Upper bound of the accessible information and lower bound of the Bayes cost in quantum signal-detection processes

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(Received 25 April 1996)

Upper bound of the accessible information and lower bound of the Bayes cost in quantum detection processes for Gaussian state signals under the influence of thermal noise are derived by means of the superoperator representation of quantum states. It is shown that the upper and lower bounds are obtained by replacing the parameters of the signal quantum state with the renormalized parameters including the thermal noise effects in the accessible information and the minimum value of the Bayes cost in quantum detection processes in the absence of thermal noise. Analytic expressions of the upper and lower bounds are given for several quantum state signals. [S1050-2947(96)02810-7]

PACS number(s): 03.65.Bz, 89.70.+c

I. INTRODUCTION

In quantum communication systems, a transmitter of information sends a receiver one of n possible messages represented by density operators $\hat{\rho}_1,\hat{\rho}_2,\ldots,\hat{\rho}_n$ with prior probabilities $p_1,p_2,\ldots,p_n$ which are normalized as $\Sigma_{j=1}^{n}p_j=1$. The receiver, on the other hand, performs a generalized quantum measurement on the received signal to infer which quantum state (message) has been sent by the transmitter. A generalized quantum measurement is described by a positive operator-valued measure (abbreviated as POM) [1,2] which is a set of non-negative Hermitian operators, $\{\hat{\Pi}_\mu|\mu\in\mathcal{S}\}$, satisfying the relation $\Sigma_{\mu\in\mathcal{S}}\hat{\Pi}_\mu=\hat{I}$, where $\hat{I}$ stands for an identity operator, μ represents an index specifying the measurement outcome, and $\mathcal{S}$ is a set of the indices of all the possible measurement outcomes.

One of the most important problems in quantum communication and information theory is to find an optimum quantum measurement on the received signal so that the mutual information I is maximized [3–7] or so that the Bayes cost C_B is minimized [1,8]. The mutual information is the appropriate measure of information successfully sent from the transmitter to the receiver [3]. On the other hand, the Bayes cost is the average cost incurred when we infer which quantum state has been received [1]. When we perform a quantum measurement on the received signal, a detection error occurs with finite probability unless the signal quantum states are orthogonal. In a communication system, we suffer a loss due to the detection error since the error damages a system. The Bayes cost represents the average amount of damage of a communication system. We minimize the cost by means of the Bayes strategy. The Bayes cost includes the average probability of error P_e in the signal-detection process as a special case.

The mutual information and the Bayes cost are important quantities to evaluate the performance of a signal-detection process. Thus our task is to obtain a POM which satisfies the requirement that the mutual information I should be maximized or that the Bayes cost C_B should be minimized in a signal-detection process. Obtaining such a POM is also important for quantum computation and quantum cryptography [9,10] since a signal-detection process is indispensable for reading a result of computation and for obtaining a key distribution. The maximum value of the mutual information is called the accessible information and the minimum value of the Bayes cost (in particular, the average probability of error) is called the Helstrom bound. In this paper we denote the accessible information as $I_{\rm opt}=\max_{\{\hat{\Pi}_\mu\}}I$ and the Helstrom bound as $C_{B\rm opt}=\min_{\{\hat{\Pi}_\mu\}}C_B$.

The upper bound of the accessible information $I_{\rm opt}$ was obtained by Holevo [4] and generalized by Yuen and Ozawa [5]. The lower bound, on the other hand, was found by Jozsa, Robb, and Wootters [6]. These upper and lower bounds are expressed in terms of the von Neumann entropy S and the subentropy Q. The results obtained in Refs. [4–6] show that the accessible information $I_{\rm opt}$ satisfies the following inequality:

$$Q(\hat{\rho})-\sum_{k=1}^{n}p_kQ(\hat{\rho}_k)\leq I_{\rm opt}\leq S(\hat{\rho})-\sum_{k=1}^{n}p_kS(\hat{\rho}_k). \tag{1.1}$$

Here $\hat{\rho}=\Sigma_{k=1}^{n}p_k\hat{\rho}_k$ is a density operator of statistical mixture of the quantum states of the signal and the von Neumann entropy $S(\hat{\sigma})$ and the subentropy $Q(\hat{\sigma})$ are given by

$$S(\hat{\sigma})=-{\rm Tr}[\hat{\sigma}\ln\hat{\sigma}]=-\sum_{\nu}\lambda_\nu\ln\lambda_\nu, \tag{1.2}$$

$$Q(\hat{\sigma})=-\sum_{\nu}\left[\prod_{\mu\neq\nu}\left(\frac{\lambda_\nu}{\lambda_\nu-\lambda_\mu}\right)\right]\lambda_\nu\ln\lambda_\nu, \tag{1.3}$$

where λ_ν is an eigenvalue of the statistical operator $\hat{\sigma}$ ($\hat{\sigma}=\hat{\rho}$ or $\hat{\sigma}=\hat{\rho}_k$). The upper bound of the accessible information is called the Holevo bound. When we consider a detection process for a pure quantum state signal, the Holevo bound becomes the von Neumann entropy $S(\hat{\rho})$. This result indicates that the amount of information that we can extract from the received signal does not exceed the amount of information that the signal carries. The meaning of the inequality (1.1) is discussed in Refs. [3,6]. Since the entropic upper and lower bounds do not depend on any quantum measurement carried out on the received signal, these bounds are general but may be fairly loose. Fuchs and Caves have recently found tighter bounds of the accessible information that strongly depend on the structure of the quantum states of the signal [7]. But their bounds are much more complicated in comparison with the entropic bounds.

The Bayes cost C_B is another quantity used to evaluate the performance of a quantum signal-detection process. We can calculate, in principle, the minimum value of the Bayes cost by means of the quantum detection theory [1,11–13]. To obtain the minimum value and the optimum POM, we have to solve the operator equations for the POM,

$$\hat{\Pi}_j(\hat{W}_j-\hat{W}_k)\hat{\Pi}_k=0, \quad \hat{W}_j-\sum_{k=1}^{n}\hat{\Pi}_k\hat{W}_k\geqslant 0 \tag{1.4}$$

where $\hat{W}_j=\Sigma_{k=1}^{n}C_{jk}\hat{\rho}_k p_k$ is the risk operator and C_{jk} is the cost incurred when we infer that the received quantum state is given by the density operator $\hat{\rho}_j$ even though the quantum state $\hat{\rho}_k$ has been actually received. In this case the set of indices becomes $\mathcal{S}=\{1,2, \ldots ,n\}$. The minimum value of the Bayes cost is calculated as $C_{B\mathrm{opt}}=\Sigma_{k=1}^{n}\mathrm{Tr}(\hat{\Pi}_k\hat{W}_k)$ [1], where $\hat{\Pi}_1,\hat{\Pi}_2, \ldots ,\hat{\Pi}_n$ satisfy Eq. (1.4). But it is too difficult to obtain the solutions of Eq. (1.4) except for simple signals. In particular, we cannot obtain an analytical expression of the minimum value of the Bayes cost in the presence of thermal noise if $[\hat{\rho}_j,\hat{\rho}_k]\neq 0$. Thus obtaining the lower bound of the Bayes cost is important for evaluating quantum communication systems.

In this paper we consider the upper bound of the accessible information and the lower bound of the Bayes cost in quantum detection processes for Gaussian state signals [2,11,14,15] under the influence of thermal noise. It will be shown that the upper bound of I_{opt} and the lower bound of $C_{B\mathrm{opt}}$ are obtained by replacing the parameters characterizing the quantum states of the signal with the renormalized parameters including the thermal noise effects in the accessible information $I_{\mathrm{opt}}^{(0)}$ and the minimum value of the Bayes cost $C_{B\mathrm{opt}}^{(0)}$ in the signal-detection process without thermal noise. In the detection process for coherent state signal, the renormalized parameters are obtained by replacing the signal power with the signal-to-noise ratio in the original parameters (see Secs. II–IV). The thermal noise effects on signal-detection processes are inevitable in practical communication systems, and obtaining $I_{\mathrm{opt}}^{(0)}$ and $C_{B\mathrm{opt}}^{(0)}$ is easier than obtaining I_{opt} and $C_{B\mathrm{opt}}$. Therefore it is important in quantum communication and information theory to obtain such upper and lower bounds. To derive the upper of the accessible information and lower bound of the Bayes cost, we use the superoperator representation [16–19] of quantum states, or equivalently thermofield dynamics [20–23], which enables us to treat mixed quantum states just like pure quantum states. The thermal noise effects on the mutual information and the average probability of error were investigated by Hall and O'Rourke [24]. But the optimization of the POM was not considered in their approach.

