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Stochastic path integrals and open quantum systems

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The general theory of path integral propagators for the solution of linear quantum state diffusion (LQSD) stochastic Schrödinger equations describing open quantum systems is developed. Both Hamiltonian and, where possible, Lagrangian path integrals are derived and their connection established. The Hamiltonian version turns out to be more suitable. The results also show how the stochastic terms in the LQSD equation introduce a weight functional under the path integral, thus restricting the set of contributing paths. The center of this weight functional is determined by the stochastic processes governing the LQSD equation. In general, this picture holds in a semiclassical limit only. Some peculiarities of stochastic path integrals are pointed out. We evaluate the stochastic path integral in closed form for soluble models, gaining further insight into the behavior of the solutions of the LQSD equation. [S1050-2947(96)10009-3]

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I. INTRODUCTION

The theoretical description of open quantum systems with the help of path integrals goes back to Feynman and Vernon [1,2]. They use the path integral propagator of the unitary time evolution of the system plus environment and trace over the irrelevant (environmental) degrees of freedom to arrive at a path integral propagator for the reduced density operator alone. This approach was further elaborated by Caldeira and Leggett [4] and others (see Grabert *et al.* [5] for more references). This approach, however, is *not* the subject of this paper.

Instead, we base the description of open quantum systems on *stochastic Schrödinger equations* as they have been introduced in a variety of circumstances during the past few years [6–28], both in linear and nonlinear versions. This approach is very much in the spirit of describing the dissipative motion of classical particles undergoing Brownian motion with the help of stochastic differential equations as introduced by Langevin [29]. Such stochastic Schrödinger equations (mainly in their nonlinear version) were used in recent years to describe the continuous measurement or continuous reduction of quantum systems by Gisin [6], Diosi [7–9], Belavkin *et al.* [10–13], Barchielli *et al.* [14,15], and Gisin and Percival [16–18]. They also appear as attempts to find an explicit description of a fundamental wave function reduction in modified quantum theories as investigated by Ghirardi *et al.* [19], Pearle *et al.* [20,21], or Percival [22,23] with the aim to overcome the difficulties with the foundations of quantum theory. In this context we also mention the recently established connection between stochastic Schrödinger equations and the decoherent histories approach to quantum mechanics by Diosi *et al.* [24]. As a computational tool for solving master equations stochastic Schrödinger equations are used extensively in quantum optics. Here, papers about the nonlinear version are numerous, see Gardiner [25] or Carmicheal [26] for more references. In Goetsch *et al.* [27,28] one can find recent investigations where *linear* stochastic Schrödinger equations, relevant for this paper, are used.

One of the most far-reaching concepts in quantum theory are path integrals as introduced by Feynman [1–3] (see also [32]), with its implications for relativistic quantum mechanics and field theory. Therefore, it seems desirable to introduce these concepts in the now well established field of stochastic Schrödinger equations. The aim of this paper is to provide general path integral expressions for their propagators. Moreover, we evaluate the path integral for soluble models to gain further insight into the behavior of the solutions of these stochastic Hilbert space differential equations.

Path integral expressions for the propagator of linear stochastic Schrödinger equations can be found immediately for simple special cases where straightforward general path integral techniques can be applied. This was investigated by Pearle and Soucek [20] for the special case of a purely position dependent diffusion term in the context of a continuous spontaneous localization theory.

In this paper, we develop the general path integral theory. This is why we devote the next section to an introduction to general linear stochastic Schrödinger equations with complex noise. We will refer to this equation (7) as the linear quantum state diffusion (LQSD) equation. We quote some of its properties, like the time dependence of the norm of the state vector and the connection to the time evolution of the corresponding density operator.

The main results of this paper appear in Secs. III–V, where we derive general path integral expressions for the LQSD propagator. Our approach starts with the derivation of a stochastic Hamiltonian path integral. Solving the LQSD equation amounts to introducing a weight functional of phase space paths under the path integral.

