Quantum analysis of the susceptibility for identical atoms subjected to an external force with a tail

Chung-In Um and In-Han Kim
Department of Physics, College of Science, Korea University, Seoul 136-701, Korea

Kyu-Hwang Yeon
Department of Physics, Chungbuk National University, Cheongju, Chungbuk 360-763, Korea

Thomas F. George
Office of the Chancellor, Department of Chemistry and Department of Physics and Astronomy, University of Wisconsin–Stevens Point, Stevens Point, Wisconsin 54481-3897

Lakshmi N. Pandey
Department of Chemistry and Department of Physics, Washington State University, Pullman, Washington 99164-4630
(Received 20 May 1996)

Using the invariant operator method, we have analytically obtained the exact wave function, propagator, and uncertainty relations for the quantum damped harmonic oscillator governed by a time-dependent external force with a tail characterizing the influence of the past. From these quantities, the frequency-dependent susceptibility, the index of refraction, and the extinction coefficient for N identical atoms in a volume V are evaluated. The susceptibility satisfies the common properties, known as Kramers-Kronig relations, and as a given parameter that determines the nature of the tail goes to infinity, the behaviors of these results are exactly reduced to the ones for a classical case. [S1050-2947(96)02310-4]

PACS number(s): 03.65.Ge, 02.90.+p

I. INTRODUCTION

Since the interaction of electromagnetic radiation with atoms was discussed by Einstein, it has evoked continuing interest from many physicists [1,2]. Einstein's theory gave a qualitative understanding of a large variety of radiation processes, such as the absorption and scattering of light by atoms and the amplification of light beams by lasers. However, the theory gave no satisfactory account of the linewidth of atomic transition frequencies and the prescription for computing the coefficients appropriate to a given atomic transition. For this one must look to the quantum-mechanical theory [3], and it is necessary to derive expressions for the frequency-dependent susceptibility.

A gas of atoms in a cavity can be regarded as a dielectric medium and the interaction of radiation with atoms has traditionally been treated by the damped harmonic oscillator with a time-dependent external force [4]. This treatment gives a good explanation of the anomalous index of refraction and absorption of light and has been reasonably applied to some linear absorbing systems. The usual approximation techniques, such as time-dependent perturbation theory, the adiabatic approximation, and the sudden approximation, have been used for treating time-dependent quantum systems, but it is not easy to deal with such systems. Lewis and Riesenfeld [5,6] developed the theory of invariants to more exactly treat such quantum systems in order to shed light on their exact solutions. The invariants have received primary concern because of their use in discussing physical problems [7–9] and their possibility in applications of classical and quantum physics [10]. For example, the exact invariant of the damped driven harmonic oscillator has been applied to the description of the motion of a charged particle in a time-dependent electromagnetic field [5,11] and the construction of coherent and squeezed states for systems with several possible physical applications [12].

In our previous papers, we obtained the wave functions, energy expectation values, uncertainty relations, transition amplitudes, and coherent states for time-dependent quantum systems [13–15]. In addition, for the bound quadratic Hamiltonian system, we have already obtained the quantum-mechanical quantities by using the invariant operator method [16]. The above results were obtained for the system without influence of the past. The purpose of this paper is to obtain the quantum-mechanical quantities for the damped harmonic oscillator governed by an external driving force with a tail. We assume that the external force $g(t)$ has the form

$$g(t)=\int_{-\infty}^{t} dt'\alpha e^{-\alpha(t-t')}f(t'), \tag{1.1}$$

where $f(t)$ is an instantaneous force at given time t and $\alpha e^{-\alpha(t-t')}$ satisfies the properties of the δ function

$$\lim_{\alpha\to\infty}\alpha e^{-\alpha(t-t')}=\begin{cases}\delta_\alpha(t-t') & (t\geqslant t')\\ 0 & (t<t').\end{cases} \tag{1.2}$$

As the time t increases, the contributions at early times may be negligible. To obtain the quantum-mechanical quantities we adopt the invariant operator method. From these quantities we evaluate the susceptibility $\chi(\omega)$, the index of refraction $n(\omega)$, and the extinction coefficient $\kappa(\omega)$ for the N identical atoms in a volume V represented by N damped driven harmonic oscillators of mass m and charge $-e$. The dependence of these quantities on the real parameter (α) is explicitly represented. The imaginary part of the susceptibility is related to a fine-grained transition rate $1/\tau(\omega)$. The transition rate is defined as the rate at which N similar atoms in a volume V are excited by absorption of radiation having sharply defined frequency ω. In addition, we show that they satisfy important properties, that is, Kramers-Kronig relations. These relations are often useful in experiments where it may be easier to measure one part of $\chi(\omega)$ than the other.

