96)04909-8]

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Fock number states are commonly used in the theoretical description of quantum fields. Nevertheless, the experimental production of such states is no trivial problem. For instance, Stoler and Yurke [1] have studied theoretically the possibility of generation of antibunched light. Hong and Mandel [2] have shown that a one-photon Fock state is produced in the parametric downconverter; they used signal photons to gate the photodetector counting the corresponding idler photon. Another way of experimental preparation of an n-photon state is based on quantum nondemolition (QND) measurement. Brune *et al.* [3,4] have suggested a method for the preparation of a Fock state based on the model in which detection of the atomic phase by the Ramsey method plays the role of a QND probe giving information on the cavity field energy. After a sequence of atomic measurements the cavity field collapses into a Fock state with an unpredictable number of photons. Moreover, the system in which two-level atoms injected into a kicked cavity gives us the possibility of generation of highly excited Fock states [5]. Quite recently Kozierowski and Chumakov [6] have shown that in the spontaneous emission of the partially inverted Dicke model Fock states are also generated.

In this paper, we propose a rather general model, in fact, a group of models, which can lead to n-photon state generation. This model is an extension of the systems discussed in [7,8]. It combines the evolution of a Kerr nonlinear medium in a cavity and a weak process of the nth order. We show that for a sufficiently weak process and for an appropriately chosen field frequency, resonance effects start to play a significant role and lead to n-photon state generation. The nonlinear quantum evolution of the cavity field in the Kerr medium is crucial for the preparation of a Fock state in such a system. The effectiveness of the preparation is, however, considerably diminished by the cavity losses. Nevertheless, it seems important to us that a cavity with a nonlinear Kerr medium, with a field initially in vacuum state, and a nonlinear process, can lead with high accuracy to an n-photon Fock state. For this situation we will derive analytical formulas for the probabilities corresponding to the Fock states we are interested in. Moreover, we will perform a numerical experiment in which we simulate the dynamics of our system and compare the results with those of an analytical attempt.

*Also at Institute for Physico-Medical Research, Skośna 1, 61-801 Poznań, Poland.

One should mention that the particular case of our system has been discussed in the context of a comparison between the classical and quantum dynamics in regions of classically regular and chaotic behaviors [9–11]. A similar system has been discussed from the point of view of the photon statistics [12,13]. Moreover, one-photon state generation for a periodically kicked Kerr medium has been discussed earlier [7,8] and the system described in this paper is a generalization of that proposed in those papers.

Thus, our model contains an anharmonic oscillator (Kerr medium) driven by a k-order process. The latter is governed by the following Hamiltonian (in the interaction picture):

$$\hat{H}_{NL}=\epsilon(\hat{a}^{\dagger k}+\hat{a}^{k}) \quad , \tag{1}$$

where ϵ denotes the strength of the nonlinear process, whereas $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators of the field, respectively. As it was mentioned previously, we deal in fact with a group of models. Thus, for $k=1$ we discuss the system involving coherent pumping [7,8]. For $k=2$ the model is equivalent to that referred to as *Cassinian oscillator* [14]. For this case the model involves the interaction of a single mode of the field with both an intensity-dependent refractive index and a parametric nonlinearity. The system is driven externally entirely by the parametric pump field. This model ($k=2$) has been discussed in numerous papers; for instance, by Tombesi and Yuen [12] and by Gerry and Rodriques [13]. Since we discuss here rather general models, we should mention about isolation of the kth order process from the processes of lower orders. Since we have involved parametric processes, the phase matching condition between the fields of the frequencies ω and ω/k should be satisfied. Moreover, our models contain resonances of the field with the cavity. In addition, the resonances between the levels generated by the Kerr Hamiltonian and the parametric process (we will mention about this problem further on) play a crucial role in our models. All of these processes are responsible for choosing of the appropriate kth-order process.