Since coordinate space path integrals are more common, we evaluate the momentum part of the phase space path integral to find the Lagrangian version of the stochastic path integral propagator in Sec. IV. This is possible only for the special—but very important—class of (effective) Hamiltonians with at most quadratic dependence on the momenta. Here, some peculiarities of the stochastic theory occur, such as indispensable stochastic prefactors of the propagator. The relation between Lagrangian and Hamiltonian description is given by a stochastic *Itô-Legendre transformation*.

Although the spirit in this paper is to derive the path integral propagator from the LQSD equation, it is instructive to repeat Feynman's original reversed approach [2,3] and establish the LQSD equation from a short-time stochastic propagator. This is accomplished in Sec. V which clarifies the connection between the different approaches.

In ordinary quantum mechanics the semiclassical propagator is exact for at most quadratic Hamiltonians. Likewise, we are able to give the analytical solution for a harmonic oscillator with linear environment operators in Sec. VI. Some further analytical results of this soluble model are discussed in Sec. VII, where we investigate the time evolution of generalized coherent states. We close with a discussion of the results and draw conclusions, pointing out further developments.

II. LINEAR QUANTUM STATE DIFFUSION (LQSD)

Linear stochastic Schrödinger equations have been introduced from many different points of view [10–15,19–21,27,28] in recent years. The aim of this section is to present the relevant terms and concepts.

The idea of describing open quantum systems with the help of stochastic Schrödinger equations is borrowed from the similar description of the Brownian motion of classical particles in a fluid with the help of stochastic differential equations as introduced by Langevin [29]. Instead of a Hamiltonian time evolution of the state vector, we consider a general Brownian-motion-like linear Itô-diffusion process

$$|d\psi\rangle=-\frac{i}{\hbar}\hat{H}_{\rm eff}|\psi\rangle dt+\sum_\mu \hat{E}_\mu|\psi\rangle d\xi_\mu \tag{1}$$

of state vectors in Hilbert space. The operator $\hat{H}_{\rm eff}$ determines the *drift* as in ordinary quantum mechanics, while the operators $\hat{E}_\mu$ determine the *diffusion* of the state vector. As in the classical Brownian motion, this diffusion is normally assigned to the random influence of an unspecified environment (*noise*), which is why we refer to the operators $\hat{E}_\mu$ as the *environment operators* throughout this paper. In this context, a measurement apparatus is just a special environment.

In the fundamental Itô stochastic equation (1) the $d\xi_\mu$s are the stochastic increments of independent complex Wiener processes $\xi_\mu(t)$ (*complex white noise*) [30] with standard properties

$$\mathcal{M}[d\xi_\mu]=0,\quad \mathcal{M}[d\xi_\mu d\xi_\nu]=0,\quad \mathcal{M}[d\xi_\mu d\xi_\nu^*]=\delta_{\mu\nu}dt. \tag{2}$$

We use the notation $\mathcal{M}[X_\xi]$ for the *ensemble average* of a stochastic variable X_ξ over the processes ξ_μ.

It is not necessary to introduce complex increments $d\xi_\mu$ in Eq. (1). Versions of linear stochastic Schrödinger equations with real noise are also common, but we will give two good reasons for the use of complex noise shortly.

The density operator is identified with the ensemble average over the one-dimensional projectors,

$$\rho_t=\mathcal{M}[|\psi_\xi(t)\rangle\langle\psi_\xi(t)|]. \tag{3}$$

To be physically sensible, we require it to be trace preserving for all times,

$$d\mathrm{Tr}\rho_t=0. \tag{4}$$

This condition puts a restriction on the anti-Hermitian part of the drift operator $\hat{H}_{\rm eff}$ in Eq. (1). From Eq. (4) we find

$$\frac{1}{2i}(\hat{H}_{\rm eff}-\hat{H}_{\rm eff}^\dagger)=-\frac{\hbar}{2}\sum_\mu \hat{E}_\mu^\dagger\hat{E}_\mu \tag{5}$$

which implies that

$$\hat{H}_{\rm eff}=\hat{H}-\frac{i\hbar}{2}\sum_\mu \hat{E}_\mu^\dagger\hat{E}_\mu \tag{6}$$

with an unspecified Hermitian Hamiltonian $\hat{H}$.