We shall begin with obtaining the quantum-mechanical quantities for the damped harmonic oscillator with a time-dependent external force influenced by the past. In Sec. II we obtain the dynamical invariant quantity and define different lowering and raising operators. By using the invariant operator, the exact wave functions satisfying the Schrödinger equation and the exact closed form of the propagator for this system are analytically evaluated and the uncertainty relations at various states are calculated. In Sec. III we obtain the frequency-dependent susceptibility, the index of refraction, the extinction coefficient, and the fine-grained transition rate. Finally, we give the results and a discussion with graphs for this system in Sec. IV.

II. QUANTUM-MECHANICAL TREATMENT

We consider a gas of one-electron atoms subjected to the time-dependent external force influenced by the past. To obtain the quantum-mechanical expression for the susceptibility, the index of refraction, and the extinction coefficient we have to evaluate the wave function satisfying the Schrödinger equation. Since the Schrödinger equation cannot be directly solved, we shall utilize the invariant operator method. By using the wave function that we will derive, the dipole moment of an atom can be given by

$$d(t)=-\int_{-\infty}^{+\infty} dq\ \psi_n^*(q,t)eq\psi_n(q,t), \tag{2.1}$$

where $\psi_n(q,t)$ is an atomic wave function. The quantity $d(t)$ represents the averaged dipole moment per atom at time t and the single-atom result is then suitably averaged to produce the analogous result for a gas of randomly oriented atoms or molecules. The macroscopic polarization of the gas is simply

$$P(t)=Nd(t)/V. \tag{2.2}$$

Consequently, we can obtain the susceptibility from the relation between the polarization and external field.

The Hamiltonian operator of this system has the form

$$\hat{H}(\hat{p},\hat{q},t)=e^{-\gamma t}\frac{\hat{p}^2}{2m}+e^{\gamma t}[\tfrac{1}{2}m\omega_0^2\hat{q}^2-g(t)\hat{q}], \tag{2.3}$$

where $\hat{p}$ and $\hat{q}$ are momentum and position operators, respectively, γ is a damping coefficient, and ω_0 is a simple harmonic-oscillator frequency. From Eq. (2.3) we know that the classical Lagrangian of this system becomes

$$L(\dot{q},q,t)=e^{\gamma t}[\tfrac{1}{2}m\dot{q}^2-\tfrac{1}{2}m\omega_0^2q^2+g(t)q]. \tag{2.4}$$

Using Eq. (2.4), the classical equation of motion is given by

$$\ddot{q}+\gamma\dot{q}+\omega_0^2q=\frac{g(t)}{m}. \tag{2.5}$$

We may find an invariant operator $\hat{I}(\hat{p},\hat{q},t)$ that satisfies Hamilton's equation

$$\frac{d\hat{I}}{dt}=\frac{\partial\hat{I}}{\partial t}+\frac{1}{i\hbar}[\hat{I},\hat{H}]=0. \tag{2.6}$$

Assuming that this invariant operator has the quadratic form of $\hat{p}$ and $\hat{q}$ and combining Eqs. (2.3) and (2.6), the invariant operator is given by

$$\hat{I}=\frac{1}{2}e^{\gamma t}\left\{m^2\left(\omega_0^2-\frac{\gamma^2}{4}\right)(\hat{q}-q_0)^2 +[e^{-\gamma t}(\hat{p}-p_0)+\tfrac{1}{2}m\gamma(\hat{q}-q_0)]^2\right\}, \tag{2.7}$$

where $p_0(t)=me^{\gamma t}\dot{q}_0(t)$ and $q_0(t)$ is the solution of the differential equation

$$\ddot{q}_0(t)+\gamma\dot{q}_0(t)+\omega_0^2q_0(t)=\frac{g(t)}{m}. \tag{2.8}$$

From Eq. (2.8), $q_0(t)$ can be regarded as the particular solution of Eq. (2.5),

$$q_0(t)=\frac{\alpha}{m\omega_d}\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''e^{-\gamma(t-t')/2}e^{-\alpha(t'-t'')}\times\sin[\omega_d(t-t')]f(t''), \tag{2.9}$$

where $\omega_d=(\omega_0^2-\gamma^2/4)^{1/2}$. Therefore, the influence of the past is involved in the time-dependent function $q_0(t)$.

