Theory of elastic scattering of particles in a static potential field

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An approximate solution of the Dirac equation is obtained for particles scattering in arbitrary static potentials. Both short-range and long-range potentials are considered. [S1050-2947(96)02210-X]

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I. INTRODUCTION

The elastic scattering of charged particles, when the external field can be considered as a perturbation, has been studied in detail in Born and eikonal approximations (see, for example, [1]). It is known that the wave function in the eikonal approximation describes the particle state only in a specific range of potentials and the eikonal solution is not valid at distances $z>a$, where z is the coordinate along the direction of particle initial momentum $\vec{p}$ and a is the range of the interaction region. The wave function of the Born approximation, in contrast to the eikonal wave function, describes the particle state at arbitrary points, particularly at large distances. However, within the potential range where both approximations are applicable, these wave functions describe the scattering by different accuracies: the Born wave function is valid when $|U|\ll\hbar v/a$ and the eikonal wave function when $|U|\ll pv$, where U is the potential energy and v the initial velocity of the particle (of course, fast particles are considered, $pa\gg\hbar$, where the eikonal approximation is generally valid. Therefore the common region where both approximations under consideration are valid is very restricted. One can find a correspondence between them by expanding the eikonal wave function into a series over the Born parameter $Ua/\hbar v\ll 1$ and the Born wave function into a series over the impact parameter $h\vec{q}^2/2\vec{p}\cdot\vec{q}\ll 1$ ($\vec{q}$ is the transfer momentum), and keeping terms to first order in U [2].

The main purpose of this paper is to derive an alternative approximate solution of the Dirac equation for a charged particle scattering in a static potential, a solution that includes the well-known Born and eikonal wave functions as limiting cases. The organization of the paper is as follows. In Sec. II we shall present the generalized eikonal approximation (GEA) for particle scattering in short-range potentials and discuss its relationship to the Born and eikonal approximations. In Sec. III we shall apply the GEA to the scattering in a long-range Coulomb field. We shall also discuss the well-known Born conditions for the Coulomb field and demonstrate the relationship of the obtained GEA wave function to Born, eikonal, and Farry-Sommerfeld-Maue wave functions. In Sec. IV, using the GEA wave functions, we shall calculate the elastic scattering cross section and compare it with the Born and eikonal approximation cross sections.

II. THE WAVE FUNCTION OF PARTICLE ELASTIC SCATTERING IN THE GENERALIZED EIKONAL APPROXIMATION

Our starting point is the Dirac equation for a charged particle in an external static field with $\hbar=c=1$

$$\{[\varepsilon-U(\vec{r})]\gamma_0+\iota\vec{\gamma}\cdot\vec{\nabla}-m\}\Psi(\vec{r})=0, \tag{1}$$

where m is the mass, ε the energy, $U(\vec{r})$ the potential energy of the particle, and γ_0, $\vec{\gamma}$ the Dirac matrices. Introducing a bispinor function $\Phi(\vec{r})$ that is related to $\Psi(\vec{r})$ by the relation

$$\Psi(\vec{r})=\frac{1}{2m}\{[\varepsilon-U(\vec{r})]\gamma_0+\iota\vec{\gamma}\cdot\vec{\nabla}+m\}\Phi(\vec{r}) \tag{2}$$

turns (1) into the quadratic equation

$$\{[\varepsilon-U(\vec{r})]^2+\Delta-m^2+\iota\gamma_0\vec{\gamma}\cdot\vec{\nabla}U(\vec{r})\}\Phi(\vec{r})=0. \tag{3}$$

We seek the solution of (3) in the form of

$$\Phi(\vec{r})=f(\vec{r})e^{\iota S(\vec{r})}, \tag{4}$$

where $\exp[\iota S(\vec{r})]$ is the solution of Klein-Gordon equation for a charged particle in a static field

$$\{[\varepsilon-U(\vec{r})]^2+\Delta-m^2\}e^{\iota S(\vec{r})}=0 \tag{5}$$

and $f(\vec{r})$ is a bispinor function.