The general linear stochastic Schrödinger equation (1) together with the trace condition (4) determines what we will call the linear quantum state diffusion (LQSD) equation

$$|d\psi\rangle=-\frac{i}{\hbar}\hat{H}|\psi\rangle dt-\frac{1}{2}\sum_\mu \hat{E}_\mu^\dagger\hat{E}_\mu|\psi\rangle dt+\sum_\mu \hat{E}_\mu|\psi\rangle d\xi_\mu. \tag{7}$$

Using the relations (2) among the Wiener increments, it is straightforward to show that the density operator (3) evolves according to the master equation

$$\dot{\rho}=-\frac{i}{\hbar}[\hat{H},\rho]+\frac{1}{2}\sum_\mu ([\hat{E}_\mu\rho,\hat{E}_\mu^\dagger]+[\hat{E}_\mu,\rho\hat{E}_\mu^\dagger]). \tag{8}$$

It has in fact been shown by Lindblad [31] that every sensible (Markovian) master equation has to be of this form. The stochastic decomposition (3) of the density operator in one-dimensional projectors is referred to as an *unraveling* of the time evolution of the density operator and starting point for quantum Monte Carlo methods.

The nice feature of Eq. (7) being a *linear* equation is overshadowed by the fact that solutions of the LQSD equation are not normalized. We find the time dependence of the norm from Eq. (7) and Itô calculus,

$$d[\langle\psi|\psi\rangle]=2\sum_\mu \mathrm{Re}\{\langle\psi|\hat{E}_\mu|\psi\rangle d\xi_\mu\}. \tag{9}$$

It follows that ensemble averages have to be taken over the *unnormalized* one-dimensional projectors $|\psi_t\rangle\langle\psi_t|$. Alternatively, the normalized projectors have to be weighted by the norm $\langle\psi_t|\psi_t\rangle$ to give the correct ensemble mean $\rho(t)$. These issues are discussed in [10,11,14,15,19–21,27] and will not be further addressed here.

We promised to give two good reasons for the use of complex noise in the LQSD equation. The first is purely practical. Since second order terms $(d\xi_\mu)^2$ can be neglected, complex Itô calculus is ordinary calculus as long as no complex conjugation is involved. This simplifies matters considerably. The second reason is related to the invariance of the master equation (8) under unitary transformations in the linear space of the environment operators $\hat{E}_\mu$. This symmetry can be preserved for the LQSD equation for complex noise only [16].

For completeness only, we mention the *nonlinear* QSD equation [16–18]

$$|d\phi\rangle = -\frac{i}{\hbar}\hat{H}|\phi\rangle dt - \frac{1}{2}\sum_\mu (\hat{E}_\mu^\dagger \hat{E}_\mu - 2e_\mu^* \hat{E}_\mu + |e_\mu|^2)|\phi\rangle dt + \sum_\mu (\hat{E}_\mu - e_\mu)|\phi\rangle d\xi_\mu, \tag{10}$$

which differs from the LQSD equation (7) by additional nonlinear terms involving the expectation values $e_\mu = \langle \phi|\hat{E}_\mu|\phi\rangle$. This nonlinear stochastic Schrödinger equation preserves the norm for individual realizations,

$$d\langle \phi_t|\phi_t\rangle = 0, \tag{11}$$

and is also an unraveling of the general master equation (8),

$$\rho_t = \mathcal{M}[|\phi_t\rangle\langle\phi_t|]. \tag{12}$$

Since it is nonlinear, it is difficult to find analytical solutions and it is primarily used for numerical purposes.