The dynamical invariant operator can be written in terms of lowering and raising operators. Let us define these operators as

$$\hat{a}(t)=\left(\frac{1}{2\hbar m\omega_d e^{\gamma t}}\right)^{1/2}\left\{me^{\gamma t}\left(\omega_d+i\frac{\gamma}{2}\right)[\hat{q}-\lambda(t)]+i\hat{p}\right\}, \tag{2.10}$$

$$\hat{a}(t)^{\dagger}=\left(\frac{1}{2\hbar m\omega_d e^{\gamma t}}\right)^{1/2}\left\{me^{\gamma t}\left(\omega_d-i\frac{\gamma}{2}\right)[\hat{q}-\lambda(t)]-i\hat{p}\right\}, \tag{2.11}$$

where

$$\lambda(t)=\left(q_0(t)+\frac{\gamma}{2\omega_0^2}\dot{q}_0(t)\right)+i\left(1-\frac{\gamma^2}{4\omega_0^2}\right)^{1/2}\frac{\dot{q}_0(t)}{\omega_0}. \tag{2.12}$$

These operators satisfy the commutation relation such as

$$[\hat{a},\hat{a}^{\dagger}]=1. \tag{2.13}$$

From Eqs. (2.10) and (2.11), the dynamical invariant operator can be represented in terms of lowering and raising operators as

$$\hat{I}=\hbar\omega_d(\hat{a}^\dagger\hat{a}+\tfrac{1}{2}). \tag{2.14}$$

Using the eigenstates of the invariant operator, we can obtain the exact wave functions satisfying the Schrödinger equation. From Eq. (2.14), we can easily obtain the nth excited state of the invariant operator as

$$\phi_n(q,t)=\frac{1}{\sqrt{n!2^n}}\left(\frac{m\omega_d e^{\gamma t}}{\hbar\pi}\right)^{1/4}\exp\left[-\frac{me^{\gamma t}\omega_d}{2\hbar}(q-q_0)^2\right]\times H_n\left[\left(\frac{m\omega_d e^{\gamma t}}{\hbar}\right)^{1/2}(q-q_0)\right]\times\exp\left\{-\frac{i}{\hbar}\frac{me^{\gamma t}}{2}\left[\frac{\gamma}{2}(q-q_0)^2-2\dot{q}_0(q-q_0)+\frac{\gamma\dot{q}_0^2}{2\omega_0^2}\right]\right\}. \tag{2.15}$$

Because the solution of the Schrödinger equation differs by only a time-dependent phase factor from the eigenstate of the invariant operator [6], we may write the wave function $\psi_n(q,t)$ as

$$\psi_n(q,t)=\phi_n(q,t)e^{(i/\hbar)R(t)}. \tag{2.16}$$

Using Eqs. (2.3), (2.15), and (2.16) along with the Schrödinger equation, we can obtain the time-dependent phase factor $R(t)$ as

$$R(t)=\int_{t_0}^{t}dt'[L_0(t')-(n+\tfrac{1}{2})\hbar\omega_d-\Lambda(t')], \tag{2.17}$$

where

$$L_0(t)=e^{\gamma t}[\tfrac{1}{2}m\dot{q}_0^2(t)-\tfrac{1}{2}m\omega_0^2q_0^2(t)+g(t)q_0(t)] \tag{2.18}$$

and

$$\Lambda(t)=\frac{m\gamma^2e^{\gamma t}}{2\omega_0^2}\left[\dot{q}_0^2(t)-\left(\frac{\gamma}{2}-\frac{\omega_0^2}{\gamma}\right)q_0(t)\dot{q}_0(t)-\frac{\dot{q}_0(t)}{m\gamma}g(t)\right]. \tag{2.19}$$

