Consistent histories and quantum reasoning

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A system of quantum reasoning for a closed system is developed by treating nonrelativistic quantum mechanics as a stochastic theory. The sample space corresponds to a decomposition, as a sum of orthogonal projectors, of the identity operator on a Hilbert space of histories. Provided a consistency condition is satisfied, the corresponding Boolean algebra of histories, called a *framework*, can be assigned probabilities in the usual way, and within a single framework quantum reasoning is identical to ordinary probabilistic reasoning. A refinement rule, which allows a probability distribution to be extended from one framework to a larger (refined) framework, incorporates the dynamical laws of quantum theory. Two or more frameworks which are incompatible because they possess no common refinement cannot be simultaneously employed to describe a single physical system. Logical reasoning is a special case of probabilistic reasoning in which (conditional) probabilities are 1 (true) or 0 (false). As probabilities are only meaningful relative to some framework, the same is true of the truth or falsity of a quantum description. The formalism is illustrated using simple examples, and the physical considerations which determine the choice of a framework are discussed. [S1050-2947(96)07910-3]

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I. INTRODUCTION

Despite its success as a physical theory, nonrelativistic quantum mechanics is beset with a large number of conceptual difficulties. While the mathematical formalism is not at issue, the physical interpretation of this formalism remains controversial. Does a wave function describe a physical property of a quantum system, or is it merely a means for calculating something? Do quantum measurements reveal preexisting properties of a measured system, or do they in some sense create the properties they reveal? These are but two of the questions which trouble both beginners and experts.

It would be wrong to dismiss these issues as mere ''philosophical problems.'' The effective use of a mathematical structure as part of a physical theory requires an intuitive understanding of what the mathematics means, both in order to relate it to the real world of laboratory experiment, and in order to motivate the approximations which must be made when the exact solution of some equation is a practical impossibility. In older domains of application of quantum theory, such as scattering theory, there is by now a well-developed set of rules, and while the justification for these is somewhat obscure, once they have been learned, they can be applied without worrying too much about ''what is really going on.'' But when quantum mechanics is applied in an unfamiliar setting, such as is happening at the present time in the field of quantum computation [1], its unresolved conceptual difficulties are a serious impediment to physical understanding, and advances which enable one to think more clearly about the problem can lead to significant improvements in algorithms, as illustrated in [2].

The principal thesis of the present paper is that the major conceptual difficulties of non-relativistic quantum theory (which, by the way, are also present in relativistic theories) can be eliminated, or at least tamed, by taking a point of view in which quantum theory is fundamentally a *stochastic* theory, in terms of its description of the time development of a physical system. The approach found in typical textbooks is that the time development of a quantum system is governed by a deterministic Schrödinger equation up to the point at which a measurement is made, the results of which can then be interpreted in a probabilistic fashion. By contrast, the point of view adopted here is that a quantum system's time evolution is fundamentally stochastic, with probabilities which can be calculated by solving Schrödinger's equation, and deterministic evolution arises only in the special case in which the relevant probability is one. This approach makes it possible to recover all the results of standard textbook quantum theory, and much else besides, in a manner which is conceptually much cleaner and does not have to make excuses of the ''for all practical purposes'' variety, justly criticized by Bell [3].

Most of the tools needed to formulate time development in quantum theory as a stochastic process have already appeared in the published literature. They include the idea that the properties of a quantum system are associated with subspaces of an appropriate Hilbert space [4], the concept of a quantum history as a set of events at a sequence of successive times [5], the use of projectors on a tensor product of copies of the Hilbert space to represent these histories [6], the notion that a collection of such histories can, under suitable conditions (''consistency''), form an event space to which quantum theory ascribes probabilities [5,7–13], and rules which restrict quantum reasoning processes to single consistent families of histories [9–11].

The present paper thus represents an extension of the ''consistent histories'' procedure for quantum interpretation. The element added to previous work is the systematic development of the concept of a *framework*, the quantum counterpart of the space of events in ordinary (''classical'') probability theory, and the use of frameworks in order to codify and clarify the process of reasoning needed to discuss the

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time development of a quantum system. A framework is a Boolean algebra of commuting projectors (orthogonal projection operators) on the Hilbert space of quantum histories, Sec. II, which satisfies certain *consistency conditions*, Sec. III. Reasoning about how a quantum system develops in time, Sec. V, then amounts to the application of the usual rules of probability theory to probabilities defined on a framework, together with an additional *refinement rule* which permits one to extend a given probability distribution to a refinement or enlargement of the original framework, Sec. IV. In particular, the standard (Born) rule for transition probabilities in a quantum system is a consequence of the refinement rule for probabilities. Logical rules of inference, in this context, are limiting cases of probabilistic rules in which (conditional) probabilities are one (true) or zero (false). Because probabilities can only be defined relative to a framework, the notions of ''true'' and ''false'' as part of a quantum description are necessarily framework dependent, as suggested in [14]; this rectifies a problem [15] with Omnès's approach [10,11] to defining ''truth'' in the context of consistent histories, and responds to certain objections raised by d'Espagnat [16–19].

The resulting structure is applied to various simple examples in Sec. VI to show how it works. These examples illustrate how the intuitive significance of a projector can depend upon the framework in which it is embedded, how certain problems of measurement theory are effectively dealt with by a consistent stochastic approach, and how the system of quantum reasoning presented here can help untangle quantum paradoxes. In particular, a recent criticism of the consistent histories formalism by Kent [20], involving the inference with probability one from the same initial data, but in two incompatible frameworks, of two events represented by mutually orthogonal projection operators, is considered in Sec. VI D with reference to a paradox introduced by Aharonov and Vaidman [21]. For reasons explained there and in Sec. VI B, such inferences do not, for the approach discussed in this paper, give rise to a contradiction.

Since the major conceptual difficulties of quantum theory are associated with the existence of *incompatible* frameworks with no exact classical analog, Sec. VII is devoted to a discussion of their significance, along with some comments on how the world of classical physics can be seen to emerge from a fundamental quantum theory. Finally, Sec. VIII contains a brief summary of the conclusions of the paper, together with a list of open questions.

II. PROJECTORS AND HISTORIES

Ordinary probability theory [22] employs a *sample space* which is, in the discrete case, a collection of *sample points*, regarded as mutually exclusive outcomes of a hypothetical experiment. To each sample point is assigned a non-negative *probability*, with the sum of the probabilities equal to one. An *event* is then a set of one or more sample points, and its probability is the sum of the probabilities of the sample points which it contains. The events, under the operations of intersection and union, form a *Boolean algebra of events*. In this and the following two sections we introduce quantum counterparts for each of these quantities. Whereas in many physical applications of probability theory only a single sample space is involved, and hence its identity is never in doubt and its basic properties do not need to be emphasized, in the quantum case one typically has to deal with many different sample spaces and their corresponding event algebras, and clear thinking depends upon keeping track of which one is being employed in a particular argument.

The quantum counterpart of a sample space is a *decomposition of the identity* on an appropriate Hilbert space. We shall always assume that the Hilbert space is *finite dimensional*; for comments on this, see Sec. VIII B. On a finite-dimensional space, such a decomposition of the identity I corresponds to a (finite) collection of orthogonal projection operators, or *projectors* $\{B_i\}$, which satisfy

$$I=\sum_i B_i\,,\quad B_i^\dagger=B_i\,,\quad B_iB_j=\delta_{ij}B_i\,. \tag{2.1}$$

The Boolean algebra $\mathcal{B}$ which corresponds to the event algebra is then the collection of all projectors of the form

$$P=\sum_i \pi_i B_i\,, \tag{2.2}$$

where π_i is either 0 or 1; different choices give rise to the 2^n projectors which make up $\mathcal{B}$ in the case in which the sum in (2.1) contains n terms. We shall refer to the $\{B_i\}$ as the *minimal elements* of $\mathcal{B}$.

For a quantum system at a single time, I is the identity operator on the usual Hilbert space $\mathcal{H}$ used to describe the system, and projectors of the form P, or the subspace of $\mathcal{H}$ onto which they project, represent *properties* of the system. (See Sec. VI for some examples.) The phase space of classical Hamiltonian mechanics provides a useful analogy in this connection. A *coarse graining* of the phase space in which it is divided up into a number of nonoverlapping cells corresponds to (2.1), where B_i is the *characteristic function* of the ith cell, that is, the function which is 1 for points of the phase space inside the cell, and 0 for points outside the cell, and I the function which is 1 everywhere. The events in the associated algebra correspond to regions which are unions of some collection of cells, and their characteristic functions P again have the form (2.2).

Projectors of the form (2.2) corresponding to a particular decomposition of the identity (2.1) commute with each other and form a Boolean algebra $\mathcal{B}$, in which the negation of a property, ''not P,'' corresponds to the complement

$$\tilde{P}=I-P \tag{2.3}$$

of the projector P, and the meet and join operations are defined by

$$P\wedge Q=PQ,\quad P\vee Q=P+Q-PQ. \tag{2.4}$$

Note that $P\wedge Q$ corresponds to the conjunction of the two properties: ''P *and* Q,'' whereas $P\vee Q$ is the disjunction, ''P *or* Q.'' Precisely the same definitions (2.3) and (2.4) apply in the case of characteristic functions for the coarse graining of a classical phase space, and the intuitive significance is much the same as in the quantum case. Of course, two quantum projectors P and Q need not commute with each other, in which case they cannot belong to the same Boolean algebra $\mathcal{B}$, and the properties "*P and Q*" and "*P or Q*" are not defined, that is, they are meaningless. (Note that at this point our treatment diverges from traditional quantum logic as based upon the ideas of Birkhoff and von Neumann [23].)

A *history* of a quantum-mechanical system can be thought of as a sequence of properties or *events*, represented by projectors $E_1, E_2, \ldots, E_n$ on the Hilbert space $\mathcal{H}$ at a succession of times $t_1 < t_2 < \cdots < t_n$. The projectors corresponding to different times are not required to belong to the same Boolean algebra, and need not commute with each other. Following a suggestion by Isham [6], we shall represent such a history as a projector

$$Y = E_1 \odot E_2 \odot \cdots \odot E_n \tag{2.5}$$

on the *history space*

$$\breve{\mathcal{H}} = \mathcal{H} \odot \mathcal{H} \odot \cdots \odot \mathcal{H} \tag{2.6}$$

consisting of the tensor product of n copies of $\mathcal{H}$. (We use $\odot$ in place of the conventional $\otimes$ to avoid confusion in the case in which $\mathcal{H}$ itself is the tensor product of two or more spaces.) The number n of times entering the history can be arbitrarily large, but will always be assumed to be finite, which ensures that $\breve{\mathcal{H}}$ is finite dimensional as long as $\mathcal{H}$ itself is finite dimensional.

The intuitive interpretation of a history of the form (2.5) is that event E_1 occurs in the closed quantum system at time t_1, E_2 occurs at time t_2, and so forth. The consistent history approach allows a realistic interpretation of such a history so long as appropriate consistency conditions, Sec. III, are satisfied. Following [6], we shall allow as a possible history *any* projector on the space (2.6), and not only those of the product form (2.5). The intuitive significance of such "generalized histories" is not clear, because most physical applications which have appeared in the literature up to the present time employ "product histories" of the form (2.5).