III. HAMILTONIAN PATH INTEGRAL PROPAGATOR FOR LQSD

In this section we derive a general *phase space* path integral expression for the stochastic propagator $G_\xi(t,t_0)$ of LQSD (7), defined by

$$|\psi_t\rangle = G_\xi(t;t_0)|\psi_{t_0}\rangle. \tag{13}$$

We stress the fluctuation dependence of the propagator by the subscript ξ. In what follows, we set $t_0=0$ thus determining $G_\xi(t;0)$.

For a given time t and independent stochastic processes $\xi_\mu(t)$ we divide the interval $[0,t]$ in N intervals of length $\Delta t = t/N$ and define N random increments $\Delta\xi_\mu(k) = \xi_\mu(k\Delta t) - \xi_\mu[(k-1)\Delta t]$, where k runs from 1 to N. From Eq. (7) with the notation (6) we deduce

$$G_\xi(\mathbf{q},t;\mathbf{q}_0,0) = \lim_{N\to\infty} \langle \mathbf{q}|\prod_{k=1}^N \left\{1 - \frac{i}{\hbar}\hat{H}_{\text{eff}}\Delta t + \sum_\mu \hat{E}_\mu \Delta\xi_\mu(k)\right\}|\mathbf{q}_0\rangle. \tag{14}$$

The first step to a path integral expression is the inclusion of $N-1$ identities of the form $\int d\mathbf{q}_k|\mathbf{q}_k\rangle\langle\mathbf{q}_k|$ to obtain

$$G_\xi(\mathbf{q},t;\mathbf{q}_0,0) = \lim_{N\to\infty}\int d\mathbf{q}_1 \cdots \int d\mathbf{q}_{N-1} \times \prod_{k=1}^N \langle \mathbf{q}_k|1 - \frac{i}{\hbar}\hat{H}_{\text{eff}}\Delta t + \sum_\mu \hat{E}_\mu \Delta\xi_\mu(k)|\mathbf{q}_{k-1}\rangle, \tag{15}$$

where we set $\mathbf{q}_N = \mathbf{q}$.

Now we introduce the Wigner transform $O(\mathbf{q},\mathbf{p})$ of the operator $\hat{O}$ [33] in $2d$-dimensional phase space,

$$O(\mathbf{q},\mathbf{p}) = 2^d(2\pi\hbar)^{(d/2)}\int d\mathbf{q}'\langle \mathbf{q}-\mathbf{q}'|\hat{O}|\mathbf{q}+\mathbf{q}'\rangle\langle\mathbf{q}'|2\mathbf{p}\rangle. \tag{16}$$

The plane waves $\langle\mathbf{q}|\mathbf{p}\rangle$ in Eq. (16) are $\langle\mathbf{q}|\mathbf{p}\rangle = (2\pi\hbar)^{-(d/2)}\exp\{i\mathbf{q}\cdot\mathbf{p}/\hbar\}$.

We use the general inverse transformation

$$\langle\mathbf{q}|\hat{O}|\mathbf{q}'\rangle = (2\pi\hbar)^{-(d/2)}\int d\mathbf{p}\, O\left(\frac{\mathbf{q}+\mathbf{q}'}{2},\mathbf{p}\right)\langle\mathbf{q}-\mathbf{q}'|\mathbf{p}\rangle \tag{17}$$

to replace the matrix elements in Eq. (15) by expression (17), thus introducing N additional momentum integrals, giving