In Sec. II we explain the quantum states considered in this paper and give their superoperator representation, which is used to derive the upper and lower bounds. We derive the upper bound of the accessible information in Sec. III and the lower bound of the Bayes cost in Sec. IV. There we obtain their analytic expressions for the binary quantum state signal under the influence of the thermal noise. In Sec. V we summarize the results.

II. SUPEROPERATOR REPRESENTATION

Quantum states of a signal that we consider in this paper belong to the class of Gaussian states [2,11,14,15]. Let $\hat{\rho}_j$ be a density operator which represents the jth quantum state of the signal in the presence of the thermal noise and let $\hat{\rho}_j^{(0)}$ be the corresponding density operator in the absence of the thermal noise. These density operators are expressed as

$$\hat{\rho}_j=\frac{\hat{V}_j\hat{\rho}_{\mathrm{th}}\hat{V}_j^{\dagger}}{\mathrm{Tr}(\hat{V}_j\hat{\rho}_{\mathrm{th}}\hat{V}_j^{\dagger})}, \quad \hat{\rho}_j^{(0)}=\frac{\hat{V}_j|0\rangle\langle 0|\hat{V}_j^{\dagger}}{\langle 0|\hat{V}_j^{\dagger}\hat{V}_j|0\rangle}=|\psi_j^{(0)}\rangle\langle\psi_j^{(0)}|, \tag{2.1}$$

where $\hat{\rho}_{\mathrm{th}}=[1/(1+\overline{n})]\Sigma_{k=0}^{\infty}[\overline{n}/(1+\overline{n})]^k|k\rangle\langle k|$ is the density operator of the thermal state, $\overline{n}$ is the average value of the photon number of the thermal noise, $|0\rangle$ is the vacuum state, and we set $|\psi_j^{(0)}\rangle=\hat{V}_j|0\rangle/\sqrt{\langle 0|\hat{V}_j^{\dagger}\hat{V}_j|0\rangle}$. In Eq. (2.1), we assume that the operator $\hat{V}_j$ is given by

$$\hat{V}_j=\exp[\gamma_j\hat{a}^2-\gamma_j^*\hat{a}^{\dagger 2}+i\phi_j\hat{a}^{\dagger}\hat{a}+\mu_j^*\hat{a}+\nu_j\hat{a}^{\dagger}], \tag{2.2}$$

where γ_j, μ_j, and ν_j are complex parameters, ϕ_j is a real parameter, and $\hat{a}$ and $\hat{a}^{\dagger}$ are bosonic annihilation and creation operators which satisfy the canonical commutation relation $[\hat{a},\hat{a}^{\dagger}]=1$. The operator $\hat{V}_j$ need not be unitary. It is easy to see that the signal quantum state represented by the density operator $\hat{\rho}_j$ includes a thermal coherent state [25] and a thermal squeezed state [26], which seem to be the most important states in quantum communication systems. The coherent state is widely used in optical communication systems. The applicability of the squeezed state [27] to optical communication systems was discussed by Yuen and Shapiro [28]. The squeezed state can also be used in a high precision measurement. The thermal noise effects are inevitable in these systems.

To obtain the superoperator representation of the quantum states given by Eqs. (2.1) and (2.2), we introduce three superoperators $\hat{\mathcal{K}}_{\pm}$ and $\hat{\mathcal{K}}_0$ by the relations [29,30]

$$\hat{\mathcal{K}}_+\hat{A}=\hat{a}^{\dagger}\hat{A}\hat{a}, \quad \hat{\mathcal{K}}_-\hat{A}=\hat{a}\hat{A}\hat{a}^{\dagger},$$

$$\hat{\mathcal{K}}_0\hat{A}=\frac{1}{2}(\hat{a}^{\dagger}\hat{a}\hat{A}+\hat{A}\hat{a}^{\dagger}\hat{a}+\hat{A}), \tag{2.3}$$

where $\hat{A}$ stands for an arbitrary operator. Here a superoperator means an operator acting on operators [16–23]. It is easy to see that the superoperators $\hat{\mathcal{K}}_{\pm}$ and $\hat{\mathcal{K}}_0$ satisfy the SU(1,1) Lie commutation relations [31],

$$[\hat{\mathcal{K}}_-,\hat{\mathcal{K}}_+]=2\hat{\mathcal{K}}_0, \quad [\hat{\mathcal{K}}_0,\hat{\mathcal{K}}_{\pm}]=\pm\hat{\mathcal{K}}_{\pm}. \tag{2.4}$$

Thus we obtain the useful decomposition formula for the SU(1,1) generators [32,33],

$$\exp[a_+\hat{\mathcal{K}}_+ + a_0\hat{\mathcal{K}}_0 + a_-\hat{\mathcal{K}}_-]$$
$$=\exp[A_+\hat{\mathcal{K}}_+]\exp[(\ln A_0)\hat{\mathcal{K}}_0]\exp[A_-\hat{\mathcal{K}}_-] \quad (2.5)$$
$$=\exp[B_-\hat{\mathcal{K}}_-]\exp[(\ln B_0)\hat{\mathcal{K}}_0]\exp[B_+\hat{\mathcal{K}}_+], \quad (2.6)$$

where the parameters $A_\pm$, A_0, $B_\pm$, and B_0 are given by

$$A_\pm = \frac{(a_\pm/\phi)\sinh\phi}{\cosh\phi-(a_0/2\phi)\sinh\phi},$$
$$A_0 = [\cosh\phi-(a_0/2\phi)\sinh\phi]^{-2}, \quad (2.7)$$
$$B_\pm = \frac{(a_\pm/\phi)\sinh\phi}{\cosh\phi+(a_0/2\phi)\sinh\phi},$$
$$B_0 = [\cosh\phi+(a_0/2\phi)\sinh\phi]^{2}, \quad (2.8)$$

with

$$\phi=\sqrt{(a_0/2)^2-a_+a_-}. \quad (2.9)$$

Equations (2.5) and (2.6) are referred to as the normally and antinormally ordered decomposition formulas.

It is important to note that the density operator of the thermal state is expressed as

$$\hat{\rho}_{\rm th}=\frac{1}{1+\bar{n}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\bar{n}}{1+\bar{n}}\right)^k(\hat{a}^\dagger)^k|0\rangle\langle 0|\hat{a}^k$$
$$=\frac{1}{1+\bar{n}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\bar{n}}{1+\bar{n}}\right)^k\hat{\mathcal{K}}_+^k(|0\rangle\langle 0|)$$
$$=\frac{1}{1+\bar{n}}\exp\left(\frac{\bar{n}}{1+\bar{n}}\hat{\mathcal{K}}_+\right)|0\rangle\langle 0|. \quad (2.10)$$

Then using the normally ordered decomposition formula and the relations $\hat{\mathcal{K}}_-(|0\rangle\langle 0|)=0$ and $\hat{\mathcal{K}}_0(|0\rangle\langle 0|)=1/2|0\rangle\langle 0|$, we can express the density operator of the thermal state in the following form:

$$\hat{\rho}_{\rm th}=\hat{\mathcal{L}}(|0\rangle\langle 0|), \quad \hat{\mathcal{L}}=\exp[\theta(\hat{\mathcal{K}}_+-\hat{\mathcal{K}}_- -1)], \quad (2.11)$$

where we have introduced $\theta=1/2\ln(1+2\bar{n})$. Therefore we find from Eqs. (2.1)–(2.3) and (2.11) that the density operator $\hat{\rho}_j$ of the signal quantum state can be expressed as

$$\hat{\rho}_j=\frac{\hat{V}_j\hat{\mathcal{L}}(|0\rangle\langle 0|)\hat{V}_j^\dagger}{{\rm Tr}[\hat{V}_j\hat{\mathcal{L}}(|0\rangle\langle 0|)\hat{V}_j^\dagger]}=\hat{\mathcal{L}}(|\psi_j\rangle\langle\psi_j|). \quad (2.12)$$

Here we set $|\psi_j\rangle=\hat{U}_j|0\rangle/\sqrt{\langle 0|\hat{U}_j^\dagger\hat{U}_j|0\rangle}$ and the operator $\hat{U}_j$ is given by

$$\hat{U}_j=\exp[\gamma_j\hat{a}^2-\gamma_j^*\hat{a}^{\dagger 2}+i\phi_j\hat{a}^\dagger\hat{a}+\tilde{\mu}_j^*\hat{a}+\tilde{\nu}_j\hat{a}^\dagger], \quad (2.13)$$

where the renormalized parameters $\tilde{\mu}_j$ and $\tilde{\nu}_j$ which include the thermal noise effects are given by

$$\tilde{\mu}_j=\frac{(1+\bar{n})\mu_j+\bar{n}\nu_j}{\sqrt{1+2\bar{n}}}, \quad \tilde{\nu}_j=\frac{(1+\bar{n})\nu_j+\bar{n}\mu_j}{\sqrt{1+2\bar{n}}}. \quad (2.14)$$

The parameters γ_j and ϕ_j remain unchanged.