Substituting (4) into (3) we get for $f(\vec{r})$ and $S(\vec{r})$ the system of equations

$$\iota\Delta S+[\varepsilon-U(\vec{r})]^2-(\vec{\nabla}S)^2-m^2=0, \tag{6}$$

$$\iota\Delta f-2\vec{\nabla}S\cdot\vec{\nabla}f-\gamma_0[\vec{\gamma}\cdot\vec{\nabla}U(\vec{r})]f=0, \tag{7}$$

where (6) is the Klein-Gordon equation [cf. with (5)] that describes the scattering of a charged particle without spin, whereas Eq. (7) describes the spinor part of the scattering. We solve this system by assuming that the scattering field is weak and we seek a solution of the form

$$S(\vec{r})=\vec{p}\cdot\vec{r}+S_1(\vec{r}),\quad f(\vec{r})=u+f_1(\vec{r}),$$

where u is the Dirac bispinor for the free particle. As a result Eqs. (6) and (7) turn into a new system of equations for the scalar function $S_1(\vec{r})$ and bispinor function $f_1(\vec{r})$,

$$\iota\Delta S_1-2\vec{p}\cdot\vec{\nabla}S_1=2\varepsilon U(\vec{r})-U^2(\vec{r})+(\vec{\nabla}S_1)^2, \tag{8}$$

$$\iota\Delta f_1-2\vec{p}\cdot\vec{\nabla}f_1=\gamma_0\vec{\gamma}\cdot\vec{\nabla}U(\vec{r})u+2\vec{\nabla}S_1\cdot\vec{\nabla}f_1+\gamma_0\vec{\gamma}\cdot\vec{\nabla}U(\vec{r})f_1. \tag{9}$$

Within the assumption of potential weakness the last two terms on the right-hand sides of Eqs. (8) and (9) are small compared to the first one and can be neglected. So instead of (8) and (9) we can write the system of equations

$$\iota\Delta S_1 - 2\vec{p}\cdot\vec{\nabla}S_1 = 2\varepsilon U(\vec{r}), \tag{10}$$

$$\iota\Delta f_1 - 2\vec{p}\cdot\vec{\nabla}f_1 = \gamma_0\vec{\gamma}\cdot\vec{\nabla}U(\vec{r})u. \tag{11}$$

This system can be solved by carrying out a Fourier transformation in (10) and (11) and taking into account that for a short-range potential the following boundary conditions are true:

$$S_1(\vec{r})=0, \quad \vec{\nabla}S_1(\vec{r})=\vec{0}; \quad f_1(\vec{r})=0, \quad \vec{\nabla}f_1(\vec{r})=\vec{0}, \tag{12}$$

when $\vec{p}\cdot\vec{r}<0$, $|\vec{r}|\to\infty$. As a result, the solutions of (10) and (11) may be written as

$$S_1(\vec{r}) = \frac{\iota\varepsilon}{4\pi^3}\int \frac{U(\vec{q})e^{\iota\vec{q}\cdot\vec{r}}}{\vec{q}^2+2\vec{p}\cdot\vec{q}-\iota 0}\,d\vec{q},$$

$$f_1(\vec{r}) = \frac{\gamma_0\vec{\gamma}\cdot\vec{\nabla}}{2\varepsilon}S_1(\vec{r})u, \tag{13}$$

where $U(\vec{q})=\int U(\vec{r})e^{-\iota\vec{q}\cdot\vec{r}}d\vec{r}$ is the Fourier transform of the potential energy, $\iota 0$ is an imaginary infinitesimal, and the path around the pole in the integral is chosen according to boundary conditions (12).

During the derivation of (13) we replaced the exact equations (8) and (9) by the approximate equations (10) and (11). Such an approximation is valid if

$$U^2(\vec{r}) \ll 2\varepsilon|U(\vec{r})|, \quad (\vec{\nabla}S_1)^2 \ll 2\varepsilon|U(\vec{r})| \tag{14}$$

and

$$|2\vec{\nabla}S_1\cdot\vec{\nabla}f_1| \ll |\gamma_0\vec{\gamma}\cdot\vec{\nabla}U(\vec{r})u|,$$
$$|\gamma_0\vec{\gamma}\cdot\vec{\nabla}U(\vec{r})f_1| \ll |\gamma_0\vec{\gamma}\cdot\vec{\nabla}U(\vec{r})u|. \tag{15}$$

Putting the explicit expression for $f_1(\vec{r})$ from the right-hand side of (13) into (15), it becomes obvious that (15) follows from (14). Using the first expression in (13) for $S_1(\vec{r})$, the second condition in (14) can be written as

$$2\varepsilon\left|\int \frac{\vec{q}U(\vec{q})e^{\iota\vec{q}\cdot\vec{r}}}{\vec{q}^2+2\vec{p}\cdot\vec{q}-\iota 0}\frac{d\vec{q}}{(2\pi)^3}\right|^2 \ll |U(\vec{r})|. \tag{16}$$