The Hamiltonian corresponding to the dynamics of the Kerr medium can be written as follows (we use units of $\hbar=1$):

$$\hat{H}_{Kerr}=\frac{\lambda}{2}\hat{n}(\hat{n}-z), \tag{2}$$

where λ is proportional to the third-order nonlinear susceptibility and $\hat{n}$ is the photon number operator. The value of the factor z appearing in (2) can be controlled by appropriately chosen field frequency. For our purposes this value should be equal to the order of the nonlinear process described by the Hamiltonian H_{NL}, i.e., $z=k$. Thus the Hamiltonian governing the whole system is

$$H=\epsilon(\hat{a}^{\dagger k}+\hat{a}^{k})+\frac{\lambda}{2}\hat{n}(\hat{n}-k). \tag{3}$$

Our aim here is to determine the time evolution of the system. We assume that the system was initially in the vacuum state $|0\rangle$. The history of our system is governed by the unitary evolution operator $\hat{U}(t)$ defined as follows:

$$\hat{U}(t)=\exp(-i\hat{H}t). \tag{4}$$

Hence, the wave function $|\Psi(t)\rangle$ for arbitrary time t can be expressed as:

$$|\Psi(t)\rangle=\hat{U}(t)|0\rangle. \tag{5}$$

Although we already can apply here the method of numerical diagonalization as performed in, [7] we shall first try to find analytical formulas for the cases of interest to us. Thus we assume here that the nonlinear process (1) is weak, i.e., $\epsilon\ll\lambda$. In consequence, we are able to treat our problem perturbatively.

Let us express the wave function in a Fock basis:

$$|\Psi(t)\rangle=\sum_{j=0}^{\infty} a_j(t)|j\rangle. \tag{6}$$

This wave function obeys a Schrödinger equation with Hamiltonians expressed by Eqs. (1) and (2):

$$i\frac{d}{dt}|\Psi(t)\rangle=(H_{Kerr}+H_{NL})|\Psi(t)\rangle. \tag{7}$$

Applying the standard procedure to the wave function (6) and the Hamiltonian (3) we obtain a set of equations for the probability amplitudes a_j. They are of the form:

$$\begin{aligned} i\frac{d}{dt}a_j(t)=&\frac{\lambda}{2}j(j-k)a_j+\epsilon\sqrt{j(j-1)\cdots(j-k+1)}a_{j-k}\\ &+\epsilon\sqrt{(j+1)(j+2)\cdots(j+k)}a_{j+k},\\ &j=0,1,2,\ldots, \end{aligned} \tag{8}$$

where k denotes the order of the nonlinear process. Obviously, one should keep in mind that for $j<0$ we have $a_j=0$. Although we see from (8) that the set of equations for a_j is infinite, the nonlinear process cuts some subspace of states out of all of the Fock states. In consequence, the dynamics of the physical process starts from the vacuum $|0\rangle$ and is restricted to the states $|mk\rangle$, where $(m=0,1,2,\ldots)$. Moreover, the nonlinear process is weak ($\epsilon\ll\lambda$) and we can apply the standard rediagonalization procedure. Nevertheless, the crucial point of our considerations is the fact that the unperturbed Hamiltonian for the Kerr process (2) produces degenerate states $|0\rangle$ and $|z\rangle$, where $(z=k)$. In consequence, we deal here with some kind of resonance between the states produced by the Hamiltonian (2) and the interaction governed by the Hamiltonian (1). Obviously, the character of this resonance differs from those for the commonly discussed situations when the field frequency is equal to the cavity frequency. This situation resembles that for degenerate atomic levels coupled to each other by the zero-frequency field, where this resonant interaction selects from the whole set of levels only two of them leading to the two-level atom dynamics. Interaction with the remaining atomic levels can be treated as a negligible perturbation [15]. In practice, we deal here with a situation analogous to that discussed above and we can write the following equations of motion for the probability amplitudes:

$$\begin{aligned} i\frac{d}{dt}a_0(t)&=\epsilon\sqrt{k!}a_k,\\ i\frac{d}{dt}a_k(t)&=\epsilon\sqrt{k!}a_0. \end{aligned} \tag{9}$$