One sometimes needs to compare two histories Y_1 and Y_2 defined on two different sets of times, say $t_1' < t_2' < \cdots t_p'$, and $t_1'' < t_2'' < \cdots t_q''$. It is then convenient to *extend* both Y_1 and Y_2 to the collection of times $t_1 < t_2 < \cdots t_n$ which is the union of these two sets, by introducing in the product (2.5) the identity operator I on $\mathcal{H}$ at every time at which the history was not originally defined. We shall use the same symbols, Y_1 and Y_2, for the extensions as for the original histories, as this causes no confusion, and the physical significance of the original history and its extension is the same, because the property I is always true.

A useful classical analogy of a quantum history is obtained by imagining a coarse graining of the phase space, and then thinking of the sequence of cells occupied by the phase point corresponding to a particular initial state, for a sequence of different times. One must allow for different coarse grainings at different times in order to have an analog of the full flexibility possible in the quantum description.

A probabilistic description of a closed quantum system as a function of time can be based upon a Boolean algebra $\mathcal{F}$ of histories generated by a decomposition of the identity operator $\breve{I}$ on $\breve{\mathcal{H}}$:

$$\breve{I} = \sum_i F_i, \quad F_i^\dagger = F_i, \quad F_i F_j = \delta_{ij} F_i, \tag{2.7}$$

where the projectors $\{F_i\}$ will be referred to as the *minimal elements* of $\mathcal{F}$. The different projectors in $\mathcal{F}$ are of the form

$$Y = \sum_i v_i F_i, \tag{2.8}$$

with each v_i either 0 or 1, and the corresponding Boolean algebra is constructed using the obvious analogs of (2.3) and (2.4). We shall refer to $\mathcal{F}$ as a *family of histories*, and, when certain additional (consistency) conditions are satisfied, as a *framework*.

III. WEIGHTS AND CONSISTENCY

Quantum dynamics is described by a collection of time evolution operators $T(t',t)$, thought of as carrying the system from time t to time t', so that a state $|\psi(t)\rangle$ evolving by Schrödinger's equation satisfies

$$|\psi(t)\rangle = T(t,0)|\psi(0)\rangle. \tag{3.1}$$

We assume that these operators satisfy the conditions

$$T(t,t) = I, \quad T(t'',t')T(t',t) = T(t'',t), \quad T(t',t)^\dagger = T(t,t'), \tag{3.2}$$

which, among other things, imply that $T(t',t)$ is unitary. If the system has a time-independent Hamiltonian, T takes the form

$$T(t',t) = \exp[-i(t'-t)H/\hbar]. \tag{3.3}$$

However, none of the results in this paper depends upon assuming the form (3.3).

Given the time transformation operators, we define the *weight operator*

$$K(Y) = E_1 T(t_1,t_2) E_2 T(t_2,t_3) \cdots T(t_{n-1},t_n) E_n \tag{3.4}$$

for the history Y in (2.5). It is sometimes convenient to define the *Heisenberg projector*

$$\hat{E}_j = T(t_r,t_j) E_j T(t_j,t_r) \tag{3.5}$$

corresponding to the event E_j at time t_j, where t_r is some arbitrary reference time independent of j, and the corresponding *Heisenberg weight operator*

$$\hat{K}(Y) = \hat{E}_1 \hat{E}_2 \cdots \hat{E}_n. \tag{3.6}$$

For histories which are not of the form (2.5), but are represented by more general projectors on $\breve{\mathcal{H}}$, one can follow the procedure in [6] and define a weight operator by noting that (3.4) also makes sense when the E_j are arbitrary operators (not just projectors), and then use linearity,

$$K(Y' + Y'' + Y''' + \cdots) = K(Y') + K(Y'') + K(Y''') + \cdots, \tag{3.7}$$

to extend K to a linear mapping from operators on $\breve{\mathcal{H}}$ to operators on $\mathcal{H}$.

Next, we define an inner product on the linear space of operators on $\mathcal{H}$ by means of

$$\langle A,B\rangle=\mathrm{Tr}[A^{\dagger}B]=\langle B,A\rangle^{*}. \tag{3.8}$$

In particular, $\langle A,A\rangle$ is positive, and vanishes only if $A=0$. In terms of this inner product we define the *weight* of a history Y as

$$W(Y)=\langle K(Y),K(Y)\rangle=\langle\hat{K}(Y),\hat{K}(Y)\rangle. \tag{3.9}$$

Intuitively speaking, the weight is like an unnormalized probability. If $W(Y)=0$, this means the history Y violates the dynamical laws of quantum theory, and thus the probability that it will occur is zero. Next, define a function

$$\theta(X|Y)=W(XY)/W(Y) \tag{3.10}$$

on pairs of histories X and Y, as long as the right side of (3.10) makes sense, that is, $XY=YX$ is a projector, and $W(Y)>0$. Under appropriate circumstances, described in Secs. IV and V, $\theta(X|Y)$, which is obviously non-negative, functions as a conditional probability of X given Y, which is why we write its arguments separated by a bar.

Let Y and Y' be projectors in the Boolean algebra $\mathcal{F}$ or histories based upon (2.7). In the analogous classical situation, where $W(Y)$ is the ''volume'' of phase space occupied at a single time by all the points lying on trajectories which pass, at the appropriate times, through all the cells specified by the history Y, the weight function is additive in the sense that

$$YY'=0 \quad \text{implies} \quad W(Y+Y')=W(Y)+W(Y'). \tag{3.11}$$

However, this equation need not hold for a quantum system, because W is defined by the quadratic expression (3.9). Indeed, in order for (3.11) to hold it is necessary and sufficient that for all Y and Y' in $\mathcal{F}$,

$$YY'=0 \quad \text{implies} \quad \mathrm{Re}\langle K(Y),K(Y')\rangle=0, \tag{3.12}$$

where Re denotes the real part. We shall refer to (3.12) as a *consistency condition*, and, in particular, as the *weak* consistency condition, in contrast to the *strong* consistency condition:

$$YY'=0 \quad \text{implies} \quad \langle K(Y),K(Y')\rangle=0. \tag{3.13}$$

Note that replacing K by $\hat{K}$ everywhere in (3.12) or (3.13) leads to an equivalent condition.

The condition (3.13) is equivalent to the requirement that

$$j\neq k \quad \text{implies} \quad \langle K(F_j),K(F_k)\rangle=0, \tag{3.14}$$

for the $\{F_j\}$ in the decomposition of the identity (2.7). In other words, strong consistency corresponds to requiring that the weight operators corresponding to the minimal elements of $\mathcal{F}$ be orthogonal to each other. This orthogonality requirement, which was pointed out in [24], is closely related to the consistency condition employed by Gell-Mann and Hartle [12,13], the vanishing of the off-diagonal elements of an appropriate ''decoherence functional.'' To express the weak consistency condition in similar terms requires that one replace (3.8) with the inner product

$$\langle\langle A,B\rangle\rangle=\mathrm{Re}(\mathrm{Tr}[A^{\dagger}B])=\langle\langle B,A\rangle\rangle, \tag{3.15}$$

which is appropriate when the linear operators on $\mathcal{H}$ are thought of as forming a *real* vector space (i.e., multiplication is restricted to real scalars). Because $\mathcal{F}$ consists of sums with real coefficients, (2.8), a real vector space is not an unnatural object to introduce into the formalism, even if it is somewhat unfamiliar. Thus the counterpart of (3.14) in the case of weak consistency is

$$j\neq k \quad \text{implies} \quad \langle\langle K(F_j),K(F_k)\rangle\rangle=0. \tag{3.16}$$

The use of a weak consistency condition has the advantage that it allows a wider class of consistent families in the quantum formalism. However, greater generality is not always a virtue in theoretical physics, and it remains to be seen whether there are ''realistic'' physical situations where it is actually helpful to employ weak rather than strong consistency. In any case, the formalism developed below works equally well if $\langle,\rangle$ is replaced by $\langle\langle,\rangle\rangle$, so that our use of the former can be regarded as simply a matter of convenience of exposition. For some further comments on the relationship of our consistency conditions and those of Gell-Mann and Hartle, see the Appendix.

Henceforth we shall refer to a consistent Boolean algebra of history projectors as a *framework*, or *consistent family*, and regard it as the appropriate quantum counterpart of the event algebra in ordinary probability theory. Since a Boolean algebra of histories is always based upon a decomposition of the (history) identity, as in (2.7), we shall say that such a decomposition is *consistent* if its minimal elements satisfy (3.14) or (3.16), as the case may be, and will occasionally, as a matter of convenience, refer to such a decomposition as a ''framework,'' meaning thereby the corresponding Boolean algebra which it generates.

While the consistency condition is not essential for defining a quantum probability, it is convenient for technical reasons, and seems to be adequate for representing whatever can be said realistically about a *closed* quantum system. (Regarding open quantum systems, see Sec. VIII B.) Note that while the concept of consistency properly applies to a Boolean algebra, or a decomposition of $\breve{I}$, an individual history Y can be inconsistent in the sense that $K(Y)$ and $K(\breve{I}-Y)$ are not orthogonal, and hence there exists no consistent family which contains Y.

It is sometimes convenient to focus one's attention on a Boolean algebra of histories for which the maximum element is not the identity $\breve{I}$ on the history space, but a smaller projector. For example, one may be interested in a family $\mathcal{G}$ of histories for which there is a fixed initial event at t_1, corresponding to the projector A. In this case it is rather natural to replace (2.7) with

$$\breve{A}=\sum_i G_i, \quad G_i^{\dagger}=G_i, \quad G_iG_j=\delta_{ij}G_i, \tag{3.17}$$

where $\breve{A}$ is defined as

$$\breve{A}=A\odot I\odot I\cdots I. \tag{3.18}$$

The largest projector or maximum element on the Boolean algebra of projectors generated by the $\{G_j\}$, in analogy with (2.8), is $\breve{A}$ rather than $\breve{I}$. If this algebra is consistent, which is to say the weight operators corresponding to the different G_i are mutually orthogonal, then one can add the projector $\breve{I}-\breve{A}$ to the algebra and the resulting family, whose maximum element is now $\breve{I}$, is easily seen to be consistent. The same comment applies to families in which there is a fixed final event B, and to those, such as in [5], with a fixed initial *and* final event. However, if an event C at an intermediate time is held fixed, the consistency of the family based upon the corresponding $\breve{C}$ is not automatic. Once again, it seems that for a description of closed quantum systems, the appropriate requirement is that an acceptable framework either be a consistent Boolean algebra whose maximum element is $\breve{I}$, or a subalgebra of such an algebra.