$$G_\xi(\mathbf{q},t;\mathbf{q}_0,0) = \lim_{N\to\infty}(2\pi\hbar)^{-(Nd/2)}\int d\mathbf{p}_1\int d\mathbf{q}_1\int d\mathbf{p}_2\cdots \times\int d\mathbf{q}_{N-1}\int d\mathbf{p}_N \prod_{k=1}^N \langle\mathbf{q}_k-\mathbf{q}_{k-1}|\mathbf{p}_k\rangle \times\left[1-\frac{i}{\hbar}H_{\text{eff}}\left(\frac{\mathbf{q}_k+\mathbf{q}_{k-1}}{2},\mathbf{p}_k\right)\Delta t + \sum_\mu E_\mu\left(\frac{\mathbf{q}_k+\mathbf{q}_{k-1}}{2},\mathbf{p}_k\right)\Delta\xi_\mu(k)\right], \tag{18}$$

the operators $\hat{H}_{\text{eff}}$ and $\hat{E}_\mu$ now being replaced by their Wigner transforms.

Just as for ordinary path integrals, in the limit $N\to\infty$, the $[1-\cdots]$ term may be replaced by an exponential $\exp\{-[\cdots]\}$. Notice that there is no additional term arising from the fluctuations since for complex noise second order terms $(d\xi_\mu)^2$ vanish according to the relations (2). If one chooses real noise, an additional term of order dt would appear in the exponent. Our final expression for the propagator is therefore

$$G_\xi(\mathbf{q},t;\mathbf{q}_0,0) = \lim_{N\to\infty}(2\pi\hbar)^{-Nd}\int d\mathbf{p}_1\int d\mathbf{q}_1\int d\mathbf{p}_2\cdots \times\int d\mathbf{q}_{N-1}\int d\mathbf{p}_N \exp\left\{\frac{i}{\hbar}\sum_{k=1}^N\left[(\mathbf{q}_k-\mathbf{q}_{k-1})\cdot\mathbf{p}_k - H_{\text{eff}}\left(\frac{\mathbf{q}_k+\mathbf{q}_{k-1}}{2},\mathbf{p}_k\right)\Delta t - i\hbar\sum_\mu E_\mu\left(\frac{\mathbf{q}_k+\mathbf{q}_{k-1}}{2},\mathbf{p}_k\right)\Delta\xi_\mu(k)\right]\right\}. \tag{19}$$

The argument in the exponent is the discrete version of a generalized Hamiltonian action integral. Here, however, in addition to the deterministic part, we find an Itô stochastic integral [30]. Notice that the use of the Wigner transform of the relevant operators through Eq. (17) leads to the well-known midpoint rule $(\mathbf{q}_k+\mathbf{q}_{k-1})/2$ [2,32] for the evaluation of the action integral.

As usual, we write thisx expression in path integral form,

$$G_{\xi}(\mathbf{q},t;\mathbf{q}_0,0)=\int_{(\mathbf{q}_0,0)}^{(\mathbf{q},t)}\mathcal{D}[\mathbf{q},\mathbf{p}]\exp\left\{\frac{i}{\hbar}S_{\xi}[\mathbf{q},\mathbf{p}]\right\}, \quad (20)$$

with the complex Itô stochastic phase space action functional

$$S_{\xi}[\mathbf{q},\mathbf{p}]=\int_0^t d\tau[\dot{\mathbf{q}}_{\tau}\cdot\mathbf{p}_{\tau}-H_{\rm eff}(\mathbf{q}_{\tau},\mathbf{p}_{\tau})] - i\hbar\sum_{\mu}\int_0^t d\xi_{\mu}(\tau)E_{\mu}(\mathbf{q}_{\tau},\mathbf{p}_{\tau}). \quad (21)$$

We see that the diffusion part $\Sigma_{\mu}\hat{E}_{\mu}|\psi\rangle d\xi_{\mu}$ of the LQSD equation is reflected as an Itô stochastic integral over the *environment functions* $E_{\mu}(\mathbf{q},\mathbf{p})$ as part of the action integral. The deterministic part, responsible for the drift, includes the Wigner transforms of the operators $\hat{E}_{\mu}^{\dagger}\hat{E}_{\mu}$ as the imaginary part of $H_{\rm eff}(\mathbf{q},\mathbf{p})$, Eq. (6). This implies that the action in Eq. (20) is complex, assigning different weights to the contribution of paths.