Assuming $a_0(t=0)=1$ and $a_k(t=0)=0$ $(k\neq 0)$ we get the following solution for the probability amplitudes

$$\begin{aligned} a_0&=i\cos(\epsilon\sqrt{k!}t),\\ a_k&=\sin(\epsilon\sqrt{k!}t). \end{aligned} \tag{10}$$

We treat Eqs. (10) as the zero-order solution. For this order the amplitude $a_{2k}=0$. To obtain the formula for a_{2k} we need higher-order solutions. We write the first-order formula for a_{2k}:

$$a_{2k}=-\frac{\epsilon\sqrt{(2k)!}}{\lambda k^2\sqrt{k!}}\sin(\epsilon\sqrt{k!}t)+O(\epsilon^2), \tag{11}$$

where we have removed all terms proportional to ϵ^2. Obviously, we are in a position to perform this perturbative procedure due to the fact that the nonlinear process (1) is weak, i.e., $\epsilon\ll\lambda$. Moreover, since we are interested in finding the time evolution of the probabilities rather than the amplitudes a_j, we neglect the influence of the dynamics of the state $|2k\rangle$ on the system as proportional to ϵ^2.

To verify these results we shall now perform a numerical experiment and compare its results with those based on our formulas. This will be done on the same basis as that described in [7]. Namely, we shall obtain the wave function for arbitrary time by computing numerically the evolution operator $\hat{U}$ [Eq. (4)] and applying it to Eq. (5). Of course, both of them, the wave function $|\Psi(t)\rangle$ and the operator $\hat{U}$, are expressed in the Fock-state basis. For our experiment we will chose the nonlinear process for $k=2$, so that we expect to obtain a pure two-photon Fock state.

Thus, Fig. 1 shows the probabilities of finding the system in the vacuum $|0\rangle$ and two-photon states $|2\rangle$. This figure corresponds to the case when $k=2$. We assume that for the time $t=0$ the field was in the vacuum state $[a_0(t=0)=1$ and $a_2=0]$. Moreover, the nonlinear process (1) is weak, $\epsilon=\pi/50$ (in units of $\lambda=1$). We see that our theoretical (analytical) results (solid and dashed lines) agree strongly with

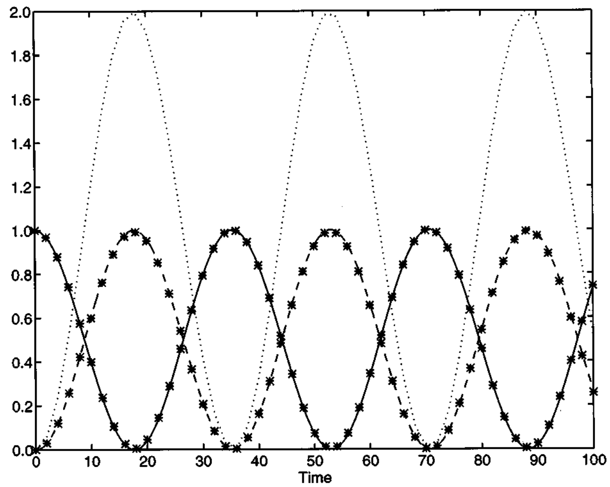

FIG. 1. Analytical solutions for the probabilities for the vacuum (solid line) and two-photon (dashed line) states, and the numerically found mean number of photons (dotted line). The parameter $\epsilon=\pi/50$ (all parameters are measured in units of $\lambda=1$). Star marks correspond to the probabilities found in the numerical experiment.

those generated in the numerical experiment (star marks). The system starts to evolve from the vacuum and after the time $t=\pi/(2\sqrt{2}\epsilon)\simeq 17.7$ the probability $|a_2|^2=1$. In practice, for this moment of time the field is in the pure two-photon state. For longer times the system returns to its initial state and starts to evolve in the same way as from $t=0$. Moreover, we have plotted in Fig. 1 the time dependence for the mean number of photons $n(t)$ (dotted line)

$$n(t)=\langle\Psi(t=0)|\hat{U}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{U}|\Psi(t=0)\rangle \tag{12}$$

found in our numerical experiment. We see that the behavior of $n(t)$ reflects the evolution of the probabilities and oscillates between 0 and 2.