From now on we shall adopt the following as a fundamental principle of quantum reasoning: *A meaningful description of a (closed) quantum-mechanical system, including its time development, must employ a single framework.*

IV. PROBABILITIES AND REFINEMENTS

Throughout this section, and in the rest of the paper, a *framework* will be understood to be a Boolean algebra of projectors on the history space, based upon a decomposition of the identity as in (2.7), and satisfying a consistency condition, either (3.12) or (3.13). In the special case where only a single time is involved, the consistency condition is not needed (or is automatically satisfied).

A *probability distribution* Pr() on a framework $\mathcal{F}$ is an assignment of a non-negative number $\Pr(Y)$ to every history Y in $\mathcal{F}$ by means of the formula

$$\Pr(Y)=\sum_i v_i \Pr(F_i)=\sum_i \theta(P|F_i)\Pr(F_i), \tag{4.1}$$

where the v_i are defined in (2.8), and the probabilities $\Pr(F_i)$ of the minimal elements are arbitrary, subject only to the conditions

$$\Pr(F_i)\geqslant 0, \quad \sum_i \Pr(F_i)=1, \tag{4.2}$$

$$W(F_i)=0 \quad \text{implies} \quad \Pr(F_i)=0. \tag{4.3}$$

Of course, (4.2) are the usual conditions of any probability theory, while (4.3), using the weight W defined in (3.9), expresses the requirement that zero probability be assigned to any history which is dynamically impossible. If $W(F_i)$ is zero, $\theta(P|F_i)$ is undefined, and we set the corresponding term in the second sum in (4.1) equal to zero, which is plausible in view of (4.3). In addition, note that, because the weights are additive for histories in a (consistent) framework, (4.3) implies that whenever $W(Y)$ is zero, $\Pr(Y)$ vanishes.

Apart from the requirement (4.3), quantum theory by itself does not specify the probability distribution on the different histories. Thus these probabilities must be assigned on the basis of various data known or assumed to be true for the quantum system of interest. A typical example is one in which a system is known, or assumed, to be in an initial state $|\psi_0\rangle$ at an initial time t_0, which would justify assigning probabilities 1 and 0, respectively, to the projectors

$$\psi_0=|\psi_0\rangle\langle\psi_0|, \quad \tilde{\psi}_0=I-\psi_0 \tag{4.4}$$

at the initial time.

The process of refining a probability distribution plays an important role in the system of quantum reasoning described in Sec. V below. We shall say that the framework $\mathcal{G}$ is a *refinement* of $\mathcal{F}$, and $\mathcal{F}$ a *coarsening* of $\mathcal{G}$, provided $\mathcal{F}\subset\mathcal{G}$, that is, provided every projector which appears in $\mathcal{F}$ also appears in $\mathcal{G}$. A collection $\{\mathcal{F}_i\}$ of two or more frameworks is said to be *compatible* provided there is a common refinement, i.e., some framework $\mathcal{G}$ such that $\mathcal{F}_i\subset\mathcal{G}$ for every i. If the collection is compatible, there is a smallest (coarsest) common refinement, and we shall call this the framework *generated by* the collection, or simply the *generated* framework. (Note that in constructing refinements it may be necessary to extend certain histories to additional times by introducing an identity operator at these times, as discussed above in Sec. II.)

Frameworks not compatible with each other are called *incompatible*. Incompatibility of $\mathcal{F}_1$ and $\mathcal{F}_2$ can arise in two somewhat different ways. First, some of the projectors in $\mathcal{F}_1$ may not commute with projectors in $\mathcal{F}_2$, and thus one cannot construct the Boolean algebra of projectors needed for a common refinement. Second, even if the common Boolean algebra can be constructed, it may not be consistent, despite the fact that the algebras for both $\mathcal{F}_1$ and $\mathcal{F}_2$ are consistent.

Given a probability distribution Pr() on $\mathcal{F}$ and a refinement $\mathcal{G}$ of $\mathcal{F}$, we can define a probability $\Pr'()$ on $\mathcal{G}$ by means of the *refinement rule*:

$$\Pr'(G)=\sum_i \theta(G|F_i)\Pr(F_i). \tag{4.5}$$

Here G is any projector in $\mathcal{G}$, and if $\Pr(F_i)$ is zero, the corresponding term in the sum is set equal to zero, thus avoiding any problems when θ is undefined. Note that (4.5) assigns zero probability to any G having zero weight, and in particular to minimal elements of $\mathcal{G}$ with zero weight. Hence $\Pr'()$ satisfies the analog of (4.3), and it is easily checked that it satisfies the conditions corresponding to (4.2). In view of (4.1) and the fact that $\mathcal{G}$ is a refinement of $\mathcal{F}$, $\Pr'(F)$ and $\Pr(F)$ are identical for any $F\in\mathcal{F}$. Consequently there is little danger of confusion if the prime is omitted from $\Pr'()$.

It is straightforward to show that if $\mathcal{G}$ is a refinement of $\mathcal{F}$, $\Pr'()$ the probability on $\mathcal{G}$ obtained by applying the refinement rule to Pr() on $\mathcal{F}$, and $\mathcal{J}$ a refinement of $\mathcal{G}$, then the same refined probability $\Pr''()$ on $\mathcal{J}$ is obtained either by applying the refinement rule to $\Pr'()$ on $\mathcal{G}$, or by regarding $\mathcal{J}$ as a refinement of $\mathcal{F}$, and applying the refinement rule directly to Pr(). Note that if A is a projector which occurs in some refinement of $\mathcal{F}$, then $\Pr(A)$ is the same in any refinement of $\mathcal{F}$ in which A occurs. This follows from noting that $\Pr(A)$ is given by (4.5), with A in the place of G, and that $\theta(A|F_i)$ is simply a ratio of weights, and thus does not depend upon the framework. (The same comment applies, of course, if A is a member of $\mathcal{F}$, and hence a member of every refinement of $\mathcal{F}$.) Thus, relative to the properties just discussed, the refinement rule is internally consistent.

The significance of the refinement rule can best be appreciated by considering some simple examples. As a first example, let $\mathcal{F}$ be the family whose minimal elements are the two projectors ψ_0 and $\tilde{\psi}_0$ at the single time t_0, see (4.4), and $\mathcal{G}$ a refinement whose minimal elements are of the form

$$\psi_0 \odot \psi_1^\alpha, \quad \tilde{\psi}_0 \odot \psi_1^\alpha, \tag{4.6}$$

where the states $|\psi_1^\alpha\rangle$, with $\alpha = 1,2, \ldots$ form an orthonormal basis of $\mathcal{H}$, and the corresponding projectors ψ_1^α, defined using dyads as in (4.4), represent properties of the quantum system at time t_1. Using the fact that

$$W(\psi_0 \odot \psi_1^\alpha) = |\langle \psi_1^\alpha | \psi_0 \rangle|^2, \tag{4.7}$$

and the assumption that $\Pr(\psi_0 \odot I) = 1$ in $\mathcal{F}$, one arrives at the conclusion that

$$\Pr(\psi_0 \odot \psi_1^\alpha) = |\langle \psi_1^\alpha | \psi_0 \rangle|^2 \tag{4.8}$$

in $\mathcal{G}$, which is just the Born rule for transition probabilities. Thus in this example the refinement rule embodies the consequences of quantum dynamics for the time development of the system.

A second example involves only a single time. Let the projector D on a subspace of dimension d be a minimal element of $\mathcal{F}$ to which is assigned a probability p. If in the refinement $\mathcal{G}$ of $\mathcal{F}$ one has two minimal elements D_1 and D_2, projectors onto subspaces of dimension d_1 and d_2, whose sum is D, then in the refined probability $\Pr'()$, D_1 is assigned a probability pd_1/d and D_2 a probability pd_2/d. That is to say, the original probability is split up according to the sizes of the respective subspaces. While in this example the refinement rule is not a consequence of the dynamical laws of quantum theory, it is at least not inconsistent with them.

The following result on conditional probabilities is sometimes useful. Let D be a minimal element of a framework $\mathcal{D}$ having positive weight, and assign to $\mathcal{D}$ the probability

$$\Pr(D) = 1, \quad \Pr(\breve{I} - D) = 0. \tag{4.9}$$

Let $\mathcal{E}$ be a refinement of $\mathcal{D}$, and E some element of $\mathcal{E}$ with positive weight such that

$$ED = E. \tag{4.10}$$

Then for E' any element of $\mathcal{E}$,

$$\Pr(E'|E) = \theta(E'|E). \tag{4.11}$$

We omit the derivation, which is straightforward. Note that it is essential that D be a minimal element of $\mathcal{D}$, and that (4.10) be satisfied; it is easy to construct examples violating one or the other of these conditions for which (4.11) does not hold.

V. QUANTUM REASONING

The type of quantum reasoning we shall focus on in this section is that in which one starts with some information about a system, known or assumed to be true, and from these *initial data* tries to reach valid *conclusions* which will be true if the initial data are correct. As is usual in logical systems, the rules of reasoning do not by themselves certify the correctness of the initial data; they merely serve to define a valid process of inference. Note that the term "initial" refers to the fact that these data represent the beginning of a logical argument, and has nothing to do with the temporal order of the data and conclusions in terms of the history of the quantum system. Thus the conclusions of the argument may well refer to a point in time prior to that of the initial data.

Since quantum mechanics is a stochastic theory, the initial data and the final conclusions will in general be expressed in the form of probabilities, and the rules of reasoning are rules for deducing probabilities from probabilities. In this context, "logical rules" for deducing true conclusions from true premises refer to limiting cases in which certain probabilities are 1 (true) or 0 (false). Since probabilities in ordinary probability theory always refer to some sample space, we must embed quantum probabilities referring to properties or the time development of a quantum system in an appropriate framework. Both the initial data and the final conclusions of a quantum argument should be thought of as labeled by the corresponding frameworks. Likewise, the truth or falsity of a quantum proposition, and more generally its probability, is relative to the framework in which it occurs.

As long as only a single framework is under discussion, the rules of quantum reasoning are the usual rules for manipulating probabilities. In particular, if the initial data are given as a probability distribution $\Pr(\ )$ on a framework $\mathcal{D}$, we can immediately say that a proposition represented by a projector D in $\mathcal{D}$ with $\Pr(D) = 1$ is true (in the framework $\mathcal{D}$ and assuming the validity of the initial data), whereas if $\Pr(D) = 0$, the proposition is false (with the same qualifications). Given a framework $\mathcal{D}$, there are certain propositions for which the probability is 1 for any probability distribution satisfying the rules (4.2) and (4.3), and we call these *tautologies*; their negations are *contradictions*. For example, given any $D \in \mathcal{D}$, the proposition "D or not D," which maps onto the projector $D \vee (\breve{I} - D) = \breve{I}$, is always true, whereas any history in $\mathcal{D}$ which has zero weight, meaning that it violates the dynamical laws, is always false.