To the integral in (16), because of the oscillations of the factor $e^{\iota\vec{q}\cdot\vec{r}}$ the main contribution gives the region where $\vec{q}\cdot\vec{r}\cong 1$. Therefore (16) can be written as

$$2\varepsilon\frac{\vec{q}_{ef}^2}{(\vec{q}_{ef}^2+2\vec{p}\cdot\vec{q}_{ef})^2}U^2(\vec{r}) \ll |U(\vec{r})|. \tag{17}$$

Finally, defining $q_{ef,z}=1/\overline{z}$, $|\vec{q}_{ef}|_\perp = 1/\overline{\rho}$, we can write the condition (14) of the replacement of the exact system of equations by the approximate equations (10) and (11) in the form

$$|U|\ll\varepsilon, \quad |U|\ll 2pv\left(\frac{\overline{\rho}}{\sqrt{\overline{\rho}^2+\overline{z}^2}}+\frac{1}{2p}\sqrt{\frac{1}{\overline{\rho}^2}+\frac{1}{\overline{z}^2}}\right)^2. \tag{18}$$

Here the initial momentum of the particle is directed along the z axis and $\overline{z},\overline{\rho}$ are, respectively, the longitudinal and transverse dimensions of the domain that give the main contribution to the integral defining $S_1(\vec{r})$ in (13) and where the interaction of the particle with the potential is the most effective.

Using (13), the approximate solution of Eq. (3) may be written as

$$\Phi(\vec{r}) = e^{\iota\vec{p}\cdot\vec{r}}\left[1-\iota\frac{\gamma_0\vec{\gamma}\cdot\vec{\nabla}}{2\varepsilon}\right]e^{\iota S_1(\vec{r})}\frac{u_{\vec{p}}}{\sqrt{2\varepsilon}}. \tag{19}$$

Inserting the expression for $\Phi(\vec{r})$ into (2) and keeping terms to first order of the potential, after simple but long calculations we obtain the solution of the Dirac equation in the applied approximation

$$\Psi(\vec{r}) = e^{\iota\vec{p}\cdot\vec{r}}\left[1-\iota\frac{\gamma_0\vec{\gamma}\cdot\vec{\nabla}}{2\varepsilon}\right]e^{\iota S_1(\vec{r})}\frac{u_{\vec{p}}}{\sqrt{2\varepsilon}}. \tag{20}$$

The wave function (20) is normalized for one particle in the unit volume and $\overline{u_{\vec{p}}}u_{\vec{p}}=2m$, where $\overline{u_{\vec{p}}}=u_{\vec{p}}^{+}\gamma_0$. Comparing (19) and (20), one can see that $\Psi(\vec{r})=\Phi(\vec{r})$. Thus, within the approximation of (14), which we call the generalized eikonal approximation, the solution (20) of the Dirac equation (1) coincides with the solution (19) of the quadratic equation (3).

Now let us clarify the relation of the obtained wave function (20) with the Born and the eikonal approximation wave functions, respectively. If $|S_1(\vec{r})|\ll 1$, then expanding the exponent in (20) into the series and keeping only terms to first order in U, we obtain

$$\Psi_B(\vec{r}) = \left[1-\frac{1}{(2\pi)^3}\int\frac{2\varepsilon+\gamma_0\vec{\gamma}\cdot\vec{q}}{\vec{q}^2+2\vec{p}\cdot\vec{q}-\iota 0}U(\vec{q})e^{\iota\vec{q}\cdot\vec{r}}d\vec{q}\right]$$
$$\times\frac{u_{\vec{p}}}{\sqrt{2\varepsilon}}e^{\iota\vec{p}\cdot\vec{r}}, \tag{21}$$

i.e., the wave function of the Born approximation.

The criterion for the condition $|S_1(\vec{r})|\ll 1$ can be found using (13) and evaluating the integral in a similar way, as done above. As a result we get

$$|U|\ll pv\left[\frac{1}{p\overline{z}}+\frac{1}{2}\frac{1}{(p\overline{z})^2}+\frac{1}{2}\frac{1}{(p\overline{\rho})^2}\right]. \tag{22}$$

This criterion generalizes the well-known Born criterion for elastic scattering. It includes both weak ($|U|\ll v/\overline{z}$) and strong ($|U|\ll 1/2\varepsilon a^2$, where $a=\max\{\overline{z},\overline{\rho}\}$) conditions of the Born approximation for fast ($p\overline{z}\gg 1$ and $p\overline{\rho}\gg 1$) and slow ($pa\ll 1$) particles, respectively.