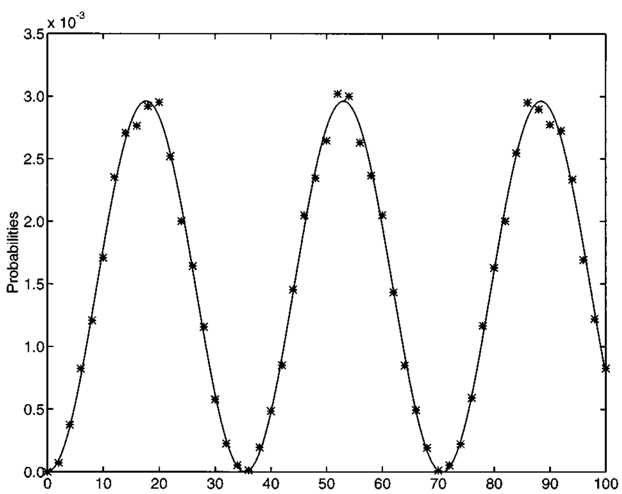

FIG. 2. Analytical solution (solid line) and numerical experiment results (star marks) for the probability corresponding to the four-photon state. All parameters are the same as in Fig. 1.

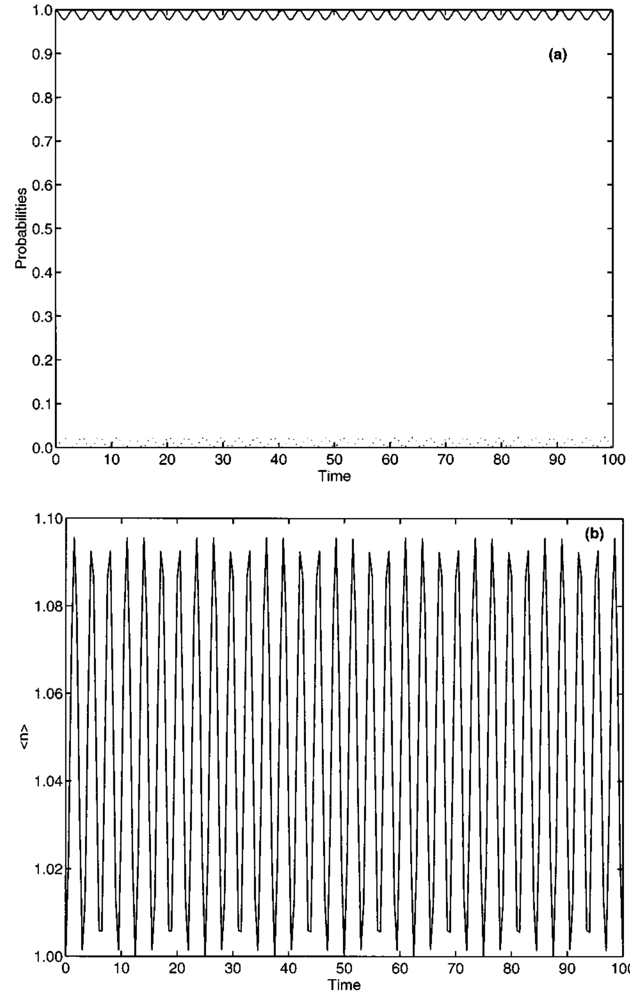

FIG. 3. Numerical experiment results for the system starting from the one-photon state: (a) shows the probabilities for $|1\rangle$ (solid line) and $|3\rangle$ (dotted line) states, (b) shows the mean number of photons. All parameters are the same as in Fig. 1.