Arguments which employ only a single framework are too restrictive to be of much use in quantum reasoning. Hence we add, as a fundamental principle, the following *refinement rule*: if a probability distribution $\Pr(\ )$ is given for a framework $\mathcal{F}$, and $\mathcal{G}$ is a refinement of $\mathcal{F}$, then one can infer the probability distribution $\Pr'(\ )$ on $\mathcal{G}$ given by the refinement rule introduced in Sec. IV, see (4.5). As noted in Sec. IV, the refinement rule embodies all the dynamical consequences of quantum theory. Replacing $\Pr'(\ )$ by $\Pr(\ )$ will generally cause no confusion, because the two are identical on $\mathcal{F}$.

Thus the general scheme for quantum reasoning is the following. One begins with data in the form of a probability distribution $\Pr(\ )$ on a framework $\mathcal{D}$, calculates the refined probability distribution on a refinement $\mathcal{E}$ of $\mathcal{D}$, and applies the standard probability calculus to the result. Note that the internal consistency of the refinement rule of Sec. IV has the following important consequence: If a history A occurs in some refinement of $\mathcal{D}$, then $\Pr(A)$ is the same in any refinement of $\mathcal{D}$ in which A occurs. In particular, it is impossible to deduce from the same initial data that some proposition

A is both true (probability 1) and false (probability 0). In this sense the scheme of quantum reasoning employed here is internally consistent.

Even in the case of ''complete ignorance,'' that is to say, in the absence of any initial data, this scheme can generate useful results. Consider the trivial framework $\mathcal{D}=\{0,\breve{I}\}$ for which the only probability assignment consistent with (4.2) and (4.3) is $\Pr(\breve{I})=1$. Let $\mathcal{E}$ be any framework which uses the same Hilbert space as $\mathcal{D}$, and which is therefore a refinement of $\mathcal{D}$. For any E' and E in $\mathcal{E}$ with $W(E)>0$, (4.11) applies, so that a logical consequence of complete ignorance is

$$\Pr(E'|E)=\theta(E'|E). \tag{5.1}$$

For example, if we apply (5.1) to the case where $\mathcal{E}$ is the framework consisting of the elements in (4.6), one consequence is

$$\Pr(\psi_1^\alpha|\psi_0)=|\langle\psi_1^\alpha|\psi_0\rangle|^2. \tag{5.2}$$

Hence while we cannot, in the absence of initial data, say what the initial state is, we can nevertheless assert that *if* the initial state is ψ_0 at t_0, *then* at t_1 the probability of ψ_1^α is given by (5.2). Thus even complete ignorance allows us to deduce the Born formula as a *conditional* probability.

In the case in which some (nontrivial) initial data are given, perhaps consisting of separate pieces of information associated with different frameworks, these must first be combined into a single probability distribution associated with a single framework before the process of refinement can begin. For example, the data may consist of a collection of pairs $\{(\mathcal{D}_i,D_i)\}$, where D_i is known or assumed to be true in framework $\mathcal{D}_i$. If the $\{\mathcal{D}_i\}$ are incompatible frameworks, the initial data must be rejected as mutually incompatible; they cannot all apply to the same physical system. If they are compatible, let $\mathcal{D}$ be the framework they generate, and let

$$D=D_1D_2D_3\cdots \tag{5.3}$$

be the projector corresponding to the simultaneous truth of the different D_i. Then we assign probability 1 to D and 0 to its complement $\breve{I}-D$ in the framework $\mathcal{D}$. [Of course, this probability assignment is impossible if $W(D)=0$, which indicates inconsistency in the initial data.] Note that if D is a minimal element of $\mathcal{D}$, then conditional probabilities are given directly in terms of the θ function, (4.11), for any E satisfying (4.10).

Of course, in general the initial data may be given not in the form of certain projectors known (or assumed) to be true, but instead as probabilities in different frameworks. If the frameworks are incompatible, the data, of course, must be rejected as mutually incompatible. If the frameworks are compatible, the data must somehow be used to generate a probability distribution on the generated framework $\mathcal{D}$. We shall not discuss this process, except to note that because it can be carried out in the single framework $\mathcal{D}$, whatever methods are applicable for the corresponding case of ''classical probabilities'' can also be applied to the quantum problem.

The requirement that the initial data be embodied in a single framework is just a particular example of the general principle already stated at the end of Sec. III: quantum descriptions, and thus quantum reasoning referring to such descriptions, must employ a single framework. This requirement is not at all arbitrary when one remembers that probabilities in probability theory only have a meaning relative to some sample space or algebra of events, and that the quantum framework is playing the role of this algebra. Probabilities in classical statistical mechanics satisfy precisely the same requirement, where it is totally uninteresting because there is never any problem combining information of various sorts into a common description using, say, a single coarse graining of the phase space (or a family of coarse grainings indexed by the time). What distinguishes quantum from classical reasoning is the presence in the former, but not in the latter, of incompatible frameworks. Thus the rules governing incompatible frameworks are necessarily part of the foundations of quantum theory itself.

Note that the system of reasoning employed here does *not* allow a ''coarsening rule'' in which, if $\mathcal{F}$ is a refinement of $\mathcal{E}$, and a probability distribution Pr( ) is given on $\mathcal{F}$, one can from this deduce a probability distribution Pr*( ) on $\mathcal{E}$ which is simply the restriction of Pr( ) to $\mathcal{E}$, i.e.,

$$E\in\mathcal{E}:\quad \Pr{}^*(E)=\Pr(E). \tag{5.4}$$

The reason such a coarsening rule is not allowed is that if it is combined with the refinement rule, the result is a system of reasoning which is internally inconsistent. For example, if we start with the probability distribution Pr( ) on $\mathcal{F}$, define Pr* on $\mathcal{E}$ by means of (5.3), and then apply the refinement rule to Pr* in order to derive a probability Pr*′ on $\mathcal{F}$, the latter will in general not coincide with the original Pr( ). Worse than this, there are cases in which successive applications of coarsening and refinement to different quantum frameworks can lead to contradictions: starting with $\Pr(A)=1$ in one framework one can eventually deduce $\Pr(A)=0$ in the same framework. To be sure, it is the combination of coarsening and refinement which gives rise to inconsistencies, and the system of reasoning would be valid if only the coarsening rule were permitted. However, such a system would not be very useful. And, indeed, there is a sense in which a coarsening rule is also not really needed. If $\mathcal{F}$ is a refinement of $\mathcal{E}$, and a probability distribution is given on $\mathcal{F}$, then it already assigns a probability to every projector E in $\mathcal{E}$, in the sense that E is already an element of $\mathcal{F}$. But once again this serves to emphasize the fact that the question of which sample space one is using, while usually a trivial and uninteresting question in classical physics, is of utmost importance in quantum theory.

One way of viewing the difference between quantum and classical reasoning is that whereas in both cases the validity of a conclusion depends upon the data from which it was derived, in the classical case one can forget about the data once the conclusion has been obtained, and no contradiction will arise when this conclusion is inserted as the premise of another argument. In the quantum case, it is safe to forget the original data *as a probability distribution*, but the fact that the data were embodied in a particular *framework* cannot be ignored: the conclusion must be expressed relative to a framework, and since that framework is either identical to, or has been obtained by refinement of the one containing the initial data, the ''framework aspect'' of the initial data has not been forgotten. The same is true, of course, in the classical case, but the framework can safely be ignored, because classical physics does not employ incompatible frameworks.

Another way in which quantum reasoning is distinctly different from its classical counterpart is that from the *same data* it is possible to draw *different conclusions* in *mutually incompatible frameworks*. Because the frameworks are incompatible, the conclusions cannot be combined, a situation which is bizarre from the perspective of classical physics, where it never arises. See the examples below, and the discussion in Sec. VII A.

VI. EXAMPLES

A. Spin-half particle

As a first example, consider a spin-one-half particle, for which the Hilbert space is two dimensional, and a framework $\mathcal{Z}$ corresponding to a decomposition of the identity:

$$I = Z^+ + Z^-, \quad Z^\pm = |Z^\pm\rangle\langle Z^\pm|, \tag{6.1}$$

where $|Z^+\rangle$ and $|Z^-\rangle$ are the states in which S_z has the values $+1/2$ and $-1/2$, respectively, in units of $\hbar$. Within this framework, the statement ''$S_z = 1/2$ or $S_z = -1/2$'' is a tautology because it corresponds to the projector I, see (2.4), which has probability 1 no matter what probability distribution is employed. Also, if $S_z = 1/2$ is true (probability 1), then $S_z = -1/2$ is false (probability 0), because $\Pr(Z^+) + \Pr(Z^-)$ is always equal to one.

Of course, we come to precisely the same type of conclusion if, instead of $\mathcal{Z}$, we use the framework $\mathcal{X}$ corresponding to

$$I = X^+ + X^-, \quad X^\pm = |X^\pm\rangle\langle X^\pm|, \tag{6.2}$$

where

$$|X^+\rangle = (|Z^+\rangle + |Z^-\rangle)/\sqrt{2}, \quad |X^-\rangle = (|Z^+\rangle - |Z^-\rangle)/\sqrt{2} \tag{6.3}$$

are states in which S_x is $+1/2$ or $-1/2$. However, the frameworks $\mathcal{Z}$ and $\mathcal{X}$ are clearly incompatible because the projectors $X^\pm$ do not commute with $Z^\pm$. Therefore, whereas $S_z = 1/2$ is a meaningful statement, which may be true or false within the framework $\mathcal{Z}$, it makes no sense within the framework $\mathcal{X}$, and, similarly, $S_x = 1/2$ is meaningless within the framework $\mathcal{Z}$. Consequently, ''$S_z = 1/2$ *and* $S_x = 1/2$'' is a meaningless statement within quantum mechanics interpreted as a stochastic theory, because a meaningful description of a quantum system must belong to some framework, and there is no framework which contains both $S_z = 1/2$ and $S_x = 1/2$ at the same instant of time.

A hint that ''$S_z = 1/2$ *and* $S_x = 1/2$'' is meaningless can also be found in elementary textbooks, where the student is told that there is no way of simultaneously *measuring* both S_z and S_z, because attempting to measure one component will disturb the other in an uncontrolled way. While this is certainly true, one should note that the fundamental reason no simultaneous measurement of both quantities is possible is that there is nothing to be measured: the simultaneous values do not exist. Even very good experimentalists cannot measure what is not there; indeed, this inability helps to distinguish them from their less talented colleagues. We return to the topic of measurement in Sec. VI C below.

As an application of the refinement rule of Sec. V, we can start with ''complete ignorance,'' expressed by assigning probability 1 to I in the framework $\mathcal{D} = \{0, I\}$, and refine this to a probability on $\mathcal{Z}$. The result is

$$\Pr(Z^+) = 1/2 = \Pr(Z^-), \tag{6.4}$$

that is, the particle is unpolarized. Were we instead to use $\mathcal{X}$ as a refinement of $\mathcal{D} = \{0, I\}$, the conclusion would be

$$\Pr(X^+) = 1/2 = \Pr(X^-). \tag{6.5}$$

Thus we have a simple example of how quantum reasoning starting from a particular datum [in this case the rather trivial $\Pr(I) = 1$] can reach two different conclusions in two different frameworks. Each conclusion is correct by itself, in the sense that it could be checked by experimental measurement, but the conclusions cannot be combined into a common description of a single quantum system.