Then the concrete form of the wave function can be obtained as

$$\psi_n(q,t)=\frac{1}{\sqrt{n!2^n}}\left(\frac{m\omega_d e^{\gamma t}}{\hbar\pi}\right)^{1/4}\exp\left\{-\frac{m\omega_d e^{\gamma t}}{2\hbar}(q-q_0)^2\right\}e^{-i(n+1/2)\omega_d(t-t')}H_n\left[\left(\frac{m\omega_d e^{\gamma t}}{\hbar}\right)^{1/2}(q-q_0)\right]\times\exp\left\{-\frac{i}{\hbar}\frac{me^{\gamma t}}{2}\left[\frac{\gamma}{2}(q-q_0)^2-2\dot{q}_0(q-q_0)+\frac{\gamma\dot{q}_0^2}{2\omega_0^2}\right]\right\}\exp\left\{\frac{i}{\hbar}\int_{t_0}^{t}dt'[L_0(t')-\Lambda(t')]\right\}. \tag{2.20}$$

An exact closed form of the propagator admits an expansion in a natural manner leading to the time-dependent wave functions $\psi_n(q,t)$ of the Schrödinger equation [17]. From Eq. (2.20) the propagator of this system is given by

$$\begin{aligned}K(q,t;q',t')&=\sum_{n=0}^{\infty}\psi_n(q,t)\psi_n^*(q',t')\\&=\left[\frac{m\omega_d(e^{\gamma t}e^{\gamma t'})^{1/2}}{2i\hbar\pi\sin\omega_d(t-t')}\right]^{1/2}\exp\left\{\frac{im}{2\hbar}\left[e^{\gamma t}\left(\omega_d\cot\omega_d(t-t')-\frac{\gamma}{2}\right)(q-q_0)^2\right.\right.\\&\quad\left.\left.+e^{\gamma t'}\left(\omega_d\cot(t-t')+\frac{\gamma}{2}\right)(q'-q_0')^2\right]\right\}\exp\left\{-\frac{im\omega_d(e^{\gamma t}e^{\gamma t'})^{1/2}}{\hbar\sin\omega_d(t-t')}(q-q_0)(q'-q_0')\right\}\\&\quad\times\exp\left\{\frac{im}{\hbar}[e^{\gamma t}\dot{q}_0(q-q_0)-e^{\gamma t'}\dot{q}_0'(q'-q_0')]\right\}\\&\quad\times\exp\left\{-\frac{i}{\hbar}\left[\frac{m\gamma}{4\omega_0^2}(e^{\gamma t}\dot{q}_0^2-e^{\gamma t'}\dot{q}_0'^2)-\int_{t'}^{t}dt''[L_0(t'')-\Lambda(t'')]\right]\right\}.\end{aligned} \tag{2.21}$$

To find the explicit form of the propagator, we make use of Mehler's formula [18].

With the use of wave functions Eq. (2.20), we can obtain the uncertainty relations for the various states as

$$(\Delta q\Delta p)_{n+2,n}=\frac{\hbar}{2}\sqrt{(n+1)(n+2)}, \tag{2.22}$$

$$(\Delta q\Delta p)_{n+1,n}=\frac{\hbar}{2}\left(n+\frac{1}{2}\right)\left\{\left[\left(\frac{8m\omega_d e^{\gamma t}}{\hbar(n+1)}\right)^{1/2}q_0(t)-\cos\omega_d t\right]^2+\sin^2\omega_d t\right\}^{1/4}$$
$$\times\left\{\left[\left(\frac{8}{m\omega_d e^{\gamma t}\hbar(n+1)}\right)^{1/2}p_0(t)+\sin\omega_d t\right]^2+\cos^2\omega_d t\right\}^{1/4}, \tag{2.23}$$

$$(\Delta q\Delta p)_{n,n}=\frac{\hbar}{2}(2n+1). \tag{2.24}$$

From Eqs. (2.22)–(2.24), we may recognize that the off-diagonal elements of the uncertainty relations $(\Delta q\Delta p)_{n\pm1,n}$ are governed by the past in terms of $q_0(t)$ and $p_0(t)$ and the relations $(\Delta q\Delta p)_{n\pm2,n}$ and $(\Delta q\Delta p)_{n,n}$ have the same form as the simple harmonic oscillator.