Comparing (18) and (22), we see that for the scattering of slow particles, when $p\overline{z}\ll 1+\overline{z}^2/\overline{\rho}^2$, the conditions of the GEA and Born approximations become the same. However, in the case of fast particles scattering, when $p\overline{z}\gg 1+\overline{z}^2/\overline{\rho}^2$, the GEA condition (18) can be written as $|U| \ll b v/\overline{z}$, where $b = 2p\overline{z}/(1+\overline{z^2}/\overline{\rho^2})$, and as far as $b \gg 1$ the condition of the GEA becomes weaker than the weak condition of the Born approximation $|U| \ll v/\overline{z}$. Thus the GEA is applicable to stronger potentials in the case of fast particle scattering rather than the Born approximation. For fast particles the wave function (16) tends to its quasiclassic limit: to the eikonal wave function. To demonstrate this we expand the expression for $S_1(\vec{r})$ in (13) over the small parameter $1/b$ and keep the term to the lowest order:

$$S_1^E(\vec{r}) = -\frac{1}{v}\int_{-\infty}^{z} U(\vec{\rho}, z')\,dz'. \tag{23}$$

Afterward, using the expansion, from the GEA wave function (20) we obtain the wave function of the eikonal approximation

$$\Psi^{(E)}(\vec{r}) = e^{\iota \vec{p}\cdot\vec{r}}\left[1 - \iota\,\frac{\gamma_0 \vec{\gamma}\cdot\vec{\nabla}}{2\varepsilon}\right] \times \exp\left(-\frac{1}{v}\int_{-\infty}^{z} U(\vec{\rho}, z')\,dz'\right)\frac{u_{\vec{p}}}{\sqrt{2\varepsilon}}. \tag{24}$$

As far as the eikonal approximation describes scattering under small angles, when $\Delta p_z \ll \Delta p_\perp$ (i.e., $\overline{z} \gg \overline{\rho}$), the condition $b \gg 1$, which we have used for deviation of (24) from the GEA wave function, is equivalent to the condition $\overline{z} \ll p\overline{\rho^2}$. Taking into account the latter, the GEA condition (18) turns to $(U/pv)(\overline{z/\rho})^2 \ll 1$, i.e., to the well-known conditions of the eikonal approximation.

Thus the obtained GEA wave function (20) in both limits—scattering of slow particles under large angles and fast particles under small angles—turns to the wave functions of the Born and eikonal approximations, respectively.

III. THE CASE OF THE COULOMB FIELD

The wave function (20) obtained above describes the particle scattering in a short-range potential, where the boundary conditions (12) are valid. For a long-range potential, generally, conditions (12) break down and, as a result, the states of a particle at infinity cannot strictly be described by a plane wave. Particularly, in the case of the Coulomb potential the particle wave function at infinity contains the well-known logarithmic divergent phase, which cannot be obtained from (20). Therefore, we separately consider the scattering in a Coulomb potential.

For a Coulomb potential $U(r) = \alpha/r$ it is more convenient to solve Eq. (10) written in the parabolic coordinates $\eta = r - z$, $\xi = r + z$, and $\varphi = \arctan(y/x)$,

$$\left[\iota\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \iota\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right) + \frac{\iota}{4}\left(\frac{1}{\xi}+\frac{1}{\eta}\right)\frac{\partial^2}{\partial\varphi^2} - p\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)\right]S_1(\xi,\eta,\varphi) - \varepsilon\alpha = 0. \tag{25}$$

Due to an axial symmetry S_1 does not depend on φ (as above the OZ axis is directed along $\vec{p}$). So seeking a solution of the form $S_1(\xi,\eta) = S_{\rm I}(\xi) + S_{\rm II}(\eta)$, the variables in (25) can be separated and for $S_{\rm I}(\xi), S_{\rm II}(\eta)$ we obtain the equations

$$\left[\iota\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) - p\left(\xi\frac{\partial}{\partial\xi}\right)\right]S_{\rm I}(\xi) = a, \tag{26}$$

$$\left[\iota\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right) + p\left(\eta\frac{\partial}{\partial\eta}\right)\right]S_{\rm II}(\eta) = b, \quad a + b = \alpha\varepsilon, \tag{27}$$

where a and b are arbitrary constants. Recalling that the particle wave function in the Coulomb field before the scattering shall describe particle plane states, we look for such a solution of (25) that is infinitely small before the scattering, when $z<0$ and $|\vec{r}| \to \infty$, compared with the phase of the wave function of free electron $S_1(\vec{r})|_{|\vec{r}|\to\infty}^{z<0} \propto o(\iota p z)$. In parabolic coordinates this condition turns to $S_1(\xi,\eta) \propto O(\iota p(\xi-\eta))$ when $\eta \to \infty$ and for all finite values of ξ, where O means "of the same order as." This requirement can be fulfilled only if $S_{\rm I}(\xi) = \text{const} = 0$ and $S_{\rm II}(\eta \to \infty) \propto O(\eta)$. Noting that $S_{\rm I}(\xi) = \text{const}$, then from (26) it follows that $a = 0$ and $b = \alpha\varepsilon$, we solve (27), and get for $S_1(\vec{r})$ the expression

$$S_1^C(\vec{r}) = -\frac{\alpha}{v}\{\text{Ei}[\iota p(r-z)] - \ln p(r-z)\}, \tag{28}$$

where Ei(x) is the integral exponential function.