It is obvious that the parametric process produces not only two-photon states but also states $|2k\rangle$ $(k=2,3,\ldots)$. However, the mentioned earlier resonance choses some of these states. As the parameter k in the Hamiltonian (3) is equal to 2 the state $|2\rangle$ is one state resonant to the vacuum state only. Thus, Fig. 2 corresponds to this situation and depicts the probability for the four-photon state $|4\rangle$, where the solid line corresponds to the formula (11), whereas the star marks originate from the numerical experiment. We again obtain good agreement between the results of the experiment and our theory. Tiny oscillations visible in the experimental results are proportional to ϵ^4 and can be neglected. We see from Fig. 2 that the influence of the coupling between the states $|2\rangle$ and $|4\rangle$ (and, in consequence, the couplings for higher states) is negligible. The oscillations of the probabilities are restricted to the subset of two states ($|0\rangle$ and $|2\rangle$) and the probability for the four-photon state is proportional to $\epsilon^2\simeq 3\times 10^{-3}$.

For cases when our system does not exhibit resonant features the situation changes drastically. Figure 3 corresponds to the same situation as that for Figs. 1 and 2, but we assume here that the field is in one-photon state for $t=0$. In consequence there are no resonant couplings between the Fock states (we are interested here mainly in the coupling between the states $|1\rangle$ and $|3\rangle$). We show in Fig. 3(a) the time dependence of the probabilities a_1 (solid line) and a_3 (dotted line) for the one-photon and three-photon states, respectively, generated in our numerical experiment. It is seen that the field remains in the one-photon state for the whole time and is only slightly perturbed by the interaction with the three-photon state. This is the result of the nonresonant interaction by way of the very weak nonlinear process. The mean number of photons [Fig. 3(b)] oscillates near its initial value 1. The amplitude of these oscillations is very small (0.05) and can be neglected in practice.

Of course, we should mention at this point about losses in our system, that may destroy the effects discussed here. This problem has been already discussed in the previous paper [7] where the similar model (with the kicked nonlinear oscillator) has been studied. It was shown that the damping constant γ should be much smaller than the nonlinearity λ. We realize that it is a very strong requirement for the experiment. Nevertheless, various experiments, for instance, the experiments where the very tiny effect of ''vacuum Rabi splitting'' [16,17] give us some hope for practical realization of our models.

We have shown here that it is possible to generate Fock states by the use of Kerr media. This generation is associated with resonant transitions between two Fock states and can be described analytically using standard perturbative procedure. Moreover, we have performed a numerical experiment that shows good agreement with our analytical solution. Of course, our considerations are of model character only and one should realize that many difficulties, for instance damping processes, can occur during the realization of practical experiments. Although the aim of this paper was not to investigate the influence of such obstacles, one should keep them in mind. A discussion of these problems was given in [7]. Thus, for instance, the ideas applied in experimental measurements of the very tiny effect of ''vacuum Rabi splitting'' [16,17] could be helpful in practical realizations of our model. Moreover, one should keep in mind that this paper describes a rather general group of models; for various values of k we deal with the models that differ completely from each other. Therefore it is difficult to propose the straightforward experimental method of realization of our system. Moreover, some nonlinear processes governed by the Hamiltonian $\hat{H}_{NL}$ are more suitable for experimental application than others. For instance, the model involving a Cassinian oscillator [9–14,18] ($k=2$) seems to be a reasonable proposal for experiment. At this point we should remind the problem of the isolation of the kth-order nonlinear process from the lower order ones. As we have mentioned earlier, the crucial role will play here phase matching and resonance effects. In addition, it is worth mention that the novel branch of optics, namely, *atomic optics*, [19] seems to be very helpful in practical realization of our models. However, this problem is not a subject of this paper, although it is worthy of further investigations.

Concluding, we believe, that we have found an interesting feature of quantum optics that can lead to the k-photon Fock-state generation.

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