III. SUSCEPTIBILITY, INDEX OF REFRACTION, AND EXTINCTION COEFFICIENT

For a gas of one-electron atoms subjected to the electric field $E(t)=E_0\cos\omega t$, suppose now that we ''turn on'' an oscillating electric field at time $t=0$. From Eqs. (2.1) and (2.20), the electric dipole moment of the atom parallel to q axis is determined by the expectation value as

$$d(t)=\frac{\alpha e^2E_0}{2m\omega_d}\{C_\omega e^{i\omega t}+C_\omega^* e^{-i\omega t}+C_{\omega_d}e^{-\gamma/2+i\omega_d t}+C_{\omega_d}^* e^{-\gamma/2-i\omega_d t}-C_\alpha e^{-\alpha t}\}, \tag{3.1}$$

where

$$C_\omega=\frac{\omega_d[\alpha(\omega_0^2-\omega^2)-\omega^2\gamma]-i\omega_d\omega[\alpha\gamma+(\omega_0^2-\omega^2)]}{(\alpha^2+\omega^2)[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]},$$

$$C_{\omega_d}=\frac{\omega_d\alpha-i\left(\frac{\gamma\alpha}{2}-\alpha^2\right)}{(\alpha^2+\omega^2)(\omega_0^2-\gamma\alpha+\alpha^2)}+\frac{2\alpha\omega_d-i(2\omega^2-\gamma\alpha)}{2(\alpha^2+\omega^2)\left[\left(\frac{\gamma}{2}\right)^2+\omega^2-\omega_j^2-i\gamma\omega_d\right]},$$

$$C_\alpha=\frac{2\alpha\omega_d}{(\alpha^2+\omega^2)(\omega_0^2-\gamma\alpha+\alpha^2)}.$$

The dipole moment of a single atom must now be related to the polarization of a gas and the single-atom result is then suitably averaged to produce the analogous result for a gas of randomly oriented atoms or molecules. The electric field is assumed to be sufficiently weak that the atomic populations suffer a negligible disturbance from their thermal equilibrium. If there are N identical atoms in a volume V, the macroscopic polarization of the gas is simply

$$P(t)=\frac{\alpha e^2E_0N}{2m\omega_dV}\{C_\omega e^{i\omega t}+C_\omega^* e^{-i\omega t}+C_{\omega_d}e^{-\gamma/2+i\omega_d t}+C_{\omega_d}^* e^{-\gamma/2-i\omega_d t}-C_\alpha e^{-\alpha t}\}. \tag{3.2}$$

We assume that the electric field and the polarization can be Fourier analyzed into frequency components $E(\eta)$ and $P(\eta)$, respectively,

$$E(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\ E(\eta)e^{i\eta t}, \tag{3.3}$$

$$P(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\ P(\eta)e^{i\eta t}. \tag{3.4}$$

The Fourier coefficients $E(\eta)$ and $P(\eta)$ are determined by the inverse transform as

$$P(\eta)=\frac{\alpha e^2E_0N}{12\pi m\omega_dV}\left[C_\omega\int_0^\infty dt\ e^{i(\eta+\omega)t}+C_\omega^*\int_0^\infty dt\ e^{i(\eta-\omega)t}+C_{\omega_d}\int_0^\infty dt\ e^{-\gamma/2+i(\eta+\omega_d)t}+C_{\omega_d}^*\int_0^\infty dt\ e^{-\gamma/2-i(\eta+\omega_d)t}-C_\alpha\int_0^\infty dt\ e^{(-\alpha+i\eta)t}\right], \tag{3.5}$$

$$E(\eta)=\frac{E_0}{4\pi}\left[\int_0^\infty dt\ e^{i(\eta+\omega)t}+\int_0^\infty dt\ e^{i(\eta-\omega)t}\right] \tag{3.6}$$

and the susceptibility $\chi(\eta)$ is defined by

$$P(\eta)=\epsilon_0\chi(\eta)E(\eta). \tag{3.7}$$

From Eqs. (3.5)–(3.7), the frequency-dependent susceptibility can be given by

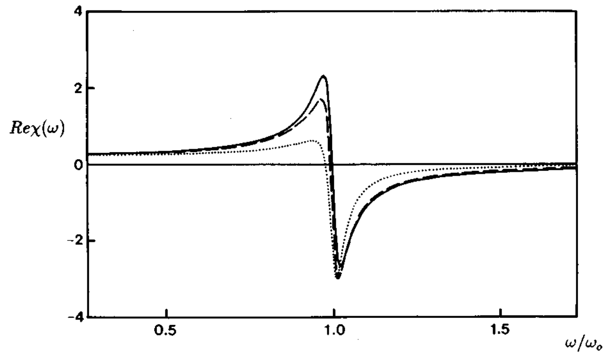

FIG. 1. Variations with frequency and parameter (α) of the real part of the susceptibility. The solid, dashed, and dotted lines correspond to the cases of α=1000, 300, and 100, respectively.