In deriving Eqs. (2.12)–(2.14) from Eqs. (2.1), (2.2), and (2.11), we have used the relations

$$e^{-\theta(\hat{\mathcal{K}}_+-\hat{\mathcal{K}}_-)}\hat{a}e^{\theta(\hat{\mathcal{K}}_+-\hat{\mathcal{K}}_-)}=\hat{a}\cosh\theta+\hat{b}^\dagger\sinh\theta, \quad (2.15)$$
$$e^{-\theta(\hat{\mathcal{K}}_+-\hat{\mathcal{K}}_-)}\hat{a}^\dagger e^{\theta(\hat{\mathcal{K}}_+-\hat{\mathcal{K}}_-)}=\hat{a}^\dagger\cosh\theta+\hat{b}\sinh\theta, \quad (2.16)$$
$$e^{-\theta(\hat{\mathcal{K}}_+-\hat{\mathcal{K}}_-)}\hat{b}e^{\theta(\hat{\mathcal{K}}_+-\hat{\mathcal{K}}_-)}=\hat{b}\cosh\theta+\hat{a}^\dagger\sinh\theta, \quad (2.17)$$
$$e^{-\theta(\hat{\mathcal{K}}_+-\hat{\mathcal{K}}_-)}\hat{b}^\dagger e^{\theta(\hat{\mathcal{K}}_+-\hat{\mathcal{K}}_-)}=\hat{b}^\dagger\cosh\theta+\hat{a}\sinh\theta, \quad (2.18)$$

where $\hat{b}$ and $\hat{b}^\dagger$ are defined as $\hat{b}\hat{A}=\hat{A}\hat{a}^\dagger$ and $\hat{b}^\dagger\hat{A}=\hat{A}\hat{a}$ for any operator $\hat{A}$ [17]. The operators $\hat{b}$ and $\hat{b}^\dagger$ satisfy the commutation relation $[\hat{b},\hat{b}^\dagger]=1$ and they commute with $\hat{a}$ and $\hat{a}^\dagger$. Using $\hat{b}$ and $\hat{b}^\dagger$, the superoperators $\hat{\mathcal{K}}_\pm$ and $\hat{\mathcal{K}}_0$ can be expressed as

$$\hat{\mathcal{K}}_+=\hat{a}^\dagger\hat{b}^\dagger, \quad \hat{\mathcal{K}}_-=\hat{a}\hat{b}, \quad \hat{\mathcal{K}}_0=\frac{1}{2}(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}+1), \quad (2.19)$$

which indicate that the superoperator representation is mathematically equivalent to the two-mode bosonic representation. In fact, it is shown that the thermal state becomes equivalent to the two-mode squeezed-vacuum state in the two-mode bosonic representation [34–36].

III. UPPER BOUND OF THE ACCESSIBLE INFORMATION

A. General formula for the upper bound

In this section we consider the upper bound of the accessible information $I_{\rm opt}$ in a quantum detection process for a signal whose quantum state is given by the density operator $\hat{\rho}_j$ in Eqs. (2.1) and (2.2). The mutual information I is calculated in terms of the POM $\hat{\Pi}_\mu$'s, density operator $\hat{\rho}_j$'s, and prior probability p_j's [3],

$$I=\sum_{j=1}^{n}\sum_{\mu\in\mathcal{S}}P(\mu|j)p_j\ln\left[\frac{P(\mu|j)}{\sum_{k=1}^{n}P(\mu|k)p_k}\right], \quad (3.1)$$

where we set the unit of information as nats and $P(\mu|j)={\rm Tr}[\hat{\Pi}_\mu\hat{\rho}_j]$ is the conditional probability that the measurement outcome is indexed by parameter μ when the quantum state of the received signal is given by the density operator $\hat{\rho}_j$. Using Eq. (2.12) and the relation ${\rm Tr}[A(\hat{\mathcal{K}}_\pm B)]={\rm Tr}[(\hat{\mathcal{K}}_\mp A)B]$ for any operators $\hat{A}$ and $\hat{B}$, we can calculate the conditional probability $P(\mu|j)$ as

$$P(\mu|j)={\rm Tr}[\hat{\Pi}_\mu\hat{\rho}_j]={\rm Tr}[\hat{\Pi}_\mu\hat{\mathcal{L}}(|\psi_j\rangle\langle\psi_j|)]$$
$$={\rm Tr}[\hat{\mathcal{L}}^\dagger(\hat{\Pi}_\mu)|\psi_j\rangle\langle\psi_j|], \quad (3.2)$$

where the superoperator $\hat{\mathcal{L}}^\dagger=\exp[-\theta(\hat{\mathcal{K}}_+-\hat{\mathcal{K}}_-+1)]$ is the Hermitian conjugate of the superoperator $\hat{\mathcal{L}}$. Since the mutual information is considered a function of the conditional probabilities, we formally write it as $I(\mathrm{Tr}[\hat{\Pi}_\mu \hat{\rho}_j])$.

To perform the optimization, let us now introduce a set $\mathcal{U}$ of all possible POMs,

$$\mathcal{U}=\left\{\hat{\Pi}=(\hat{\Pi}_1,\ldots,\hat{\Pi}_\mu,\ldots)\,\middle|\,\hat{\Pi}_\mu\geqslant 0,\sum_{\mu\in\mathcal{S}}\hat{\Pi}_\mu=\hat{I}\right\}. \quad (3.3)$$

Thus our task is to find an element $\hat{\Pi}$ of the set $\mathcal{U}$ such that the mutual information should be maximized. The optimum POM $\hat{\Pi}$ is determined by the requirement that $\max_{\hat{\Pi}\in\mathcal{U}} I(\mathrm{Tr}[\hat{\Pi}_\mu\hat{\rho}_j])$. When the POM $\hat{\Pi}$ maximizes the mutual information, $\hat{\Pi}$ has to satisfy certain conditions [11,37]. However, the necessary and sufficient condition for the POM to maximize the mutual information I has never been found. Using Eq. (3.2), we can calculate the accessible information I_{opt} as follows:

$$\begin{aligned} I_{\mathrm{opt}} &= \max_{\hat{\Pi}\in\mathcal{U}} I(\mathrm{Tr}[\hat{\Pi}_\mu\hat{\rho}_j]) \\ &= \max_{\hat{\Pi}\in\mathcal{U}} I\{\mathrm{Tr}[\hat{\Pi}_\mu\hat{\mathcal{L}}(|\psi_j\rangle\langle\psi_j|)]\} \\ &= \max_{\hat{\Pi}\in\mathcal{U}} I\{\mathrm{Tr}[\hat{\mathcal{L}}^\dagger(\hat{\Pi}_\mu)|\psi_j\rangle\langle\psi_j|]\} \\ &= \max_{\hat{\Pi}'\in\tilde{\mathcal{U}}} I(\mathrm{Tr}[\hat{\Pi}'_\mu|\psi_j\rangle\langle\psi_j|]), \end{aligned} \quad (3.4)$$

where we have introduced the set $\tilde{\mathcal{U}}=\{\hat{\Pi}'=\hat{\mathcal{L}}^\dagger(\hat{\Pi})|\hat{\Pi}\in\mathcal{U}\}$ which is the range of the superoperator $\hat{\mathcal{L}}^\dagger$ when the domain is restricted to the set $\mathcal{U}$.