B. Harmonic oscillator

The intuitive or ''physical'' meaning of a projector on a subspace of the quantum Hilbert space depends to some extent on the framework in which this projector is embedded, as illustrated by the following example.

Let $|n\rangle$ with energy $(n + 1/2)\hbar\omega$ denote the nth energy eigenstate of a one-dimensional oscillator. (In order to have a finite-dimensional Hilbert space, we must introduce an upper bound for n; say $n < 10^{80}$.) Throughout the following discussion it will be convenient to assume that the energy is expressed in units of $\hbar\omega$, or, equivalently, $\hbar\omega = 1$.

Define the projectors

$$B_n = |n\rangle\langle n|, \quad P = B_1 + B_2, \quad \tilde{P} = I - P. \tag{6.6}$$

In any framework which contains it, P can be interpreted to mean that ''the energy is less than 2,'' but in general it is *not* correct to think of P as meaning ''the energy is 1/2 or 3/2.'' The latter is a correct interpretation of P in the framework based on

$$I = B_0 + B_1 + \tilde{P}, \tag{6.7}$$

because the projectors B_0 and B_1 can be interpreted as saying that the energy is 1/2 and 3/2, respectively, and P is their sum; see (2.4). However, it is totally incorrect to interpret P to mean ''the energy is 1/2 or 3/2'' when P is an element in the framework based on

$$I = C^+ + C^- + \tilde{P}, \tag{6.8}$$

where C^+ and C^- are projectors onto the states

$$|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}, \quad |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}. \tag{6.9}$$

Because C^+ and C^- do not commute with B_0 and B_1, the assertion that ''the energy is 1/2'' makes no sense if we use (6.8), and the same is true of ''the energy is 3/2.'' Combining them with ''or'' does not help the situation unless one agrees that ‘‘the energy is 1/2 or 3/2’’ is a sort of shorthand for the correct statement that ‘‘the energy is not greater than 3/2.’’ And since even the latter can easily be misinterpreted, it is perhaps best to use the projector P itself, as defined in (6.6), rather than an ambiguous English phrase, if one wants to be very careful and avoid all misunderstanding.

The meaning of P in the smallest framework which contains it, the one based upon

$$I = P + \widetilde{P}, \tag{6.10}$$

involves an additional subtlety. Since neither B_0 nor B_1 are part of this framework, it is, at least formally, incorrect to say that within this framework P means ‘‘the energy is 1/2 or 3/2.’’ On the other hand, the (assumed) truth of P in (6.10) corresponds to $\Pr(P)=1$, and since (6.7) is a refinement of (6.10), the refinement rule allows us to conclude that the probability of $B_0 + B_1$ in (6.7) is also equal to one, and therefore ‘‘B_0 or B_1’’ is true in the framework (6.7). And since, at least in informal usage, the ‘‘meaning’’ of a physical statement includes various logical consequences which the physicist regards as more or less intuitively obvious, part of the informal meaning or ‘‘aura’’ of P in the framework (6.10) is ‘‘B_0 or B_1.’’ However, because of the possibility of making alternative logical deductions from the truth of P, such as ‘‘ C^+ or C^-,’’ the best policy, if one wants to be precise, is to pay attention to the framework, and say that the truth of P in (6.10) means that ‘‘the energy is 1/2 or 3/2 *in the framework based upon* (6.7).’’ To be sure, in informal discourse one might omit the final qualification on the grounds that the phrase ‘‘the energy is 1/2 or 3/2’’ itself singles out the appropriate framework. The point, in any case, is that quantum descriptions necessarily take place inside frameworks, and clear thinking requires that one be able to identify which framework is being used at any particular point in an argument.

As another example of a possible pitfall, suppose that we know that the energy is 5/2. Can we conclude from this that the energy is *not* equal to 1/2? There is an almost unavoidable temptation to say that the second statement is an immediate consequence of the first, but in fact it is or is not depending upon the framework one is using. To say that the energy is 5/2 means that we are employing a framework which includes B_2 as one of its elements. If this framework also includes B_0, the fact that B_0 is false (probability 0) follows at once from the assumption that B_1 is true (probability 1), by an elementary argument of probability theory, so that, indeed, the energy is not equal to 1/2. If the framework does not include B_0, but has some refinement which does include B_0, we can again conclude that *within this refined framework*—which, note, is not the original framework—the energy is not equal to 1/2. However, if the original framework is incompatible with B_0 (e.g., it might contain C^+), then the fact that the energy is 5/2 does *not* imply that the energy is not equal to 1/2. Ignoring differences between different frameworks quickly leads to paradoxes, as in the example in Sec. VI D below.

C. Measurement of spin

Textbook discussions of quantum measurement suffer from two distinct but related ‘‘measurement problems.’’ The first is that the use of unitary time development can result in MQS (macroscopic quantum superposition) or ‘‘Schrödinger’s cat’’ states, which must then somehow be explained away in a manner which has been justly criticized by Bell [3]. The second is that many measurements of properties of quantum particles, such as energy or momentum, when actually carried out in the laboratory result in large changes in the measured property. Since one is generally interested in the property of the particle before its interaction with the measurement apparatus, the well-known von Neumann ‘‘collapse’’ description of the measurement is unsatisfactory (quite aside from the never-ending debate about what such a ‘‘collapse’’ really means). The system of quantum reasoning developed in Sec. V resolves both problems through the use of appropriate frameworks, as illustrated in the following discussion of the measurement of the spin of a spin-half particle.

The particle and the measuring apparatus should be thought of as a single closed quantum system, with Hilbert space

$$\mathcal{H} = \mathcal{S} \otimes \mathcal{A}. \tag{6.11}$$

Here $\mathcal{S}$ is the two-dimensional spin space for the spin-half particle, and $\mathcal{A}$ is the Hilbert space for all the remaining degrees of freedom: the particles constituting the apparatus, and the center of mass of the spin-half particle. We consider histories involving three times, $t_0 < t_1 < t_2$, and suppose that the relevant unitary time development, indicated by $\mapsto$, has the form

$$\begin{aligned} |Z^+A\rangle &\mapsto |Z^+A'\rangle \mapsto |P^+\rangle, \\ |Z^-A\rangle &\mapsto |Z^-A'\rangle \mapsto |P^-\rangle, \end{aligned} \tag{6.12}$$

where $|Z^+\rangle$ and $|Z^-\rangle$ are the spin states for S_z equal to $\pm 1/2$, as in (6.1), $|A\rangle$ is a state on $\mathcal{A}$ at time t_0 in which the particle is traveling towards the apparatus, and the apparatus is ready for the measurement, $|A'\rangle$ is the corresponding state at t_1, with the particle closer to, but still not at the apparatus, and $|P^+\rangle$ and $|P^-\rangle$ are states on $\mathcal{H}$ at t_2, after the measurement is complete, which correspond to the apparatus indicating, through the position of a pointer, the results of measuring S_z for the particle. Note that the spin state of the particle at t_2 is included in $|P^+\rangle$ and $|P^-\rangle$, and we do *not* assume that it remains unchanged during the measuring process. Such a description using only pure states is oversimplified, but we will later indicate how essentially the same results come out of a more realistic discussion.

To keep the notation from becoming unwieldy, we use the following conventions. A letter outside a ket indicates the dyad for the corresponding projector; e.g., A stands for $|A\rangle\langle A|$. Next, we make no distinction in notation between A as a projector on $\mathcal{A}$ and as the projector $I \otimes A$ on $\mathcal{S} \otimes \mathcal{A}$; similarly, Z^+ stands both for the projector on $\mathcal{S}$ and for $Z^+ \otimes I$ on $\mathcal{H}$. Finally, projectors on the history space $\breve{\mathcal{H}}$ carry subscripts which indicate the time, as in the following examples:

$$A_0 = A \odot I \odot I, \quad P_2^+ = I \odot I \odot P^+. \tag{6.13}$$

We first consider a framework associated with the decomposition

$$\breve{I}=\tilde{A}_0+\{Z_0^+A_0+Z_0^-A_0\}\{P_2^++P_2^-+P_2^*\}, \qquad (6.14)$$

containing seven minimal elements, of the identity on $\breve{\mathcal{H}}$, where

$$\tilde{A}=I-A, \quad P^*=I-(P^++P^-). \qquad (6.15)$$

The family generated by (6.14) is easily shown to be consistent, and the following weights are a consequence of (6.12):

$$W(Z_0^+A_0P_2^+)=1=W(Z_0^-A_0P_2^-),$$

$$W(Z_0^-A_0P_2^+)=0=W(Z_0^+A_0P_2^-). \qquad (6.16)$$

In addition, weights of histories which include both A_0 and P_2^* vanish. Note that the weights are additive, so that, for example,

$$W(A_0P_2^+)=W(Z_0^+A_0P_2^+)+W(Z_0^-A_0P_2^+)=1. \quad (6.17)$$

If we assume that the initial data correspond either to ''complete ignorance,'' see the remarks preceding (5.1), or to probability 1 for A_0 in the framework corresponding to $\breve{I}=A_0+\tilde{A}_0$, see (4.9), we can equate probabilities which include A_0 as a condition with the corresponding θ functions, (4.11), and the latter can be computed using (3.10). The results include

$$\Pr(P_2^+|Z_0^+A_0)=1, \quad \Pr(P_2^-|Z_0^+A_0)=0, \qquad (6.18)$$

$$\Pr(P_2^+|A_0)=1/2=\Pr(P_2^-|A_0), \qquad (6.19)$$

$$\Pr(Z_0^+|P_2^+A_0)=1, \quad \Pr(Z_0^-|P_2^+A_0)=0. \qquad (6.20)$$

The probabilities in (6.18) are certainly what we would expect: if at t_0 we have $S_z=1/2$, then at t_2 the apparatus pointer will surely be in state P^+ and not in state P^-. On the other hand, if we are ignorant of S_z at t_0, the results in (6.19) are those appropriate for an unpolarized particle. Equally reasonable is the result (6.20), which tells us that if at t_2 the pointer is at P^+, the spin of the particle at t_0 was given by $S_z=1/2$, not $S_z=-1/2$; that is, the measurement reveals a property which the particle had before the measurement took place.

Next consider, as an alternative to (6.14), the framework based upon

$$\breve{I}=\tilde{A}_0+\{X_0^+A_0+X_0^-A_0\}\{P_2^++P_2^-+P_2^*\}, \qquad (6.21)$$

where X^+ and X^- are projectors associated with $S_x=\pm 1/2$, see (6.3). It is straightforward to check consistency and calculate the weights:

$$W(X_0^+A_0P_2^+)=1/2=W(X_0^-A_0P_2^-),$$

$$W(X_0^-A_0P_2^+)=1/2=W(X_0^+A_0P_2^-). \qquad (6.22)$$

Once again, weights of histories which include both A_0 and P_2^* vanish. With the same assumptions as before (ignorance, or A_0 at t_0), we obtain

$$\Pr(P_2^+|X_0^+A_0)=1/2, \quad \Pr(P_2^-|X_0^+A_0)=1/2, \qquad (6.23)$$

$$\Pr(X_0^+|P_2^+A_0)=1/2, \quad \Pr(X_0^-|P_2^+A_0)=1/2. \qquad (6.24)$$

In addition, the probabilities in (6.19) are the same in the new framework as in the old, which is not surprising, since they make no reference to S_z or S_x at t_0.