$$\chi(\omega)=\frac{e^2N}{3\epsilon_0 mV}\times\frac{\alpha^2(\omega_d^2-\omega^2)-\omega^2\gamma\alpha+i\omega[\alpha^2\gamma+\alpha(\omega_d^2-\omega^2)]}{(\alpha^2+\omega^2)[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}. \quad (3.8)$$

A real field must give rise to a real polarization, and for $P(t)$ to be real the susceptibility must satisfy the relation

$$\chi(-\omega)=\chi^*(\omega). \quad (3.9)$$

Using this expression for the susceptibility, we can also obtain the variations of the experimentally more accessible refractive index n and extinction coefficient κ (and in turn the absorption coefficient related to κ by $K=2\omega\kappa/c$):

$$n^2-\kappa^2=1+\frac{e^2N}{3\epsilon_0 mV}\frac{\alpha^2(\omega_0^2-\omega^2)-\omega^2\gamma\alpha}{(\alpha^2+\omega^2)[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}, \quad (3.10)$$

$$2n\kappa=\frac{e^2N}{3\epsilon_0 mV}\frac{\omega[\alpha^2\gamma+\alpha(\omega_0^2-\omega^2)]}{(\alpha^2+\omega^2)[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}. \quad (3.11)$$

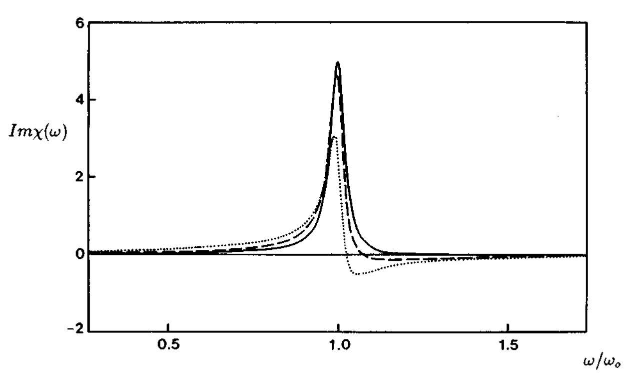

FIG. 2. Variations with frequency and parameter (α) of the imaginary part of the susceptibility. The solid, dashed, and dotted lines correspond to the cases of α=1000, 300, and 100, respectively.

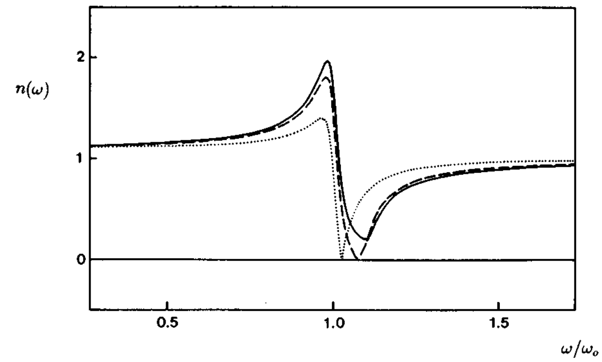

FIG. 3. Variations with frequency and parameter (α) of the refractive index n. The solid, dashed, and dotted lines correspond to the cases of α=1000, 300, and 100, respectively.

Since a fine-grained transition rate $1/\tau$ is proportional to the imaginary part of the susceptibility, using the imaginary part of the susceptibility, we can show the dependence of a fine-grained transition rate on the parameter α as

$$\frac{1}{\tau}=\frac{e^2NE_0^2}{6\hbar m}\frac{\omega[\alpha^2\gamma+\alpha(\omega_d^2-\omega^2)]}{(\alpha^2+\omega^2)[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}. \quad (3.12)$$

The susceptibility measures the response of the atoms to an external electric field, that is, it belongs to a class of functions known as response functions. Such functions have some general properties that are independent of any particular theoretical model of the system that they describe. Some important properties, known as Kramers-Kronig relations or dispersion relations, can be proved very generally for the susceptibility. We can easily show that the susceptibility obtained in this paper satisfies the Kramers-Kronig relations

$$\mathrm{Im}\chi(\omega)=-\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}d\omega'\,\frac{\mathrm{Re}\chi(\omega')}{\omega'-\omega},$$

$$\mathrm{Re}\chi(\omega)=\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}d\omega'\,\frac{\mathrm{Im}\chi(\omega')}{\omega'-\omega},$$

where P denotes the Cauchy principal-value integral.