Let $\hat{\Pi}'=(\hat{\Pi}'_1,\ldots,\hat{\Pi}'_\mu,\ldots)$ be an arbitrary element of the set $\tilde{\mathcal{U}}$. Then there is some element $\hat{\Pi}=(\hat{\Pi}_1,\ldots,\hat{\Pi}_\mu,\ldots)$ of the set $\mathcal{U}$ such that $\hat{\Pi}'_\mu=\hat{\mathcal{L}}^\dagger(\hat{\Pi}_\mu)$ for all $\mu\in\mathcal{S}$. Using the relations $\Sigma_{\mu\in\mathcal{S}}\hat{\Pi}_\mu=\hat{I}$ and $\hat{\mathcal{L}}^\dagger(\hat{I})=\hat{I}$, we can show that the relation $\Sigma_{\mu\in\mathcal{S}}\hat{\Pi}'_\mu=\hat{I}$ is satisfied. Furthermore, using the decomposition formula for the SU(1,1) generators, we can also show from the inequality $\hat{\Pi}_\mu\geqslant 0$ that the operator $\hat{\Pi}'_\mu$ is non-negative definite; that is, $\hat{\Pi}'_\mu\geqslant 0$ for all $\mu\in\mathcal{S}$. Thus $\hat{\Pi}'$ becomes a POM and belongs to the set $\mathcal{U}$. Therefore we have proved that $\hat{\Pi}'\in\tilde{\mathcal{U}}\rightarrow\hat{\Pi}'\in\mathcal{U}$. This means that $\tilde{\mathcal{U}}$ is a subset of $\mathcal{U}$; namely, $\tilde{\mathcal{U}}\subseteq\mathcal{U}$. This result provides the following inequality:

$$\max_{\hat{\Pi}'\in\tilde{\mathcal{U}}} I(\mathrm{Tr}[\hat{\Pi}'_\mu|\psi_j\rangle\langle\psi_j|])\leqslant\max_{\hat{\Pi}'\in\mathcal{U}} I(\mathrm{Tr}[\hat{\Pi}'_\mu|\psi_j\rangle\langle\psi_j|]). \quad (3.5)$$

Here the equality holds if and only if the optimum POM $\hat{\Pi}'_{\mathrm{opt}}$ chosen among the elements of $\mathcal{U}$ belongs to the subset $\tilde{\mathcal{U}}$; that is, $\hat{\Pi}_{\mathrm{opt}}\in\tilde{\mathcal{U}}\subseteq\mathcal{U}$. It is important to note that the quantity on the right-hand side in the inequality (3.5) is equal to the accessible information calculated in the quantum detection process, where the quantum states of the received signal are given by the pure states $|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_n\rangle$.

We denote the accessible information for the signal given by Eqs. (2.1) and (2.2) in the presence of the thermal noise as $I_{\mathrm{opt}}(\gamma,\phi,\mu,\nu)$ and the accessible information for the same signal in the absence of the thermal noise as $I^{(0)}_{\mathrm{opt}}(\gamma,\phi,\mu,\nu)$. Then we finally obtain the following inequality from Eqs. (3.4) and (3.5):

$$I_{\mathrm{opt}}(\gamma,\phi,\mu,\nu)\leqslant I^{(0)}_{\mathrm{opt}}(\gamma,\phi,\tilde{\mu},\tilde{\nu}), \quad (3.6)$$

where the renormalized parameters $\tilde{\mu}_j$ and $\tilde{\nu}_j$ which include the thermal noise effects are related to the original parameters μ_j and ν_j by Eq. (2.14). It should be noted that the inequality $I^{(0)}_{\mathrm{opt}}(\gamma,\phi,\tilde{\mu},\tilde{\nu})\leqslant I^{(0)}_{\mathrm{opt}}(\gamma,\phi,\mu,\nu)$ is satisfied. Of course, the inequality $I_{\mathrm{opt}}(\gamma,\phi,\mu,\nu)\leqslant I^{(0)}_{\mathrm{opt}}(\gamma,\phi,\mu,\nu)$ is trivial. The inequality (3.6) indicates that the upper bound of the accessible information for the signal in the presence of the thermal noise is obtained by replacing the signal parameters with the renormalized parameters including the thermal noise effects in the accessible information obtained for the same signal in the absence of the thermal noise.

B. Binary signal-detection process

We now consider a binary signal detection to obtain an analytic expression of the upper bound of the accessible information, $I^{(0)}_{\mathrm{opt}}(\gamma,\phi,\tilde{\mu},\tilde{\nu})$, under the influence of the thermal noise. To this end, we have to obtain the accessible information $I^{(0)}_{\mathrm{opt}}(\gamma,\phi,\mu,\nu)$ in the absence of the thermal noise. We first investigate a detection process for quantum states whose density operators are given by $\hat{\rho}_1$ and $\hat{\rho}_2$. To simplify the discussion, we confine ourselves to the case of $\mathcal{S}=\{1,2\}$. Thus the binary detection process we consider is described by POMs $\hat{\Pi}_1$ and $\hat{\Pi}_2$ which satisfy

$$\hat{\Pi}_1+\hat{\Pi}_2=\hat{I}, \quad \hat{\Pi}_j\geqslant 0\ (j=1,2). \quad (3.7)$$

In this case, the quantity $P(j|k)=\mathrm{Tr}(\hat{\Pi}_j\hat{\rho}_k)$ is the conditional probability that the measurement outcome indicates the quantum state j when the quantum state k has actually been received. Suppose that the signal-detection process described by $\hat{\Pi}_1$ and $\hat{\Pi}_2$ maximizes the mutual information. Then the POMs $\hat{\Pi}_1$ and $\hat{\Pi}_2$ satisfy the two equivalent conditions derived by Holevo [11],

$$\hat{\Pi}_1(\hat{F}_1-\hat{F}_2)\hat{\Pi}_2=0, \quad (3.8)$$

$$(\hat{F}_j-\hat{\Gamma})\hat{\Pi}_j=0\ (j=1,2), \quad \hat{\Gamma}=\sum_{j=1,2}\hat{\Pi}_j\hat{F}_j=\sum_{j=1,2}\hat{F}_j\hat{\Pi}_j \quad (3.9)$$

where $\hat{\Gamma}$ is called the Lagrange operator and $\hat{F}_j$ is given by

$$\hat{F}_j=\sum_{k=1,2}\hat{\rho}_k p_k \ln\left[\frac{P(j|k)}{\Sigma_{m=1,2}P(j|m)p_m}\right]. \quad (3.10)$$

Then using Eqs. (3.7) and (3.9), if the operator $\hat{F}_1-\hat{F}_2$ does not have zero eigenvalue, we can derive the relations

$$\hat{\Pi}_1\hat{\Pi}_2=\hat{\Pi}_2\hat{\Pi}_1=0, \quad \hat{\Pi}_j^2=\hat{\Pi}_j(j=1,2), \quad (3.11)$$

where the derivation is given in the Appendix. This result indicates that the accessible information in the binary signal detection is obtained by a quantum measurement described by projection operators. It should be noted that Eq. (3.11) is valid for any density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ as long as the operator $\hat{F}_1-\hat{F}_2$ does not have zero eigenvalue.

Suppose that the signal quantum states are pure and linearly independent. Let $|\psi_1^{(0)}\rangle$ and $|\psi_2^{(0)}\rangle$ be the signal quantum states, where $|\psi_j^{(0)}\rangle$ is given in Eqs. (2.1) and (2.2). In this case, we find from Eq. (3.11) that $\hat{\Pi}_1$ and $\hat{\Pi}_2$ become one-dimensional projection operators. Thus after some calculation (see the Appendix), we obtain the accessible information [7,38,39],

$$\begin{aligned} I_{\rm opt}^{(0)}(\gamma,\phi,\mu,\nu)=I_0+&\frac{1}{2}(1+\sqrt{1-4p_1p_2\kappa^2})\\ &\times\ln(1+\sqrt{1-4p_1p_2\kappa^2})\\ &+\frac{1}{2}(1-\sqrt{1-4p_1p_2\kappa^2})\\ &\times\ln(1-\sqrt{1-4p_1p_2\kappa^2}), \end{aligned} \tag{3.12}$$

where we set $\kappa=|\langle\psi_1^{(0)}|\psi_2^{(0)}\rangle|$ and $I_0=-\ln 2-p_1\ln p_1-p_2\ln p_2$.