Separating the action (21) into its real and imaginary parts, the propagator becomes

$$G_{\xi}(\mathbf{q},t;\mathbf{q}_0,0)=\int_{(\mathbf{q}_0,0)}^{(\mathbf{q},t)}\mathcal{D}[\mathbf{q},\mathbf{p}] \times\exp\left\{\frac{i}{\hbar}(S_{\rm cl}[\mathbf{q},\mathbf{p}]+\hbar\Phi_{\xi}[\mathbf{q},\mathbf{p}])\right\} \times\exp\left\{-\frac{1}{2}D_{\xi}[\mathbf{q},\mathbf{p}]\right\}, \quad (22)$$

with the classical action functional

$$S_{\rm cl}[\mathbf{q},\mathbf{p}]=\int_0^t d\tau\{\dot{\mathbf{q}}_{\tau}\cdot\mathbf{p}_{\tau}-H(\mathbf{q}_{\tau},\mathbf{p}_{\tau})\} \quad (23)$$

and a stochastic phase functional

$$\Phi_{\xi}[\mathbf{q},\mathbf{p}]=\sum_{\mu}\ \mathrm{Im}\left\{\int_0^t d\xi_{\mu}(\tau)E_{\mu}(\mathbf{q}_{\tau},\mathbf{p}_{\tau})\right\}. \quad (24)$$

The main effect of the environmental terms in the LQSD equation is the appearance of a weight functional $\exp\{-\frac{1}{2}D_{\xi}[\mathbf{q},\mathbf{p}]\}$ under the path integral. If we denote the Wigner transform of $\hat{E}_{\mu}^{\dagger}\hat{E}_{\mu}$ by $|\Xi_{\mu}(\mathbf{q},\mathbf{p})|^2$ we find

$$\begin{aligned}D_{\xi}[\mathbf{q},\mathbf{p}]&=\sum_{\mu}\int_0^t d\tau|\Xi_{\mu}(\mathbf{q}_{\tau},\mathbf{p}_{\tau})|^2 -2\sum_{\mu}\ \mathrm{Re}\left\{\int_0^t d\xi_{\mu}(\tau)E_{\mu}(\mathbf{q}_{\tau},\mathbf{p}_{\tau})\right\}\\ &=\sum_{\mu}\int_0^t d\tau\left|E_{\mu}(\mathbf{q}_{\tau},\mathbf{p}_{\tau})-\frac{d\xi_{\mu}^{*}(\tau)}{d\tau}\right|^2-\left|\frac{d\xi_{\mu}(\tau)}{d\tau}\right|^2 +O(\hbar). \end{aligned}\quad (25)$$

For the second expression we used the fact that the Wigner transform of $\hat{E}_{\mu}^{\dagger}\hat{E}_{\mu}$ is equal to the squared modulus of the Wigner transform of $\hat{E}_{\mu}$ to lowest order in $\hbar$,

$$|\Xi_{\mu}(\mathbf{q}_{\tau},\mathbf{p}_{\tau})|^2=|E_{\mu}(\mathbf{q}_{\tau},\mathbf{p}_{\tau})|^2+O(\hbar). \quad (26)$$

Notice that the $O(\hbar)$ term is a mere constant for the important cases of linear or Hermitian quadratic environment operators, but can in general only be neglected in the semiclassical limit ($\hbar\rightarrow 0$).

The last line in Eq. (25) should be regarded as a formal expression since the integral over the squared derivative of the fluctuations itself does not exist. Still, it shows that the weight functional (25) is peaked around Brownian-motion-like phase space paths with

$$E_{\mu}(\mathbf{q}_{\tau},\mathbf{p}_{\tau})dt\approx d\xi_{\mu}(\tau). \quad (27)$$

We see that the stochastic processes $\xi_{\mu}(t)$ determine the center of this Gaussian-type weight functional (25). Moreover, it is apparent that paths with the same values of the environment functions $E_{\mu}(\mathbf{q}_{\tau},\mathbf{p}_{\tau})$ contribute equally to the path integral.