Everyone agrees that (6.23), assigning equal probability to the pointer states P^+ and P^- if at t_0 the spin state is $S_x=1/2$, is the right answer. What is interesting is that, with the formalism used here, the right answer emerges without having to make the slightest reference to a MQS state, and thus there is no need to make excuses of the ''for all practical purposes'' type in order to get rid of it. How have we evaded the problem of Schrödinger's cat?

The answer is quite simple: there is no MQS state at t_2 in the decomposition of the identity (6.21), and therefore there is no reference to it in any of the probabilities. To be sure, we could have investigated an alternative framework based upon

$$\breve{I}=\tilde{A}_0+\{X_0^+A_0+X_0^-A_0\}\{Q_2^++Q_2^-+P_2^*\}, \qquad (6.25)$$

where

$$|Q^+\rangle=(|P^+\rangle+|P^-\rangle)/\sqrt{2}, \quad |Q^-\rangle=(|P^+\rangle-|P^-\rangle)/\sqrt{2} \qquad (6.26)$$

are MQS states. Using this framework one can calculate, for example,

$$\Pr(Q_2^+|X_0^+A_0)=1, \quad \Pr(Q_2^-|X_0^+A_0)=0. \qquad (6.27)$$

Note that there is no contradiction between (6.27) and (6.23), because they have been obtained using mutually incompatible frameworks. Here is another illustration of the fact that quantum reasoning based upon the same data will lead to different conclusions, depending upon which framework is employed. However, conclusions from incompatible frameworks cannot be combined, and the overall consistency of the reasoning scheme is guaranteed, see the discussion in Sec. V, by the fact that only refinements of frameworks are permitted and coarsening is not allowed.

Also note that the framework generated by

$$\breve{I}=\tilde{A}_0+\{Z_0^+A_0+Z_0^-A_0\}\{Q_2^++Q_2^-+P_2^*\} \qquad (6.28)$$

is just as acceptable as that based upon (6.14), and one can perfectly well calculate various probabilities, such as $\Pr(Q_2^+|Z_0^+A_0)$, by means of it. In this case the initial state corresponds to a definite value of S_z, and yet the states at t_2 are MQS states. What this shows is that the real ''measurement problem'' is not the presence of MQS states in certain frameworks; instead, it comes about because one is attempting to address a particular question—P^+ or P^-?—by means of a framework in which this question has no meaning, and hence no answer. Trying to claim that the projector Q^+ is somehow equivalent to the density matrix $(P^++P^-)/2$ for all practical (or any other) purposes is simply making a second mistake in order to correct the results of a more fundamental mistake: using the wrong framework for discussing pointer positions. A major advantage of treating quantum mechanics as a stochastic theory from the outset, rather than adding a probabilistic interpretation as some sort of addendum, is that it frees one from having to think that a quantum system ''must'' develop unitarily in time, and then being forced to make a thousand excuses when the corresponding framework is incompatible with the world of everyday experience.

While the framework based upon (6.21) solves the first measurement problem in the case of a particle which at t_0 has $S_x=1/2$, and is traveling towards an apparatus which will measure S_z, it does not solve the second measurement problem, that of showing that if the apparatus is in the P^+ state at t_2, then the particle actually was in the state $S_z=1/2$ before the measurement. Indeed, we cannot even introduce the projectors Z_0^+ and Z_0^- into the family based on (6.21), because they do not commute with X_0^+ and X_0^-. However, nothing prevents us from introducing them at the later time t_1, and considering the following refinement of (6.21):

$$\breve{I}=\widetilde{A}_0+\{X_0^+A_0+X_0^-A_0\}\{Z_1^++Z_1^-\}\{P_2^++P_2^-+P_2^*\}. \tag{6.29}$$

After checking consistency, one can calculate the following weights:

$$W(X_0^+A_0Z_1^+P_2^+)=1/2=W(X_0^-A_0Z_1^-P_2^-),$$
$$W(X_0^-A_0Z_1^+P_2^+)=1/2=W(X_0^+A_0Z_1^-P_2^-). \tag{6.30}$$

In addition, all the weights with Z_1^+ followed by P_2^-, or Z_1^- followed by P_2^+, vanish. Conditional probabilities can then be computed in the same way as before, with (among others) the following results:

$$\Pr(Z_1^+|P_2^+X_0^+A_0)=1,\quad \Pr(Z_1^-|P_2^+X_0^+A_0)=0. \tag{6.31}$$

That is, given the initial condition X^+A at t_0, and the pointer state P^+ at t_2, one can be certain that S_z was equal to 1/2 and not $-1/2$ at the time t_1 before the measurement took place.

It may seem odd that we can discuss a history in which the particle has $S_x=1/2$ at t_0 and $S_z=1/2$ at t_1 in the absence of a magnetic field which could reorient its spin. To see why there is no inconsistency in this, note that whereas in the two-dimensional Hilbert space $\mathcal{S}$ appropriate for a spin-half particle at a single time there is no way to describe a particle which simultaneously has $S_x=1/2$ and $S_z=1/2$, the same is not true in the history space $\mathcal{S}\odot\mathcal{S}$ for the two times t_0 and t_1, which is four dimensional, and hence analogous to the tensor product space appropriate for describing two (nonidentical) spin-half particles. The fact that the ''incompatible'' spin states occur at different times is the reason that all 13 projectors on the right side of (6.29) commute with one another. To be sure, spin directions cannot be chosen arbitrarily at a sequence of different times without violating the consistency conditions, but in the case of (6.29) these conditions are satisfied. It is also useful to remember that were we applying classical mechanics to a spinning body, there would be no problem in ascribing a definite value to the x component of its angular momentum at one time, and to the z component of its angular momentum at a later time. That this is (sometimes) possible in the quantum case should therefore not be too surprising, as long as one can make sense of this in the appropriate Hilbert space (of histories).

In place of (6.29) we could, of course, use a framework

$$\breve{I}=\widetilde{A}_0+\{X_0^+A_0+X_0^-A_0\}\{X_1^++X_1^-\}\{P_2^++P_2^-+P_2^*\} \tag{6.32}$$

appropriate for discussing the value of S_x at t_1, and from it deduce the results

$$\Pr(X_1^+|P_2^+X_0^+A_0)=1,\quad \Pr(X_1^-|P_2^+X_0^+A_0)=0, \tag{6.33}$$

in place of (6.31). Note, however, that (6.32) and (6.29) are incompatible frameworks, so that one cannot combine (6.31) and (6.33) in any way.

What is the physical significance of two conclusions, (6.31) and (6.33), based upon the same initial data, which are incompatible because the deductions were carried out using incompatible frameworks? One way of thinking about this is to note that (6.31) could be verified by an appropriate idealized measurement which would determine the value of S_z at t_1 without perturbing it, and similarly (6.33) could be checked by a measurement of S_x at t_1 which did not perturb that quantity [25]. However, carrying out both measurements at the same time is not possible.

In summary, the solution of quantum measurement problems, which has hitherto led to a never-ending debate, consists in choosing an appropriate framework. If one wants to find out what the predictions of quantum theory are for the position of a pointer at the end of a measurement, it is necessary (and sufficient) to use a framework containing projectors corresponding to the possible positions. If one wants to know how the pointer position is correlated with the corresponding property of the particle before the measurement took place, it is necessary (and sufficient) to employ a framework containing projectors corresponding to this property at the time in question. While these criteria do not define the framework uniquely, they suffice, because the consistency of the quantum reasoning process as discussed in Sec. V ensures that the same answers will be obtained in any framework in which one can ask the same questions.

As noted above, a description of the measurement process based solely upon pure states, as in (6.12), is not very realistic. It would be more reasonable to replace the one-dimensional projectors A, A', with projectors of very high dimension (corresponding to a macroscopic entropy). This can, indeed, be done without changing the main conclusions. Thus let A be a projector onto a subspace of $\mathcal{A}$ of arbitrarily large (but finite) dimension spanned by an orthonormal basis $|a_j\rangle$, and replace the unitary time evolution (6.12) with

$$|Z^+a_j\rangle\mapsto|Z^+a_j'\rangle\mapsto|b_j^+\rangle,$$
$$|Z^-a_j\rangle\mapsto|Z^-a_j'\rangle\mapsto|b_j^-\rangle, \tag{6.34}$$

where the $|a_j'\rangle$ are, again, a collection of orthonormal states in $\mathcal{A}$, while the $|b_j^{\pm}\rangle$ are orthonormal states on $\mathcal{H}$, the exact nature of which is of no particular interest aside from the fact that they satisfy (6.35) below. Note in particular that nothing is said about the spin of the particle at t_2, as that is entirely irrelevant for the measuring process. Next we assume that P^+ and P^- are projectors onto enormous subspaces of $\mathcal{H}$

(macroscopic entropy) corresponding to the physical property that the apparatus pointer is pointing in the $+$ and the $-$ direction, respectively. As in all cases where one associates quantum projectors with macroscopic events, there will be some ambiguity in the precise definition, but all that matters for the present discussion is that, for all j,

$$P^+|b_j^+\rangle=|b_j^+\rangle, \quad P^+|b_j^-\rangle=0,$$
$$P^-|b_j^-\rangle=|b_j^-\rangle, \quad P^-|b_j^+\rangle=0. \tag{6.35}$$

Using these definitions, one can work out the weights corresponding to the families (6.14), (6.21), (6.29), and (6.32). From them one obtains the same conditional probabilities as before: (6.18) to (6.20), (6.23) and (6.24), (6.31), and (6.33), respectively. Nor are these probabilities altered if, instead of assuming complete ignorance, or an initial state A at t_0, one introduces an initial probability distribution which assigns to each $|a_j\rangle$ a probability p_j in such a way that the total probability of A is 1. Thus, while the simplifications employed in (6.12) and the following discussion make it easier to do the calculations, they do not affect the final conclusions.

As a final remark, it may be noted that we have made no use of *decoherence*, in the sense of the interaction of a system with its environment [27], in discussing measurement problems. This is not to suggest that decoherence is irrelevant to the theory of quantum measurement; quite the opposite is the case. For example, the fact that certain physical properties, such as pointer positions in a properly designed apparatus, have a certain stability in the course of time despite perturbations from a random environment, while other physical properties do not, is a matter of both theoretical and practical interest. However, the phenomenon of decoherence does not, in and of itself, specify which framework is to be employed in describing a measurement; indeed, in order to understand what decoherence is all about, one needs to use an appropriate framework. Hence decoherence is not the correct conceptual tool to disentangle conceptual dilemmas brought about by mixing descriptions from incompatible frameworks.