Using (4) and (28) and recalling that in the applied approximation $\Psi(\vec{r}) = \Phi(\vec{r})$, $S(\vec{r}) = \vec{p}\cdot\vec{r} + S_1^C(\vec{r})$, and $f(\vec{r}) = u + f_1(\vec{r})$, where $f_1(\vec{r})$ is again defined through $S_1^C(\vec{r})$ in accordance with (13), we obtain the particle wave function in a Coulomb field

$$\Psi^C(\vec{r}) = e^{\iota \vec{p}\cdot\vec{r}}\left[1 - \iota\,\frac{\gamma_0 \vec{\gamma}\cdot\vec{\nabla}}{2\varepsilon}\right] \times \exp\left\{-\iota\frac{\alpha}{v}[\text{Ei}(\iota p(r-z)) - \ln p(r-z)]\right\}\frac{u_{\vec{p}}}{\sqrt{2\varepsilon}}. \tag{29}$$

Note that this approximate solution of the Dirac equation is valid if the following condition of the GEA holds:

$$\frac{\alpha}{v} \ll p(r-z) = pr(1-\cos\theta), \tag{30}$$

where θ is the angle between the directions of observation and the particle's initial momentum. Let us clarify the relation of the obtained wave function with the wave functions of the Born, Farry-Sommerfeld-Maue (FSM) and eikonal approximations for a particle in the Coulomb field.

A. Relation between the GEA and the Born approximation

Solving the Dirac equation (1) by the perturbation theory over Coulomb potential in parabolic coordinates, we obtain the particle wave function in the first Born approximation

$$\Psi_B^C(\vec{r}) = e^{\iota \vec{p}\cdot\vec{r}}\left[1-\iota\,\frac{\gamma_0\vec{\gamma}\cdot\vec{\nabla}}{2\varepsilon}\right] \times\left\{1-\iota\,\frac{\alpha}{v}\,\{\mathrm{Ei}[\iota p(r-z)]-\ln p(r-z)\}\right\}\,\frac{u_{\vec{p}}}{\sqrt{2\varepsilon}} \tag{31}$$

and the conditions when this solution is valid

$$\frac{\alpha}{v}\,|\mathrm{Ei}(\iota p(r-z))-\ln p(r-z)|\ll 1,\quad \frac{\alpha}{r}\ll\varepsilon. \tag{32}$$

As seen from (32), the first one of these conditions differs from the well-known condition of the Born approximation for the Coulomb field $\alpha/v\ll 1$. At first sight this difference may be viewed with suspicion. However, it is noteworthy that the traditional derivation of the Born condition for the Coulomb field is based on the model of the short-range potential; therefore the Born condition does not contain spatial restrictions and seems as if it allows the description of the particle states at arbitrary distances. Conversely, for a field of long-range action such a restriction exists, just as condition (32) indicates. In particular, when $p(r-z)\gg 1$ the Born approximation condition for the Coulomb wave function takes the form of

$$\frac{\alpha}{v}\ln p(r-z)\ll 1. \tag{33}$$

As follows from (32) the well-known Born condition for Coulomb field

$$\frac{\alpha}{v}\ll 1 \tag{34}$$

is valid in the spatial region, where $p(r-z)\leqslant 1$. However, when $p(r-z)\ll 1$ one also has to consider the second condition in (32), which has a relativistic origin. Then the condition of validity of the Born approximation becomes

$$\frac{\alpha}{v}\ll pr\,\frac{c^2}{v^2}. \tag{35}$$

The condition $pr\ll(v/c)^2$ is stricter than $\alpha/v\ll 1$. Note that $p(r-z)$ is just the parameter that should be confined to obtain the wave function of the Born approximation either from the exact Coulomb wave function or from the FSM wave function in the relativistic domain by expanding them over the small parameter α/v.

Now let us compare the Coulomb wave functions (29) and (31) in the GEA and the Born approximation, respectively. When the conditions (32) hold, one can obtain the wave function of the first Born approximation (31) by expanding the wave function (29) into series and keeping the term of first order over the Coulomb field.