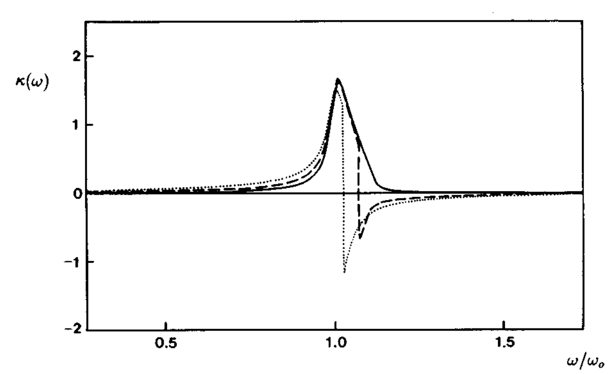

FIG. 4. Variations with frequency and parameter (α) of the extinction coefficient κ. The solid, dashed, and dotted lines correspond to the cases of α=1000, 300, and 100, respectively.

IV. RESULTS AND DISCUSSION

In this section we discuss the results of the previous sections. We have considered the damped harmonic oscillator with a time-dependent external force governed by the past. In Sec. II we have analyzed this system from a quantum-mechanical point of view and then obtained the dynamical invariant operator. To obtain the eigenstates of invariant operator, we defined different lowering and raising operators. By using these operators, the invariant operator of this system can be represented in Fock space. The wave function satisfying the Schrödinger equation differs by only a time-dependent phase factor from the eigenstate of the invariant operator and the exact closed form of the propagator admits an expansion in a natural manner leading to the time-dependent wave functions. Thus, from the eigenstates of the invariant operator, we obtained the exact wave functions and the exact closed form of the propagator. By using these wave functions, the uncertainty relations at various states were evaluated. Only the off-diagonal elements of the uncertainty relations $(\Delta q \Delta p)_{n\pm1,n}$ are influenced by the external driving force $g(t)$. In Sec. III we obtained the susceptibility, the index of refraction, the extinction coefficient, and the fine-grained transition rate for N identical atoms in a volume V.

The susceptibility $\chi(\omega)$ belongs to a class of functions known overall as the response functions, which measure the response of the atoms to a stimulus in the form of an applied electric field. Our result is the case of the time-dependent external force governed by the past. The influence of the past depends on the parameter α. If α increases towards infinity, the influence of the past can be negligible. Therefore, in the limit of α going to infinity, the susceptibility is given by

$$\lim_{\alpha\to\infty} \chi(\omega) = \frac{e^2 N}{3\epsilon_0 m V} \frac{(\omega_0^2-\omega^2)+i\omega\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}. \tag{4.1}$$

This means that the susceptibility is exactly reduced to the classical case [4]. Figures 1 and 2 illustrate the variations with frequency of the real and imaginary parts of the susceptibility for the parameter values $S=e^2N/3\epsilon_0 m V\omega_0^2=\frac{1}{4}$ and $\gamma=\omega_0/20$. The parameter S characterizes the strength of the interaction between the oscillator and the electromagnetic wave. The solid, dashed, and dotted lines correspond to the cases of $\alpha=1000$, 300, and 100, respectively. As α increases, the behaviors have the same form as the classical result [4].

The transmission of an electromagnetic wave through an atomic gas is governed by the refractive index n and extinction coefficient κ. As α goes to infinity, the refractive index n and the extinction coefficient κ are given by

$$\lim_{\alpha\to\infty} n^2-\kappa^2 = 1+\frac{e^2 N}{3\epsilon_0 m V} \frac{(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}, \tag{4.2}$$

$$\lim_{\alpha\to\infty} 2n\kappa = \frac{e^2 N}{3\epsilon_0 m V} \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}. \tag{4.3}$$

Figures 3 and 4 display the variations with frequency of the refractive index and the extinction coefficient for the parameters S and γ. The solid, dashed, and dotted lines correspond to the cases of $\alpha=1000$, 300, and 100, respectively. As α goes to infinity, the behaviors have the same form as the classical result. The quantity of experimental interest is often the absorption coefficient $K=2\omega\kappa/c$. These quantities are functions of the frequency ω in the vicinity of an atomic transition frequency.