D. Three state paradox

Aharonov and Vaidman [21] (also see Kent [20]) have introduced a class of paradoxes, of which the following is the simplest example, in which a particle can be in one of three states: $|A\rangle$, $|B\rangle$, or $|C\rangle$, and in which the unitary dynamics for a set of three times $t_0<t_1<t_2$ is given by the identity operator: $|A\rangle\mapsto|A\rangle$, etc. Define

$$|\Phi\rangle=(|A\rangle+|B\rangle+C\rangle)/\sqrt{3},$$
$$|\Psi\rangle=(|A\rangle+|B\rangle-|C\rangle)/\sqrt{3}, \tag{6.36}$$

and, consistent with our previous notation, let a letter outside a ket denote the corresponding projector, and a tilde its complement, thus

$$A=|A\rangle\langle A|, \quad \tilde{A}=I-A=B+C. \tag{6.37}$$

Let us begin with the framework based upon

$$\breve{I}=\{\Phi_0+\tilde{\Phi}_0\}\{\Psi_2+\tilde{\Psi}_2\}, \tag{6.38}$$

and consider two refinements. In the first, generated by

$$\breve{I}=\{\Phi_0+\tilde{\Phi}_0\}\{A_1+\tilde{A}_1\}\{\Psi_2+\tilde{\Psi}_2\}, \tag{6.39}$$

and easily shown to be consistent, an elementary calculation yields the result

$$\Pr(A_1|\Phi_0\Psi_2)=1. \tag{6.40}$$

The second refinement is generated by

$$\breve{I}=\{\Phi_0+\tilde{\Phi}_0\}\{B_1+\tilde{B}_1\}\{\Psi_2+\tilde{\Psi}_2\}, \tag{6.41}$$

and within this framework,

$$\Pr(B_1|\Phi_0\Psi_2)=1. \tag{6.42}$$

The paradox comes about by noting that the product of the projectors A and B, and thus A_1 and B_1, is zero. Consequently, were B_1 an element of the framework (6.39), (6.40) would imply that $\Pr(B_1|\Phi_0\Psi_2)=0$, in direct contradiction to (6.42). But of course there is no contradiction when one follows the rules of Sec. V, because B_1 and A_1 can never belong to the same refinement of (6.38). Thus this paradox is a good illustration of the importance of paying attention to the framework in order to avoid contradictions when reasoning about a quantum system, and provides a nice illustration of the pitfall pointed out at the end of Sec. VI B.

VII. SOME ISSUES OF INTERPRETATION

A. Incompatible frameworks

The central conceptual difficulty of quantum theory, expressed in the terminology used in this paper, is the existence of mutually incompatible frameworks, any one of which can, at least potentially, apply to a particular physical system, whereas two (or more) cannot be applied to the same system. Whereas the reasoning procedures described in Sec. V provide an internally consistent way of dealing with this ‘‘framework problem,’’ it is, as is always the case in quantum theory, very easy to become confused through habits of mind based upon classical physics. The material in this section is intended to address at least some of these sources of confusion at a more intuitive level, assuming that Sec. V is sound at the formal level.

It will be useful to consider the explicit example discussed in Sec. VI C, in which a spin-half particle with $S_x=1/2$ at time t_0 is later, at t_2, subjected to a measurement of S_z, and this measurement yields the result $S_z=1/2$. There is then a framework $\mathcal{Z}$, (6.29), in which one can conclude Z_1^+ with probability one: that is, the particle was in a state $S_z=1/2$ at the intermediate time t_1. And there is another, incompatible, framework $\mathcal{X}$, (6.32), in which, on the basis of the same initial data, one can conclude X_1^+ with probability one: that is, the particle was in a state $S_x=1/2$ at t_1.

The first issue raised by this example is the following. The rules of reasoning in Sec. V allow us to infer the truth of Z_1^+ in framework $\mathcal{Z}$, and the truth of X_1^+ in framework $\mathcal{X}$, but we cannot infer the truth of Z_1^+ *and* X_1^+, because they do not belong to the same framework. This is quite different from a classical system, in which we are accustomed to think that whenever an assertion E is true about a physical system, in the sense that it can be correctly inferred from some known (or assumed) data, and F is true in the same sense, then E *and* F must be true. As d'Espagnat has emphasized [16,17,19], this is always a valid conclusion in standard systems of logic. But in quantum theory, as interpreted in this paper, such is no longer the case. Note that there is no formal difficulty involved: once we have agreed that quantum mechanics is a stochastic theory in which the concept of ''true'' corresponds to ''probability one,'' then precisely because probabilities (classical or quantum) only make sense within some algebra of events, the truth of a quantum proposition is necessarily labeled, at least implicitly, by that algebra, which in the quantum case we call a framework. The existence of incompatible quantum frameworks is no more or less surprising than the existence of noncommuting operators representing dynamical variables; indeed, there is a sense in which the former is a direct consequence of the latter. Thus physicists who are willing to accept the basic mathematical framework employed in quantum theory, with its nonclassical noncommutativity, should not be shocked that incompatible frameworks arise when quantum probabilities are incorporated into the theory in a consistent, rather than an *ad hoc*, manner. If the dependence of truth on a framework violates classical intuition, the remedy is to revise that intuition by working through examples, as in Sec. VI.

Precisely the same point can be made using the example in Sec. VI D. Indeed, the importance of using the correct framework is perhaps even clearer in this case, where the projectors A and B commute with each other.

A second issue raised by the approach of Sec. V can be stated in the form of a question: does quantum theory itself specify a unique framework? And if the answer is ''no,'' as maintained in this paper, does this mean the interpretation of quantum theory presented here is subjective? Or that it somehow implies that physical reality is influenced by the choices made by a physicist [17,19]?

In response, the first thing to note is that while the choice of framework is not specified by quantum theory, it is also far from arbitrary. Thus in our example, given the initial data in the form of $S_x = 1/2$ at t_0 and the results of the measurement of S_z at t_2, $\mathcal{Z}$ is the unique coarsest framework which contains the data and allows us to discuss the value of S_z at the time t_1. To be sure, any refinement of this framework would be equally acceptable, but it is also the case that any refinement would lead to precisely the same probability of S_z at the time t_1, conditional upon the initial data. The same holds for the more general situation discussed in Sec. V: any refinement of the smallest (coarsest) framework which contains the data and conclusions will lead to the same probability for the latter, conditional upon the former. This is also the case for various sorts of quantum reasoning constantly employed in practice in order to calculate, for example, a differential cross section.

In a certain sense, the very fact that incompatible frameworks are incompatible is what brings about the quasi-uniqueness in the choice of frameworks just mentioned. Certain questions are meaningless unless one uses a framework in which they mean something, and the same is true of initial data. Differential scattering cross sections require one type of framework, whereas the discussion of interference between two parts of a wave going off in different directions, but later united by a system of mirrors, requires another. While this fact is appreciated at an intuitive level by practicing physicists, they tend to find it confusing, because the general principles of Sec. V are not as yet contained in standard textbooks.

A classical analogy, that of ''coarse graining'' in classical statistical mechanics, is helpful in seeing why the physicist's freedom in choosing a quantum framework does not make quantum theory subjective, or imply that this choice influences physical reality. As noted in Sec. II, coarse graining means dividing the classical phase space into a series of cells of finite volume. From the point of view of classical mechanics, such a coarse graining is, of course, arbitrary; cells are chosen because they are convenient for discussing certain problems, such as macroscopic (thermodynamic) irreversibility. But this does not make classical statistical mechanics a subjective theory. And, in addition, no one would ever suppose that by choosing a particular coarse graining, the theoretical physicist is somehow influencing the system. If, because it is convenient for his calculations, he chooses one coarse graining for times t preceding a certain t_0, and a different coarse graining for later times, it would be bizarre to suppose that this somehow induced a ''change'' in the system at t_0.

To be sure, no classical analogy can adequately represent the quantum world. In particular, any two classical coarse grainings are compatible: a common refinement can always be constructed by using the intersections of cells from the two families. And one can always imagine replacing the coarse grainings by an exact specification of the state of the system. An analogy which comes a bit closer to the quantum situation can be constructed by imposing the rule that one can only use coarse grainings in which the cells have ''volumes'' which are integer multiples of h^P, for a classical system with P degrees of freedom. Two coarse grainings which satisfy this condition will not, in general, have a common refinement which also satisfies this condition.

While classical analogies cannot settle things, they are useful in suggesting ways in which the formalism of Sec. V can be understood in an intuitive way. Eventually, of course, quantum theory, because it is distinctly different from classical physics, must be understood on its own terms, and an intuitive understanding of the quantum world must be developed by working through examples, such as those in Sec. VI, interpreted by means of a sound and consistent mathematical formalism, such as that of Sec. V.

B. Emergence of the classical world

Both Gell-Mann and Hartle [13], and Omnès [26] have discussed how classical physics expressed in terms of suitable ''hydrodynamic'' variables emerges as an approximation to a fully quantum-mechanical description of the world when the latter is carried out using suitable frameworks. While these two formulations differ somewhat from each other, and from the approach of the present paper, both are basically compatible with the point of view found in Secs. II–V. It is not our purpose to recapitulate or even summarize the detailed technical discussions by these authors, but instead to indicate the overall strategy, as viewed from the perspective of this paper, and comment on how it relates to the problem of incompatible frameworks discussed above.

The basic strategy of Gell-Mann and Hartle can be thought of as the search for a suitable ''quasiclassical'' framework, a consistent family whose Boolean algebra includes projectors appropriate for representing coarse-grained variables, such as average density and average momentum inside volume elements which are not too small, variables which can plausibly be thought of as the quantum counterparts of properties which enter into hydrodynamic and other macroscopic descriptions of the world provided by classical physics. Hence it is necessary to first find suitable commuting projectors representing appropriate histories, and then show that the consistency conditions are satisfied for the corresponding Boolean algebra. Omnès states his strategy in somewhat different terms which, however, are generally compatible with the point of view just expressed.

Both Gell-Mann and Hartle, and Omnès, employ consistency conditions which, unlike those in the present paper, involve a density matrix; see the discussion in the Appendix. However, the difference is probably of no great importance when discussing ''quasiclassical'' systems involving large numbers of particles, for the following reason. In classical statistical mechanics one knows (or at least believes) that for macroscopic systems the choice of ensemble—microcanonical, canonical, or grand—is for many purposes unimportant, and, indeed, the average behavior of the ensemble will be quite close to that of a ''typical'' member. Stated in other words, the use of probability distributions is a convenience which is not ''in principle'' necessary. Presumably an analogous result holds for quantum systems of macroscopic size: the use of a density matrix, both as an ''initial condition'' and as part of the consistency requirement may be convenient, but it is not absolutely necessary when one is discussing the behavior of a closed system. For an example in which the final results are to a large degree independent of what one assumes about the initial conditions, see the discussion at the end of Sec. VI C.