A comparison of the validity conditions (30) and (32) of the GEA and the Born approximation shows that in the domain where $p(r-z)\leqslant 1$, the wave function of the Born approximation describes the scattering in the field of a heavier nuclei and/or of slower particles than of the GEA wave function (indeed, if one can consider the scattering field as a perturbation for slow particles, then it could be done for fast particles too). In the domain where $p(r-z)\approx 1$ the GEA wave function describes the particle scattering of the same velocities and fields as the Born wave function. Conversely, in the domain where $p(r-z)\gg 1$ there is an essential difference between the GEA and the Born approximation. As seen from (33), in this domain the condition of the Born approximation is very strict (the values of the parameter α/v are strongly depressed), whereas according to the condition (30), the above obtained GEA wave function describes the particle scattering in a wider enough region of α/v values. In this case $[p(r-z)\gg 1]$ the wave function (29) goes to the quasiclassic limit, coinciding with the eikonal wave function of the Coulomb field

$$\Psi_E^C(\vec{r}) = e^{\iota\vec{p}\cdot\vec{r}}\left[1-\iota\,\frac{\gamma_0\vec{\gamma}\cdot\vec{\nabla}}{2\varepsilon}\right]e^{\iota(\alpha/v)\ln p(r-z)}\,\frac{u_{\vec{p}}}{\sqrt{2\varepsilon}}. \tag{36}$$

B. Relation between the GEA and the FSM approximation

Let us compare the wave function (29) with the well-known wave function of a relativistic particle in the Coulomb field in the FSM approximation, which is valid when $\alpha^2/p\rho\ll 1$ (approximation of large momenta), where ρ is the impact parameter [1]:

$$\Psi^{\mathrm{FSM}}(\vec{r}) = e^{-\pi\alpha/2v}\Gamma\left(1+\iota\,\frac{\alpha}{v}\right)e^{\iota\vec{p}\cdot\vec{r}}\left[1-\iota\,\frac{\gamma_0\vec{\gamma}\cdot\vec{\nabla}}{2\varepsilon}\right] \times F\left(-\iota\,\frac{\alpha}{v},1;-\iota p(r-z)\right)\frac{u_{\vec{p}}}{\sqrt{2\varepsilon}}. \tag{37}$$

Here $\Gamma(t)$ is the Gamma function and $F(v,1;t)$ is the hypergeometric function.

In the quasiclassic limit, where $p(r-z)\gg 1$, using the asymptote formula of the hypergeometric function $F(-\iota a,1;\iota y)$ for $y\gg 1$,

$$F(-\iota a,1;\iota y)\cong\frac{e^{\pi a/2}}{\Gamma(1+\iota a)}\,e^{\iota a\,\ln y}\left[1+O\left(\frac{1}{y}\right)\right],$$

we obtain from (37) the Coulomb wave function in the eikonal approximation (26). Thus, where $p(r-z)\gg 1$, the GEA and FSM wave functions coincide, turning into the eikonal wave function of a particle in the Coulomb field.

These wave functions also coincide in the region where $p(r-z)\leqslant 1$, if $\alpha/v\ll 1$ too. In fact, in this limit, using the asymptotic formula for the hypergeometric function

$$F\left(-\iota\,\frac{\alpha}{v},1;\iota y\right)\cong 1-\iota\,\frac{\alpha}{v}\sum_{\kappa=1}^{\infty}\frac{y^\kappa}{\kappa\kappa!} = 1-\iota\,\frac{\alpha}{v}\,[\mathrm{Ei}(y)-\ln y-C+\iota\pi]$$

($C=0.577\,215\ldots$ is Euler constant), we obtain from (37) the Coulomb wave function in the Born approximation (31). As in the case of the derivation from the GEA wave function, the Born wave function can be obtained from the FSM wave function if only in addition to the well-known condition $\alpha/v\ll 1$, the condition $p(r-z)\leqslant 1$ is fulfilled too.