Using Eq. (3.12), we can obtain the analytic expression of the upper bound of the accessible information in the presence of the thermal noise. When the density operators of the signal quantum states are given by $\hat{\rho}_1=\hat{V}_1\hat{\rho}_{\rm th}\hat{V}_1^\dagger/{\rm Tr}(\hat{V}_1\hat{\rho}_{\rm th}\hat{V}_1^\dagger)$ and $\hat{\rho}_2=\hat{V}_2\hat{\rho}_{\rm th}\hat{V}_2^\dagger/{\rm Tr}(\hat{V}_2\hat{\rho}_{\rm th}\hat{V}_2^\dagger)$, the upper bound of the accessible information is obtained by substituting $\kappa=|\langle\psi_1|\psi_2\rangle|$ into Eq. (3.12), where $|\psi_j\rangle=\hat{U}_j|0\rangle/\sqrt{\langle 0|\hat{U}_j^\dagger\hat{U}_j|0\rangle}$ and the operator $\hat{U}_j$ is given by Eq. (2.13). In particular, when we consider the binary detection process for thermal coherent states, $\hat{\rho}_1=\hat{D}(\alpha)\hat{\rho}_{\rm th}\hat{D}^\dagger(\alpha)$ and $\hat{\rho}_2=\hat{D}(\beta)\hat{\rho}_{\rm th}\hat{D}^\dagger(\beta)$, where $\hat{D}(\alpha)=e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}$ is the displacement operator, we can obtain the inequality

$$\begin{aligned} I_{\rm opt}\leqslant I_0+&\frac{1}{2}\left[1+\sqrt{1-4p_1p_2\exp\left(-\frac{|\alpha-\beta|^2}{1+2\bar{n}}\right)}\right]\\ &\times\ln\left[1+\sqrt{1-4p_1p_2\exp\left(-\frac{|\alpha-\beta|^2}{1+2\bar{n}}\right)}\right]\\ &+\frac{1}{2}\left[1-\sqrt{1-4p_1p_2\exp\left(-\frac{|\alpha-\beta|^2}{1+2\bar{n}}\right)}\right]\\ &\times\ln\left[1-\sqrt{1-4p_1p_2\exp\left(-\frac{|\alpha-\beta|^2}{1+2\bar{n}}\right)}\right]. \end{aligned} \tag{3.13}$$

In optical communication systems where $|\alpha-\beta|^2\geqslant\bar{n}$ is satisfied, the right-hand side of this inequality becomes a much tighter bound of the accessible information than the Holevo bound.

IV. LOWER BOUND OF THE BAYES COST

A. General formula for the lower bound

In this section we consider the lower bound of the Bayes cost C_B [1,8,11–13] in the quantum detection process for the Gaussian state signal under the influence of the thermal noise. In this case the number of the indices μ is equal to the number of the quantum states of the signal, so we can set $\mathcal{S}=\{1,2,\ldots,n\}$. The Bayes cost C_B to be minimized is given by

$$C_B=\sum_{j=1}^{n}\sum_{k=1}^{n}p_kC_{jk}{\rm Tr}[\hat{\Pi}_j\hat{\rho}_k], \tag{4.1}$$

where the quantity C_{jk} is the cost incurred when we infer that the received quantum state is described by the density operator $\hat{\rho}_j$ even though the quantum state $\hat{\rho}_k$ has actually been received [1]. Since the cost is reduced by the correct detection, the inequality $C_{jj}<C_{jk}$ is satisfied. In the following, we denote the Bayes cost as $C_B({\rm Tr}[\hat{\Pi}_j\hat{\rho}_k])$. The necessary and sufficient condition for the POM $\hat{\Pi}$ to minimize the Bayes cost is given by Eq. (1.4). But it is difficult to obtain the optimum POM from Eq. (1.4).

From the same argument used for deriving the upper bound of the accessible information, we find the following relation:

$$\begin{aligned} C_{B\rm opt}&=\min_{\hat{\Pi}\in\mathcal{U}}C_B\{{\rm Tr}[\hat{\Pi}_j\hat{\mathcal{L}}(|\psi_k\rangle\langle\psi_k|)]\}\\ &=\min_{\hat{\Pi}\in\mathcal{U}}C_B\{{\rm Tr}[\hat{\mathcal{L}}^\dagger(\hat{\Pi}_j)|\psi_k\rangle\langle\psi_k|]\}\\ &=\min_{\hat{\Pi}'\in\tilde{\mathcal{U}}}C_B({\rm Tr}[\hat{\Pi}_j'|\psi_k\rangle\langle\psi_k|])\\ &\geqslant\min_{\hat{\Pi}'\in\mathcal{U}}C_B({\rm Tr}[\hat{\Pi}_j'|\psi_k\rangle\langle\psi_k|]). \end{aligned} \tag{4.2}$$

In the last inequality we have used the fact that $\tilde{\mathcal{U}}$ is a subset of $\mathcal{U}$, and the equality holds if and only if the optimum POM $\hat{\Pi}_{\rm opt}$ chosen among the elements of the set $\mathcal{U}$ such that the Bayes cost in the detection process for the quantum states $(|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_n\rangle)$ should be minimized belongs to the subset $\tilde{\mathcal{U}}$; namely, $\hat{\Pi}_{\rm opt}\in\tilde{\mathcal{U}}\subseteq\mathcal{U}$. Let $C_{B\rm opt}(\gamma,\phi,\mu,\nu)$ be the minimum value of the Bayes cost in the quantum detection process for the signal $(\hat{\rho}_1,\hat{\rho}_2,\ldots,\hat{\rho}_n)$ and let $C_{B\rm opt}^{(0)}(\gamma,\phi,\mu,\nu)$ be the minimum value of the Bayes cost in the detection process for the signal $(|\psi_1^{(0)}\rangle,|\psi_2^{(0)}\rangle,\ldots,|\psi_n^{(0)}\rangle)$, where $\hat{\rho}_j$ and $|\psi_j^{(0)}\rangle$ are given in Eqs. (2.1) and (2.2). Then from Eq. (4.2), we can obtain the following inequality:

$$C_{B\rm opt}(\gamma,\phi,\mu,\nu)\geqslant C_{B\rm opt}^{(0)}(\gamma,\phi,\tilde{\mu},\tilde{\nu}), \tag{4.3}$$

where the renormalized parameters $\tilde{\mu}_j$ and $\tilde{\nu}_j$ including the thermal noise effects are given by Eq. (2.14). It is obvious that the inequalities $C_{B\rm opt}^{(0)}(\gamma,\phi,\tilde{\mu},\tilde{\nu})\geqslant C_{B\rm opt}^{(0)}(\gamma,\phi,\mu,\nu)$ and $C_{B\rm opt}(\gamma,\phi,\mu,\nu)\geqslant C_{B\rm opt}^{(0)}(\gamma,\phi,\mu,\nu)$ are satisfied. Thus the lower bound of the Bayes cost $C_{B\rm opt}$ is obtained from $C_{B\rm opt}^{(0)}$ by substituting the renormalized parameters.

The average probability of error P_e which is one of the most important quantities to evaluate a performance of a quantum communication system is obtained as $P_e=1+C_B$ by substituting $C_{jk}=-\delta_{jk}$ into Eq. (4.1). Let us denote as $P_{\rm opt}(\gamma,\phi,\mu,\nu)$ $[P_{\rm opt}^{(0)}(\gamma,\phi,\mu,\nu)]$ the minimum value of the average probability of error in the signal-detection process in the presence (absence) of the thermal noise. Then we obtain from the inequality (4.3)

$$P_{\rm opt}(\gamma,\phi,\mu,\nu)\geqslant P^{(0)}_{\rm opt}(\gamma,\phi,\tilde{\mu},\tilde{\nu}). \tag{4.4}$$

In the absence of the thermal noise, the quantum states of the signal considered here become pure and linearly independent. Thus we can apply Kennedy's lemma [1] to obtain the minimum value of the average probability of error and the optimum POM . The lemma ensures that the minimum value of the average probability of error for a linearly independent pure quantum state signal is attained by a quantum measurement described by a set of one-dimensional projection operators. This lemma greatly simplifies calculation of the minimum values of the average probability of error $P^{(0)}_{\rm opt}(\gamma,\phi,\mu,\nu)$. The analytic expressions of $P^{(0)}_{\rm opt}(\gamma,\phi,\mu,\nu)$ have been obtained for several coherent state signals [1,40,41]. Thus we can obtain the lower bound $P^{(0)}_{\rm opt}(\gamma,\phi,\tilde{\mu},\tilde{\nu})$ under the influence of the thermal noise.