The task of finding an appropriate quasiclassical consistent family is made somewhat easier by two facts. The first is that decoherence [27], in the sense of the interaction of certain degrees of freedom with an ''environment,'' can be quite effective in rendering the weight operators corresponding to minimal elements of a suitably chosen family almost orthogonal, in the sense discussed in Sec. III. (In the present context one should think of the relevant degrees of freedom as those represented by the hydrodynamic variables, and the ''environment'' as consisting of the remaining ''microscopic'' variables which are smoothed out, or ignored, in order to obtain a hydrodynamic description.) The second is that the weight operators depend continuously on projectors which form their arguments, and hence it is at least plausible that if the former are almost orthogonal, small changes in the projectors can be made in order to achieve exact orthogonality [15]. Since there is in any case some arbitrariness in choosing the quantum projectors which represent particular coarse-grained hydrodynamic variables, small changes in these projectors are unimportant for their physical interpretation. Thus exact consistency does not seem difficult to achieve ''in principle,'' even if in practice theoretical physicists are unlikely to be worried by small deviations from exact orthogonality, as long as these do not introduce significant inconsistencies into the probabilities calculated from the weights. To be sure, there are issues here which deserve further study.

There are likely to be many different frameworks which are equally good for the purpose of deriving hydrodynamics from quantum theory, and among these a number which are mutually incompatible. Is this a serious problem? Not unless one supposes that quantum theory must single out a single framework, a possibility entertained by Dowker and Kent [15]. If, on the contrary, the analogy of classical coarse grainings introduced earlier is valid, one would expect that the same ''coarse-grained'' classical laws would emerge from any framework which is compatible with this sort of ''quasiclassical'' description of the world. The internal consistency of the reasoning scheme of Sec. V, which can be thought of as always giving the same answer to the same question, points in this direction, although this is another topic which deserves additional study.

There are, of course, many frameworks which are *not* quasiclassical and are incompatible with a ''hydrodynamic'' description of the world, and there is no principle of quantum theory which excludes the use of such frameworks. However, the existence of alternative frameworks does not invalidate conclusions based upon a quasiclassical framework. Again, it may help to think of the analogy of coarse grainings of the classical phase space. The existence of coarse grainings in which a classical system exhibits no irreversible behavior—they can be constructed quite easily if one allows the choice of cells to depend upon the time—does not invalidate conclusions about thermodynamic irreversibility drawn from a coarse graining chosen to exhibit this phenomenon. Similarly, in the quantum case, if we are interested in the ''hydrodynamic'' behavior of the world, we are naturally led to employ quasiclassical frameworks in which hydrodynamic variables make sense, rather than alternative frameworks in which such variables are meaningless.

This suggests an answer to a particular concern raised by Dowker and Kent [15]: If we, as human beings living in a quantum world, have reason to believe (based upon our memories and the like) that this world has been ''quasiclassical'' up to now, why should we assume that it will continue to be so tomorrow? In order not to be trapped in various philosophical subtleties such as whether (and if so, how) human thought and belief can be represented by physical processes, let us consider an easier problem in which there is a computer inside a closed box, which we as physicists (outside the box) have been describing up till now in quasiclassical terms. Suppose, further, that one of the inputs to the computer is the output of a detector, also inside the box, measuring radioactive decay of some atoms. What would happen if, ten minutes from now, we were to abandon the quasiclassical framework for one in which, say, there is a coherent quantum superposition of the computer in distinct macroscopic states? Of course, nothing particular would happen to anything inside the box; we, on the other hand, would no longer be able to describe the object in the box as a computer, because the language consistent with such a description would be incompatible with the framework we were using for our description. The main point can be made using an even simpler example: consider a spin-half particle in zero magnetic field, and a history in which $S_x = 1/2$ at a time $t_0 < t_1$, and $S_z = 1/2$ at a time $t_2 > t_1$. Nothing at all is happening to the particle at time t_1; the only change is in our manner of describing it. Additional criticisms of consistent history ideas with reference to quasiclassical frameworks will be found in [15,28]; responding to them is outside the scope of the present paper.

VIII. CONCLUSION

A. Summary

The counterpart for a closed quantum system of the event space of classical probability theory is a framework: a Boolean algebra of commuting projectors on the space $\breve{\mathcal{H}}$, (2.6), of quantum histories chosen in such a way that the weight operators of its minimal elements are orthogonal, (3.14) or (3.16). This ensures that the corresponding weights are additive, (3.11). A refinement of a framework is an enlarged Boolean algebra which again satisfies the consistency conditions. Two or more frameworks with a common refinement are called compatible, but in general different quantum frameworks are incompatible with one another, a situation which has no classical analog.

Given some framework and an associated probability distribution, the rules for quantum reasoning, Sec. V, are the usual rules for manipulating probabilities, with ''true'' and ''false'' corresponding to (conditional) probabilities equal to 1 and 0, respectively. In addition, a probability distribution defined on one framework can be extended to a refinement of this framework using (4.5). This refinement rule incorporates the laws of quantum dynamics into the theory: for example, the Born formula emerges as a conditional probability, (5.2), even in the absence of any initial data.

The refinement rule allows descriptions in compatible frameworks to be combined, or at least compared, in a common refinement. However, there is no way of comparing or combining descriptions belonging to incompatible frameworks, and it is a mistake to think of them as simultaneously applying to the same physical system.

Quantum reasoning allows one, on the basis of the same initial data, to reach different conclusions in different, sometimes mutually incompatible, refinements. However, the system is internally consistent in the sense that the probability assigned to any history on the basis of some initial data (which must be given in a single framework) is independent of the refinement in which that history occurs. Hence it is impossible to conclude that some consequence of a given set of initial data is both true and false. Nevertheless, probabilities are only meaningful with reference to particular frameworks, and the same is the case for ''true'' and ''false'' regarded as limiting cases in which a probability is 1 or 0. Hence a basic condition for sound quantum reasoning is keeping track of the framework employed at a particular point in an argument.

B. Open questions

The entire technical discussion in Secs. II–V is based upon a finite-dimensional Hilbert space $\mathcal{H}$ for a quantum system at a single time, and likewise a finite-dimensional history space $\breve{\mathcal{H}}$. This seems satisfactory for exploring those conceptual difficulties which are already present in the finite-dimensional case, and allows a simple exposition with a minimal number of technical conditions and headaches. And, as a practical matter, in any situation in which a finite physical system can be thought of as possessing a finite entropy S, it is reasonable to suppose that the ''right physics'' will emerge when one restricts one's attention to a subspace of $\mathcal{H}$ with dimension of order $\exp[S/k_B]$. Nonetheless, introducing such a cutoff, even for the case of a single particle in a finite box, is mathematically awkward, and for this reason alone it would be worthwhile to construct the appropriate extension of the arguments given in this paper to the (or at least some) infinite-dimensional case. For some steps in this direction, see the work of Isham and his collaborators [6,29,30].

It is not necessary to require that the Boolean algebra of histories introduced in Sec. II satisfy the consistency conditions of Sec. III in order to introduce a probability distribution on the former. Consistency becomes an issue only when one considers refinements of a framework, and wants to define a refined probability. Even so, one can introduce refinements of an inconsistent framework $\mathcal{F}$, with probabilities given by (4.5), by demanding that for each i, the weight operator associated with F_i be a sum of mutually orthogonal weight operators of those minimal elements G_j of the refinement $\mathcal{G}$ whose sum is F_i. The open question is whether there is some physical application for such a generalized system of frameworks and refinement rules. Consistent frameworks seem to be sufficient for describing closed quantum systems, but it is possible that generalized frameworks would be of some use in thinking about an open system: a subsystem of a closed system in which the remainder of the closed system is regarded as forming some sort of ''environment'' of the system of interest.

While the scheme of quantum reasoning presented in this paper has wide applicability, there are certain to be situations not covered by the rules given in Sec. V. One of these is the case of counterfactuals, such as ''*if* the counter had not been located directly behind the slit, *then* the particle would have . . . .'' Analyzing these requires comparing two situations which differ in some specific way—e.g., in the position occupied by some counter—and it is not clear how to embed this in the scheme discussed in Sec. V. Inasmuch as many quantum paradoxes, including some of the ones associated with double-slit diffraction, and certain derivations of Bell's inequality and analogous results, make use of counterfactuals, analyzing them requires considerations which go beyond those in the present paper. As philosophers have yet to reach general agreement on a satisfactory scheme for counterfactual reasoning applied to the classical world [31], an extension which covers all of quantum reasoning is likely to be difficult. On the other hand, one sufficient to handle the special sorts of counterfactual reasoning found in common quantum paradoxes is perhaps a simpler problem.

Can the structure of reasoning developed in this paper for nonrelativistic quantum mechanics be extended to relativistic quantum mechanics and quantum field theory? Various examples suggest that the sort of peculiar nonlocality which is often thought to arise from violations of Bell's inequality and various EPR paradoxes will disappear when one enforces the rules of reasoning given in Sec. V. While this is encouraging, it is also the case that locality (or the lack thereof) in nonrelativistic quantum theory has yet to be carefully analyzed from the perspective presented in this paper, and hence must be considered among the open questions. And, of course, getting rid of spurious nonlocalities is only a small step along the way towards a fully relativistic theory.

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APPENDIX: CONSISTENCY USING A DENSITY MATRIX

The consistency condition introduced in Sec. III differs in a small but not insignificant way from the one introduced by Gell-Mann and Hartle [12,13], based upon a decoherence functional. The latter employs a density matrix and amounts, in effect, to replacing the operator inner product (3.8) by

$$\langle A,B\rangle = \mathrm{Tr}[A^\dagger \rho B], \tag{A1}$$

where ρ is a density matrix (positive operator with unit trace) or, [32], by

$$\langle A,B\rangle = \mathrm{Tr}[A^\dagger \rho B \rho'], \tag{A2}$$

where both ρ and ρ' are density matrices, thought of as associated with the initial and final time, respectively. Still more general possibilities have been proposed by Isham *et al.* [30]. While Omnès's approach [33] is somewhat different, his consistency condition also employs a density matrix in a manner similar to (A1).

Certainly one cannot object to either (A1) or (A2), or some completely different definition, on purely mathematical grounds. If, on the other hand, ρ is to be interpreted as representing something like a probability distribution for the physical system at an initial time, the following considerations favor (3.8). First, given that an arbitrary probability distribution can be introduced once a framework has been specified, Sec. IV, and this can refer to the initial time, or the final time, or to anything in between, there is no (obvious) gain in generality from introducing a density matrix into the operator inner product. Second, in the scheme outlined in Secs. II–IV, the conditions for choosing a framework are independent of the probability one chooses to assign to the corresponding histories, whereas employing (A1) or (A2) couples the acceptability of a framework and the probability assigned to its histories in a somewhat awkward way. Third, (3.8) is obviously a simpler construction than either (A1) or (A2), and there seems to be no physical situation in nonrelativistic quantum mechanics in which it is not perfectly adequate. To be sure, all of these considerations have a certain aesthetic character, and elegance is not always a good guide to developing a physical theory, even when there is agreement as to what is most elegant. The reader will have to make up his own mind.

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