IV. ELASTIC SCATTERING CROSS SECTION IN THE GENERALIZED EIKONAL APPROXIMATION

The knowledge of the wave function of a charged particle in the scattering field enables one to calculate the amplitude of the elastic scattering, and hence the differential scattering cross sections. When the wave function also describes the particle states at large distances and has an asymptote at $r\to\infty$ that is a superposition of a plane and spherical convergent waves

$$\Psi(\vec{r})\approx u^{\mu}_{\vec{p}} e^{\iota \vec{p}\cdot\vec{r}}+G^{+}(\hat{\vec{r}})\ \frac{e^{\iota pr}}{r}, \tag{38}$$

the scattering amplitude can be defined as (see, for example, [1])

$$f^{\mu}(\hat{\vec{r}})=\frac{1}{2m}\ \overline{u^{\mu}_{\vec{p}}}G^{+}(\hat{\vec{r}}), \tag{39}$$

where $u^{\mu}_{\vec{p}}, \overline{u^{\mu}_{\vec{p}\,'}}$, are bispinors describing the state of a free charged particle with polarization μ and momenta $\vec{p}$ and $\vec{p}\,'=p\hat{\vec{r}}$, respectively, and $G^{+}(\hat{\vec{r}})$ is a bispinor depending on $\hat{\vec{r}}=\vec{r}/r$.

However, when the wave function describes the particle states only in the region where the particle potential energy $U(\vec{r})$ is not zero, then it is impossible to determine the scattering amplitude by the asymptote of the wave function. Although the scattering amplitude can be linked with such a wave function [1],

$$f^{\mu}(\hat{\vec{r}})=-\frac{1}{4\pi}\int e^{-\iota\vec{p}\,'\cdot\vec{r}\,'}\overline{u^{\mu}_{\vec{p}\,'}}\gamma_0\Psi(\vec{r}\,')U(\vec{r}\,')d^3r'. \tag{40}$$

As far as the wave function obtained in Secs. I and II in the GEA describes the particle states either within the range of a scattering field or at asymptotic large distances, then both approaches can be applied to calculate the scattering amplitude in the GEA. At first, we shall find the asymptote of the GEA wave function (20) for a particle scattering in a short-range potential. To calculate the asymptote of the function $S_1(\vec{r})$ in (20), temporarily we direct the OZ coordinate axis along $\vec{r}$ and change the integration variable in (13) to $\vec{Q}=\vec{p}+\vec{q}$. Turning to spherical coordinates, we carry out the integration over the variable cos $\theta=\vec{Q}\cdot\hat{\vec{r}}$ by parts. As a result, at $r\to\infty$ we obtain

$$S_1(\vec{r})=\frac{\iota\varepsilon}{2\pi}\ e^{-\iota\vec{p}\cdot\vec{r}}U(\vec{p}\,'-\vec{p})\ \frac{e^{\iota pr}}{r}+O\left(\frac{1}{r^2}\right). \tag{41}$$

Using (41), we obtain from (20) an asymptote of the wave function of the form (38), where

$$G^{+}(\hat{\vec{r}})=-\frac{1}{4\pi}\left[2\varepsilon+\gamma_0\vec{\gamma}\cdot(\vec{p}\,'-\vec{p})\right]u_{\vec{p}}U(\vec{p}\,'-\vec{p}). \tag{42}$$

In addition, taking into account that $\overline{u_{\vec{p}\,'}}(\varepsilon\gamma_0-\vec{\gamma}\cdot\vec{p}\,'-m)=0$ and $(\varepsilon\gamma_0-\vec{\gamma}\cdot\vec{p}-m)u_{\vec{p}}=0$, the scattering amplitude (39) takes the form

$$f^{\mu}(\hat{\vec{r}})=-\frac{1}{4\pi}\ \overline{u_{\vec{p}\,'}}\gamma_0 u_{\vec{p}}U(\vec{p}\,'-\vec{p}), \tag{43}$$

which coincides with the amplitude of the elastic scattering in the first Born approximation.

The GEA wave function in the Coulomb field (29) describes the particle states at large distances too. In fact, at large distance, taking into account the asymptote of the integral exponential function

$$\mathrm{Ei}(\iota x)\approx\frac{e^{\iota x}}{\iota x}\quad \text{when}\ \ x\gg 1,$$

we obtain from (29) the asymptote of the wave function

$$\Psi^{C}(\vec{r})\approx\left[1+\frac{\alpha}{pv}\frac{1}{r(1-\cos\theta)}\frac{\gamma_0}{2\varepsilon}\ \vec{\gamma}\cdot(\vec{p}\,'-\vec{p})\right]\times u_{\vec{p}}e^{\iota\vec{p}\cdot\vec{r}+\iota(\alpha/v)\ln pr(1-\cos\theta)}+G^{+}(\hat{\vec{r}})\ \frac{e^{\iota pr+\iota(\alpha/v)\ln pr}}{r}, \tag{44}$$