B. Lower bounds for binary and coherent state signals

We now consider a binary signal-detection process which is very important in practical digital communication systems. The signal takes quantum states $\hat{\rho}_1$ and $\hat{\rho}_2$ with prior probabilities p_1 and p_2, where the density operator $\hat{\rho}_j$ $(j=1,2)$ is given in Eqs. (2.1) and (2.2). In the absence of the thermal noise, since the quantum state of the signal becomes pure, the minimum value of the average probability of error is obtained [1,39–41] as

$$P^{(0)}_{\rm opt}=\frac{1}{2}[1-\sqrt{1-\Gamma_{12}(\gamma,\phi,\mu,\nu)}], \tag{4.5}$$

where $\Gamma_{12}(\gamma,\phi,\mu,\nu)=|\langle\psi^{(0)}_1|\psi^{(0)}_2\rangle|^2$ and $|\psi^{(0)}_j\rangle=\hat{V}_j|0\rangle/\sqrt{\langle 0|\hat{V}^\dagger_j\hat{V}_j|0\rangle}$. Then the minimum value of the average probability of error $P_{\rm opt}$ in the binary signal-detection process under the influence of the thermal noise satisfies the inequality

$$P_{\rm opt}\geqslant\frac{1}{2}[1-\sqrt{1-\Gamma_{12}(\gamma,\phi,\tilde{\mu},\tilde{\nu})}], \tag{4.6}$$

where $\Gamma_{12}(\gamma,\phi,\tilde{\mu},\tilde{\nu})=|\langle\psi_1|\psi_2\rangle|^2$ and $|\psi_j\rangle=\hat{U}_j|0\rangle/\sqrt{\langle 0|\hat{U}^\dagger_j\hat{U}_j|0\rangle}$ and the operator $\hat{U}_j$ is given by Eq. (2.13).

Let us consider, for instance, the binary phase-shift keyed coherent state signal under the influence of the thermal noise, where the density operators of the two signal states are given by $\hat{\rho}_1=\hat{D}(\alpha)\hat{\rho}_{\rm th}\hat{D}^\dagger(\alpha)$ and $\hat{\rho}_2=\hat{D}(-\alpha)\hat{\rho}_{\rm th}\hat{D}^\dagger(-\alpha)$ with $\hat{D}(\alpha)=e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}$. For simplicity, we assume the equal prior probabilities, $p_1=p_2=1/2$. Since $\hat{V}_1=\hat{D}(\alpha)$ and $\hat{V}_2=\hat{D}(-\alpha)$, we obtain $\hat{U}_1=\hat{D}(\alpha/\sqrt{1+2\overline{n}})$ and $\hat{U}_2=\hat{D}(-\alpha/\sqrt{1+2\overline{n}})$. Thus the lower bound of the average probability of error is obtained by substituting $\Gamma_{12}(\gamma,\phi,\tilde{\mu},\tilde{\nu})=\exp[-4\overline{n_s}/(1+2\overline{n})]$ into Eq. (4.6), where $n_s=|\alpha|^2$ is the average value of the photon number in each bit of the signal. Among the conventional optical measurements, the lower average probability of error for the binary phase-shift keyed coherent state signal is attained by the homodyne detection [42], and it is given by $P_{\rm hom}=1/2\{1-{\rm erf}[\sqrt{2n_s/(1+2\overline{n})}]\}$, where ${\rm erf}(x)=(2/\sqrt{\pi})\int_0^x dt e^{-t^2}$ is the error function. Since the minimum value of the average probability of error obtained by the quantum detection theory is smaller than the average probability of error in any conventional optical measurement, we can obtain the inequality

$$\frac{1}{2}\left[1-{\rm erf}\left(\frac{D}{\sqrt{2}}\right)\right]\geqslant P_{\rm opt}\geqslant\frac{1}{2}[1-\sqrt{1-\exp(-D^2)}], \tag{4.7}$$

where the parameter $D=\sqrt{4n_s/(1+2\overline{n})}$ represents the equivalent signal-to-noise ratio [1]. When the signal has a large number of photons $(D\gg 1)$, Eq. (4.7) is approximated as $e^{-D^2/2}/D\sqrt{2\pi}\geqslant P_{\rm opt}\geqslant 1/4e^{-D^2}$. On the other hand, when the signal is very noisy $(D\ll 1)$, Eq. (4.7) becomes $1/2(1-D/\sqrt{2\pi})\geqslant P_{\rm opt}\geqslant 1/2(1-D)$.

For an M-ary coherent state signal, where the quantum states are given by the density operators $\hat{\rho}_1,\hat{\rho}_2,\ldots,\hat{\rho}_M$ with $\hat{\rho}_j=\hat{D}(\alpha_j)\hat{\rho}_{\rm th}\hat{D}^\dagger(\alpha_j)$, the lower bound of the average probability of error can be obtained by replacing the complex amplitude α_j with $\tilde{\alpha}_j=\alpha_j/\sqrt{1+2\overline{n}}$ in the minimum value of the average probability of error $P^{(0)}_{\rm opt}(\alpha_1,\alpha_2,\ldots,\alpha_M)$ for the thermal noise-free M-ary coherent state signal whose quantum states are given by pure coherent states $|\alpha_1\rangle,|\alpha_2\rangle,\ldots,|\alpha_M\rangle$; that is, the lower bound of the average probability of error is given by $P^{(0)}_{\rm opt}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_M)$. In the absence of the thermal noise, since the analytic expressions of the minimum value of the average probability of error for the several coherent state signals have been obtained by the present authors [40,41], we can use the results to obtain the lower bounds of the average probability of error in the presence of the thermal noise. For the ternary phase-shift keyed coherent state signal, where the quantum states are $|\alpha\rangle$, $|\alpha e^{(2/3)i\pi}\rangle$, and $|\alpha e^{-(2/3)i\pi}\rangle$, the minimum value of the average probability of error becomes

$$P^{(0)}_{\rm opt}(\alpha)=\frac{2}{9}\{3-\sqrt{(1-\kappa_c)^2-3\kappa_s^2}-\sqrt{2(1+2\kappa_c)[1-\kappa_c+\sqrt{(1-\kappa_c)^2-3\kappa_s^2}]}\}, \tag{4.8}$$

where the parameters κ_c and κ_s are given by

$$\kappa_c=\exp\left(-\frac{3}{2}\overline{n_s}\right)\cos\left(\frac{\sqrt{3}}{2}\overline{n_s}\right),$$

$$\kappa_s=\exp\left(-\frac{3}{2}\overline{n_s}\right)\sin\left(\frac{\sqrt{3}}{2}\overline{n_s}\right). \tag{4.9}$$

Then the lower bound of the average probability of error for the ternary phase-shift coherent state signal under the influence of the thermal noise is obtained by substituting the following expressions into Eq. (4.8):

$$\kappa_c=\exp\left[-\frac{3\overline{n_s}}{2(1+2\overline{n})}\right]\cos\left[\frac{\sqrt{3}\overline{n_s}}{2(1+2\overline{n})}\right], \tag{4.10}$$

$$\kappa_s = \exp\left[-\frac{3\overline{n_s}}{2(1+2\overline{n})}\right]\sin\left[\frac{\sqrt{3}\overline{n_s}}{2(1+2\overline{n})}\right]. \tag{4.11}$$

In the same way, we can obtain the lower bound of the average probability of error for thermal coherent state signals.

V. SUMMARY

In this paper, by means of the superoperator representation of quantum states, we have obtained the upper bound of the accessible information and the lower bound of the Bayes cost in the quantum detection processes for the Gaussian state signals under the influence of the thermal noise. The analytic expressions of the upper and lower bounds for the binary quantum state signals have been given. The method we applied in this paper can be used for estimating any quantity that is a function of the conditional probabilities, $P(\mu|j)=\mathrm{Tr}[\hat{\Pi}_\mu\hat{\rho}_j]$. Let $\mathcal{F}(x)$ be an analytic function of x. Then for the Gaussian quantum state given in Eqs. (2.1) and (2.2), we can obtain the inequalities

$$\max_{\hat{\Pi}\in\mathcal{U}}\mathcal{F}(\mathrm{Tr}[\hat{\Pi}_\mu\hat{\rho}_j])\leqslant\max_{\hat{\Pi}\in\mathcal{U}}\mathcal{F}(\mathrm{Tr}[\hat{\Pi}_\mu|\psi_j\rangle\langle\psi_j|]), \tag{5.1}$$

$$\min_{\hat{\Pi}\in\mathcal{U}}\mathcal{F}(\mathrm{Tr}[\hat{\Pi}_\mu\hat{\rho}_j])\geqslant\min_{\hat{\Pi}\in\mathcal{U}}\mathcal{F}(\mathrm{Tr}[\hat{\Pi}_\mu|\psi_j\rangle\langle\psi_j|]), \tag{5.2}$$

where the set $\mathcal{U}$ is given by Eq. (3.3). The superoperator representation of quantum states that we have used here seems to be very useful for investigating quantum communication and information systems under the influence of the thermal noise. It is shown [43] that when bounded operators are considered, the superoperator method or thermofield dynamics is mathematically equivalent to C^* algebra [44]. Therefore when we use C^* algebra, the inequalities (5.1) and (5.2) will be proved more rigorously.