In addition, we represented the dependence of the fine-grained transition rate on the parameter α. In the same manner, as α goes to infinity, the fine-grained transition rate is given by

$$\lim_{\alpha\to\infty} \frac{1}{\tau} = \frac{e^2 N E_0^2}{6\hbar m} \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}. \tag{4.4}$$

Some important properties, known as Kramers-Kronig relations, can be derived for the susceptibility. We can easily show that our results satisfy these relations. The Kramers-Kronig relations show that the real and imaginary parts of the susceptibility are very intimately connected. Indeed, a knowledge of one part at all positive frequencies provides a complete knowledge of the other part at all frequencies. This is often useful in experiments where it may be easier to measure one part of $\chi(\omega)$ than the other.

The results obtained in this paper will be useful in other studies such as quantum optics and atomic and molecular physics. In future works we shall more directly evaluate the exact solutions (invariant quantity, exact wave functions, propagator, minimum-uncertainty states, susceptibility, refractive index, and extinction coefficient) for the damped harmonic oscillator in the presence of an electric and a magnetic field.

ACKNOWLEDGMENTS

This work was supported by the Basic Science Research Institute Program, Ministry of Education, Republic of Korea, and by KOSEF through the Center for Theoretical Physics, Seoul National University.

[1] Y. M. Hua, C. Hu, and G. P. Li, Appl. Phys. Lett. **64**, 1277 (1994).
[2] P. W. Fenimore and M. B. Weissman, J. Appl. Phys. **76**, 6192 (1994).
[3] B. R. Johnson, J. O. Hirschfelder, and K. H. Yang, Rev. Mod. Phys. **55**, 109 (1983).
[4] R. London, *The Quantum Theory of Light* (Oxford University Press, London, 1973), Chap. 4.
[5] H. R. Lewis, Jr., J. Math. Phys. **9**, 1976 (1968).
[6] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[7] H. J. Korsch, Phys. Lett. **74A**, 294 (1979).

[8] H. Kohl and R. M. Dreizler, Phys. Lett. **98A**, 95 (1983).

[9] D. C. Khandekar and S. V. Lawande, J. Math. Phys. **20**, 1870 (1979).

[10] A. K. Dhara and S. V. Lawande, Phys. Rev. A **30**, 560 (1984).

[11] H. R. Lewis, Jr., Phys. Rev. **172**, 1313 (1968).

[12] C. C. Gerry, Phys. Lett. **109A**, 149 (1985).

[13] C. I. Um, K. H. Yeon, and W. H. Kahng, J. Korean Phys. Soc. **19**, 1 (1986); **19**, 8 (1986); J. Phys. A **20**, 611 (1987); K. H. Yeon, C. I. Um, and T. F. George, Phys. Rev. A **36**, 5287 (1987).

[14] K. H. Yeon, C. I. Um, W. H. Kahng, and T. F. George, Phys. Rev. A **38**, 6224 (1988); K. H. Yeon, C. I. Um, and W. H. Kahng, J. Korean Phys. Soc. **23**, 82 (1990); K. H. Yeon, C. I. Um, T. F. George, V. V. Dodonov, and O. V. Manko, J. Sov. Laser Res. **12**, 385 (1991); **13**, 219 (1992).

[15] K. H. Yeon, K. K. Lee, C. I. Um, T. F. George, and L. N. Pandey, Phys. Rev. A **48**, 2716 (1993).

[16] K. H. Yeon, C. I. Um, and T. F. George, in *Proceedings of the Third International Workshop on Squeezed States and Uncertainty Relations*, edited by D. Han, Y. S. Kim, N. H. Rubin, Y. Shih, and W. W. Zachary, NASA Conf. Publ. No. 3270, 1994, p. 381.

[17] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), Chap. 3.; D. C. Khandekar, S. V. Lawande, and K. V. Bhagwat, *Path-Integral Methods and Their Applications* (World Science, Singapore, 1993).

[18] A. Erdelyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 2, p. 194.