where

$$G^{+}(\hat{\vec{r}})=-\frac{\alpha}{pv}\left[1+\frac{\gamma_0}{2\varepsilon}\ \vec{\gamma}\cdot(\vec{p}\,'-\vec{p})\right]u_{\vec{p}}\ \frac{e^{\iota(\alpha/v)\ln(1-\cos\theta)}}{1-\cos\theta} \tag{45}$$

and θ is the scattering angle. The first term in (44) is the falling wave with the logarithmic distortion in the phase that occurs because of the slow decrease of the Coulomb field at the distance. There is a such kind of distortion in the scattered spherical wave described by the second term in (44) too. However, these deviations from the usual asymptotic form of the wave function (38) are not essential for the definition of the scattering amplitude and using (39) and (45) we obtain

$$f^{\mu}(\hat{\vec{r}})=-\frac{\alpha}{2p^2}\frac{\overline{u_{\vec{p}\,'}}\gamma_0 u_{\vec{p}}}{1-\cos\theta}\ e^{\iota(\alpha/v)\ln(1-\cos\theta)}. \tag{46}$$

The expression (46) differs from the well-known scattering amplitude of the Coulomb field in the first Born approximation only by a phase factor. So using the amplitude of the scattering in a short-range field in the first Born approximation (43) for a long-range Coulomb field gives the same scattering cross sections.

Now let us calculate the scattering amplitude by the formula (40). We carry out integration by parts by substituting the wave function (20) into (40) and using Eqs. (10) and (11). Taking into account that for a short-range potential $U=0$, $\partial U/\partial a=0$, and $\partial^2 U/\partial a^2=0$ at $a\to\infty$, where $a=x,y,z$, we obtain the scattering amplitude in the lowest order of the GEA

$$f^{\mu}(\hat{\vec{r}})=-\frac{\iota}{4\pi}\frac{\overline{u^{\mu}_{\vec{p}\,'}}\gamma_0}{2\varepsilon}\ |\vec{p}+\vec{p}\,'|\int e^{-\iota\vec{Q}\cdot\vec{\rho}}[e^{\iota\widetilde{S}_1(\vec{\rho})}-1]u_{\vec{p}}d^2\rho, \tag{47}$$

where OZ axis is taken along $\vec{p}+\vec{p}\,'$, $\vec{Q}=\vec{p}\,'-\vec{p}$ is the transfer momentum, $\vec{\rho}$ is two-dimensional radius vector in the plane XOY, and we have denoted by $\widetilde{S}_1(\vec{\rho})$ the function $S_1(\vec{r})$ at $z=+\infty$, which is defined in (13).

The expression (47) defines the amplitude of an elastic scattering in the generalized eikonal approximation. From it one can easily derive the elastic scattering amplitudes of the Born and eikonal approximations. In fact, when the condition (22) of the Born approximation holds, then $\widetilde{S}_1(\vec{\rho})\ll 1$ and in (47) one can expand the exponent into the series. Keeping terms to first order in U and substituting the $S_1(\vec{r})$ from (13), we obtain the scattering amplitude in the first Born approximation (43). If in addition to the condition (18) also $b=2p\overline{z/(1+\overline{z^2}/\rho^2)}\gg 1$, then instead of $\widetilde{S}_1(\vec{\rho})$ its approximate value $\widetilde{S}_1^E(\vec{\rho})\equiv\widetilde{S}_1^E(\vec{\rho},z=+\infty)$ can be substituted from (23) into (47). Doing that we obtain from (47) the scattering amplitude in the eikonal approximation. So the obtained amplitude of the electron elastic scattering in an arbitrary static potential differs from the known results in eikonal approximation. For the scattering of nonpolarized particles, after summing over the final and averaging over initial polarization, we obtain from (47) the differential cross sections in generalized eikonal approximation

$$d\sigma=\tfrac{1}{2}\sum_{\mu}|f^{\mu}(\hat{\vec{r}})|^2 do=|f(\hat{\vec{r}})|^2 do, \tag{48}$$

where

$$f(\hat{\vec{r}})=-\frac{\iota}{8\pi}|\vec{p}+\vec{p}\,'|\left(1+\frac{\vec{p}\cdot\vec{Q}}{2\varepsilon^2}\right)^{1/2}\int e^{-\iota\vec{Q}\cdot\vec{\rho}}[e^{\iota\widetilde{S}_1(\vec{\rho})}-1]d^2\rho$$

and do is the solid angle along the $\hat{\vec{r}}$.

[1] A. E. Akhiezer and V. B. Berestetzki, *Quantum Electrodynamics* (Nauka, Moscow, 1981), pp. 67–79.

[2] A. E. Akhiezer and V. B. Berestetzki, Theor. Math. Phys. **23**, 11 (1975).