ACKNOWLEDGMENTS

One of the authors (M.B.) would like to thank Dr. M. Hirokawa of Tokyo Gakugei University for his useful comments on operator algebra. He is also grateful to the members of the quantum communication group of Tamagawa University for their hospitality.

APPENDIX: OPTIMIZATION OF THE BINARY QUANTUM DETECTION PROCESS

In this appendix we first derive Eq. (3.11) from Eqs. (3.7) and (3.9). It is easy to see from the relation $\sum_{j=1,2}\hat{\Pi}_j=\hat{I}$ that the operators $\hat{\Pi}_1$ and $\hat{\Pi}_2$ commute with each other, $\hat{\Pi}_1\hat{\Pi}_2=\hat{\Pi}_2\hat{\Pi}_1$. Furthermore the relations $\sum_{j=1,2}\hat{\Pi}_j=\hat{I}$ and $\hat{\Gamma}=\sum_{j=1,2}\hat{F}_j\hat{\Pi}_j$ give $\hat{F}_1-\hat{\Gamma}=(\hat{F}_1-\hat{F}_2)\hat{\Pi}_2$. Multiplying this relation by the operator $\hat{\Pi}_1$ from the right and using Eq. (3.9), we obtain

$$(\hat{F}_1-\hat{F}_2)\hat{\Pi}_1\hat{\Pi}_2=(\hat{F}_1-\hat{F}_2)\hat{\Pi}_2\hat{\Pi}_1=0. \tag{A1}$$

On the other hand, the operator $\hat{F}_1-\hat{F}_2$ is calculated from Eq. (3.10) as

$$\hat{F}_1-\hat{F}_2=A_1\hat{\rho}_1-A_2\hat{\rho}_2, \tag{A2}$$

with

$$A_1=p_1\ln\left[\frac{P(1|1)P_{\mathrm{out}}(2)}{P(2|1)P_{\mathrm{out}}(1)}\right],\quad A_2=p_2\ln\left[\frac{P(2|2)P_{\mathrm{out}}(1)}{P(1|2)P_{\mathrm{out}}(2)}\right], \tag{A3}$$

where we set $P_{\mathrm{out}}(j)=\sum_{k=1,2}P(j|k)p_k$. Let $|\omega_\mu\rangle$ and ω_μ be an eigenstate and eigenvalue of the operator $\hat{F}_1-\hat{F}_2$. Then using the spectral decomposition $\hat{F}_1-\hat{F}_2=\sum_{\mu\in\mathcal{V}}|\omega_\mu\rangle\omega_\mu\langle\omega_\mu|$, we obtain from Eq. (A1)

$$\sum_{\mu\in\mathcal{V}}|\omega_\mu\rangle\omega_\mu\langle\omega_\mu|\hat{\Pi}_1\hat{\Pi}_2=\sum_{\mu\in\mathcal{V}}|\omega_\mu\rangle\omega_\mu\langle\omega_\mu|\hat{\Pi}_2\hat{\Pi}_1=0, \tag{A4}$$

where $\mathcal{V}$ stands for a set of indices of the eigenvalues. Since $\hat{F}_1-\hat{F}_2$ is a Hermitian operator, the set $\{|\omega_\mu\rangle|\mu\in\mathcal{V}\}$ becomes a complete orthonormal system. Thus Eq. (A4) is equivalent to

$$\omega_\mu\langle\omega_\mu|\hat{\Pi}_1\hat{\Pi}_2=\omega_\mu\langle\omega_\mu|\hat{\Pi}_2\hat{\Pi}_1=0. \tag{A5}$$

If $\omega_\mu\neq0$ for all $\mu\in\mathcal{V}$, the equality $\langle\omega_\mu|\hat{\Pi}_1\hat{\Pi}_2=\langle\omega_\mu|\hat{\Pi}_2\hat{\Pi}_1=0$ is obtained. Therefore we find from $\sum_{j=1,2}\hat{\Pi}_j=\hat{I}$ and the completeness of $\{|\omega_\mu\rangle|\mu\in\mathcal{V}\}$ that $\hat{\Pi}_1\hat{\Pi}_2=\hat{\Pi}_2\hat{\Pi}_1=0$ and $\hat{\Pi}_j^2=\hat{\Pi}_j$ $(j=1,2)$ are satisfied. We have obtained Eq. (3.11).

Next we will derive Eq. (3.12). When the density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ represent linearly independent pure quantum states, $\hat{\Pi}_1$ and $\hat{\Pi}_2$ become one-dimensional projection operators. Here we set $\hat{\rho}_j=|\phi_j\rangle\langle\phi_j|$ and $\hat{\Pi}_j=|\hat{\phi}_j\rangle\langle\hat{\phi}_j|$ $(j=1,2)$. It should be noted that $\{|\hat{\phi}_1\rangle,|\hat{\phi}_2\rangle\}$ is a complete orthonormal system in the two-dimensional Hilbert space $\mathcal{H}_2$. Since the signal quantum states $|\phi_1\rangle$ and $|\phi_2\rangle$ are linearly independent, a set of two states given by

$$|\phi_1'\rangle=\frac{|\phi_1\rangle+e^{-i\phi}|\phi_2\rangle}{\sqrt{2(1+\kappa)}},\quad|\phi_2'\rangle=\frac{|\phi_2\rangle-e^{i\phi}|\phi_1\rangle}{\sqrt{2(1-\kappa)}} \tag{A6}$$

becomes a complete orthonormal system in the Hilbert space $\mathcal{H}_2$. In Eq. (A6), we set $\langle\phi_1|\phi_2\rangle=\kappa e^{i\phi}$ $(\kappa>0)$. Thus there must be a 2×2 unitary matrix which represents a unitary transformation between $(|\hat{\phi}_1\rangle,|\hat{\phi}_2\rangle)$ and $(|\phi_1'\rangle,|\phi_2'\rangle)$. Such a transformation can be expressed [39] as

$$\begin{pmatrix}|\hat{\phi}_1\rangle\\|\hat{\phi}_2\rangle\end{pmatrix}=\begin{pmatrix}e^{i\alpha/2}\cos(\beta/2) & e^{-i\alpha/2}\sin(\beta/2)\\-e^{i\alpha/2}\sin(\beta/2) & e^{-i\alpha/2}\cos(\beta/2)\end{pmatrix}\begin{pmatrix}|\hat{\phi}_1'\rangle\\|\hat{\phi}_2'\rangle\end{pmatrix}, \tag{A7}$$

where α and β are real parameters to be determined by the optimization of the signal-detection process. Using Eqs. (A6) and (A7), the mutual information is calculated as

$$I(\alpha,\beta)=[1-p(\alpha,\beta)]p_1\ln\left[\frac{1-p(\alpha,\beta)}{[1-p(\alpha,\beta)]p_1+q(\alpha,\beta)p_2}\right]+p(\alpha,\beta)p_1\ln\left[\frac{p(\alpha,\beta)}{p(\alpha,\beta)p_1+[1-q(\alpha,\beta)]p_2}\right]+[1-q(\alpha,\beta)]p_2\ln\left[\frac{1-q(\alpha,\beta)}{p(\alpha,\beta)p_1+[1-q(\alpha,\beta)]p_2}\right]+q(\alpha,\beta)p_2\ln\left[\frac{q(\alpha,\beta)}{[1-p(\alpha,\beta)]p_1+q(\alpha,\beta)p_2}\right], \tag{A8}$$

with

$$p(\alpha,\beta)=\frac{1}{2}[1-\kappa\cos\beta+\sqrt{1-\kappa^2}\cos(\alpha-\phi)\sin\beta], \tag{A9}$$

$$q(\alpha,\beta)=\frac{1}{2}[1+\kappa\cos\beta+\sqrt{1-\kappa^2}\cos(\alpha-\phi)\sin\beta]. \tag{A10}$$

Then the maximization condition of the mutual information $I(\alpha,\beta)$ is given by

$$\frac{\partial}{\partial\alpha}I(\alpha,\beta)=\frac{\partial}{\partial\beta}I(\alpha,\beta)=0, \tag{A11}$$

$$\begin{pmatrix} \partial^2 I(\alpha,\beta)/\partial\alpha^2 & \partial^2 I(\alpha,\beta)/\partial\alpha\partial\beta \\ \partial^2 I(\alpha,\beta)/\partial\beta\partial\alpha & \partial^2 I(\alpha,\beta)/\partial\beta^2 \end{pmatrix}<0, \tag{A12}$$

where the matrix inequality means that the Hesse matrix calculated from $I(\alpha,\beta)$ is negative definite. Substituting Eqs. (A8)–(A10) into Eqs. (A11) and (A12), we obtain the following solutions of Eqs. (A8)–(A12):

$$\alpha=\phi, \quad \sin\beta=\mp\sqrt{\frac{1-\kappa^2}{1-4p_1p_2\kappa^2}}, \quad \cos\beta=\pm\frac{(p_1-p_2)\kappa}{\sqrt{1-4p_1p_2\kappa^2}}. \tag{A13}$$

Substituting these solutions into Eqs. (A9) and (A10), we can obtain the accessible information from Eq. (A8),

$$I_{\rm opt}=-\ln 2-p_1\ln p_1-p_2\ln p_2+\frac{1}{2}(1+\sqrt{1-4p_1p_2\kappa^2})\times\ln(1+\sqrt{1-4p_1p_2\kappa^2})+\frac{1}{2}(1-\sqrt{1-4p_1p_2\kappa^2})\times\ln(1-\sqrt{1-4p_1p_2\kappa^2}), \tag{A14}$$

for any linearly independent quantum states $|\phi_1\rangle$ and $|\phi_2\rangle$. In particular, when we set $|\phi_j\rangle=|\psi_j^{(0)}\rangle=\hat{V}_j|0\rangle/\sqrt{\langle 0|\hat{V}_j^\dagger\hat{V}_j|0\rangle}$ we obtain the accessible information given by Eq. (3.12). For reference, we write down the explicit expressions of the optimum POMs $\hat{\Pi}_1$ and $\hat{\Pi}_2$,

$$\hat{\Pi}_1=\frac{R\pm\frac{1}{2}(1+\lambda)\mp\lambda\kappa^2}{2(1-\kappa^2)R}|\phi_1\rangle\langle\phi_1|+\frac{R\mp\frac{1}{2}(1+\lambda)\pm\kappa^2}{2(1-\kappa^2)R}|\phi_2\rangle\langle\phi_2|-\kappa\frac{R\pm\frac{1}{2}(1-\lambda)}{2(1-\kappa^2)R}(e^{i\phi}|\phi_1\rangle\langle\phi_2|+e^{-i\phi}|\phi_2\rangle\langle\phi_1|), \tag{A15}$$

$$\hat{\Pi}_2=\frac{R\mp\frac{1}{2}(1+\lambda)\pm\lambda\kappa^2}{2(1-\kappa^2)R}|\phi_1\rangle\langle\phi_1|+\frac{R\pm\frac{1}{2}(1+\lambda)\mp\kappa^2}{2(1-\kappa^2)R}|\phi_2\rangle\langle\phi_2|-\kappa\frac{R\mp\frac{1}{2}(1-\lambda)}{2(1-\kappa^2)R}(e^{i\phi}|\phi_1\rangle\langle\phi_2|+e^{-i\phi}|\phi_2\rangle\langle\phi_1|), \tag{A16}$$

where the parameters λ and R are given by

$$\lambda=p_1/p_2, \quad R=\sqrt{\frac{1}{4}(1+\lambda)^2-\lambda\kappa^2}. \tag{A17}$$

Since the Bayes cost in the binary signal-detection process is maximized by a quantum measurement described by projection operators [1], we can obtain the minimum value of the Bayes cost in the same way. Using Eqs. (A6) and (A7), the Bayes cost in the binary detection process for the pure quantum state signal is calculated as

$$C_B(\alpha,\beta)=p_1C_{11}[1-p(\alpha,\beta)]+p_1C_{21}p(\alpha,\beta)+p_2C_{22}[1-q(\alpha,\beta)]+p_2C_{12}q(\alpha,\beta), \tag{A18}$$

where $p(\alpha,\beta)$ and $q(\alpha,\beta)$ are given by Eqs. (A9) and (A10). The minimization condition of the Bayes cost $C_B(\alpha,\beta)$ is given by

$$\frac{\partial}{\partial\alpha}C_B(\alpha,\beta)=\frac{\partial}{\partial\beta}C_B(\alpha,\beta)=0, \tag{A19}$$

$$\begin{pmatrix} \partial^2 C_B(\alpha,\beta)/\partial\alpha^2 & \partial^2 C_B(\alpha,\beta)/\partial\alpha\partial\beta \\ \partial^2 C_B(\alpha,\beta)/\partial\beta\partial\alpha & \partial^2 C_B(\alpha,\beta)/\partial\beta^2 \end{pmatrix}>0. \tag{A20}$$

Substituting Eq. (A18) into Eqs. (A19) and (A20), we obtain the solutions

$$\alpha=\phi, \quad \cos\beta=-\frac{(1-\chi)\kappa}{2S},$$

$$\sin\beta=-\frac{(1+\chi)\sqrt{1-\kappa^2}}{2S}, \tag{A21}$$

where the parameters χ and S are given by

$$\chi=\frac{p_1(C_{21}-C_{11})}{p_2(C_{12}-C_{22})}, \quad S=\sqrt{\frac{1}{4}(1+\chi)^2-\chi\kappa^2}. \tag{A22}$$

Thus the minimum value of the Bayes cost and the optimum POM become

$$C_{B\text{opt}}=C_0+\frac{1}{2}[p_1(C_{21}-C_{11})+p_2(C_{12}-C_{22})]\times\left\{1-\sqrt{1-\frac{4\chi}{(1+\chi)^2}\kappa^2}\right\}, \tag{A23}$$

$$\hat{\Pi}_1=\frac{S+\frac{1}{2}(1+\chi)-\chi\kappa^2}{2(1-\kappa^2)S}|\phi_1\rangle\langle\phi_1| + \frac{S-\frac{1}{2}(1+\chi)+\kappa^2}{2(1-\kappa^2)S}|\phi_2\rangle\langle\phi_2| - \kappa\frac{S+\frac{1}{2}(1-\chi)}{2(1-\kappa^2)S}(e^{i\phi}|\phi_1\rangle\langle\phi_2|+e^{-i\phi}|\phi_2\rangle\langle\phi_1|), \tag{A24}$$

$$\hat{\Pi}_2=\frac{S-\frac{1}{2}(1+\chi)+\chi\kappa^2}{2(1-\kappa^2)S}|\phi_1\rangle\langle\phi_1| + \frac{S+\frac{1}{2}(1+\chi)-\kappa^2}{2(1-\kappa^2)S}|\phi_2\rangle\langle\phi_2| - \kappa\frac{S-\frac{1}{2}(1-\chi)}{2(1-\kappa^2)S}(e^{i\phi}|\phi_1\rangle\langle\phi_2|+e^{-i\phi}|\phi_2\rangle\langle\phi_1|), \tag{A25}$$

where we set $C_0=p_1C_{11}+p_2C_{22}$. It is easy to check that the optimum POMs $\hat{\Pi}_1$ and $\hat{\Pi}_2$ satisfy Eq. (1.4). When we set $C_{jk}=-\delta_{jk}$ and use $P_e=1+C_B$, we obtain the minimum value of the average probability of error,

$$P_{\text{opt}}=\frac{1}{2}(1-\sqrt{1-4p_1p_2\kappa^2}), \tag{A26}$$

from which Eq. (4.5) is derived by setting $|\phi_j\rangle=\hat{V}_j|0\rangle/\sqrt{\langle 0|\hat{V}_j^\dagger\hat{V}_j|0\rangle}$.

It is important to note that the maximum value of the mutual information and the minimum value of the average probability of error are simultaneously attained in the quantum measurement described by Eqs. (A15) and (A16) with the upper sign. In this measurement, the following relation is established:

$$I_{\text{opt}}=H_S-H_{\text{error}}, \tag{A27}$$

with

$$H_S=-p_1\ln p_1-p_2\ln p_2, \tag{A28}$$

$$H_{\text{error}}=-P_{\text{opt}}\ln P_{\text{opt}}-(1-P_{\text{opt}})\ln(1-P_{\text{opt}}), \tag{A29}$$

where H_S is the Shannon entropy [45] of the signal and H_{error} represents the amount of information loss due to the detection error.

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