Equation (19) and its formal version (20) with Eq. (21) and the more explicit expression (22) for the LQSD propagator are the main results of this section. They serve as a starting point for the application of further methods to solve the LQSD equation, like semiclassical methods or perturbation theory. Next, however, we want to proceed to the Lagrangian version of the path integral, which, as will be seen shortly, shows some unexpected features.

IV. LAGRANGIAN PATH INTEGRAL PROPAGATOR FOR LQSD

In this section we restrict ourselves to a one-degree-of-freedom system, in order to keep the expressions simple. Historically, [3] Lagrangian path integrals were introduced as a direct route to quantization given the classical Lagrangian description of the theory. The Lagrangian version of the path integral follows from the Hamiltonian version by integrating out the momentum path integral.

For Hamiltonians that depend at most quadratically on the momenta, the Legendre transformation between coordinate space and phase space corresponds to a linear relation between the canonical momenta and the velocities. In terms of path integrals, this linear Legendre transformation between Hamiltonian and Lagrangian description is simply performed by doing the Gaussian momentum integrals of the phase space path integrals in Eq. (19). Thus, in this section we assume a standard Hamiltonian of the form

$$\hat{H}=\frac{\hat{p}^2}{2m}+V(\hat{q}). \quad (28)$$

In order to get at most quadratic momentum contributions from the environment operators $\hat{E}_{\mu}$, we have to assume their linear dependence on the momenta. We want to keep things simple which is why, in this section, we also assume a linear dependence on the coordinates,

$$\hat{E}_{\mu}=\frac{\beta_{\mu}\hat{q}+\gamma_{\mu}\hat{p}}{\sqrt{\hbar}}. \quad (29)$$

This restriction is not necessary and will be dropped in the next section, where we investigate the more general form $\hat{E}_\mu=[\beta_\mu(\hat{q})+\gamma_\mu\hat{p}]/\sqrt{\hbar}$ with an arbitrary function $\beta_\mu(q)$.

In Eq. (29), the β_μ's and γ_μ's are complex constants and the overall division by $\sqrt{\hbar}$ is introduced for convenience. Keep in mind, however, that most investigations on open quantum systems are covered by such linear environment operators, as in quantum optics (homodyne and heterodyne measurement [27,28]) or in the high temperature and Markovian limit of the quantum Brownian motion model of Caldeira and Leggett [4,35,36]. In order to shorten notation, we set

$$\sum_\mu |\beta_\mu|^2=|\beta|^2, \quad \sum_\mu \beta_\mu\gamma_\mu^*=\overline{\omega}-i\Gamma,$$

and (30)

$$\sum_\mu |\gamma_\mu|^2=|\gamma|^2,$$

with real $\overline{\omega}$ and Γ.

From Eqs. (28) and (29) we deduce that the Wigner transform of $\hat{H}_{\rm eff}$ is

$$H_{\rm eff}(q,p)=\frac{(p-iM\overline{\omega}q)^2}{2M}+V_{\rm eff}(q), \tag{31}$$

where we used the following abbreviations for effective, complex mass M and potential $V_{\rm eff}$:

$$M=\frac{m}{1-im|\gamma|^2}, \tag{32}$$

$$V_{\rm eff}(q)=V(q)+\frac{1}{2}\{M\overline{\omega}^2-i|\beta|^2\}q^2+\frac{i\hbar}{2}\Gamma. \tag{33}$$

We are now in the position to derive the Lagrangian version of the path integral by doing the multiple Gaussian momentum integrals in Eq. (19). The value of such an integral is given by the value of the integrand (the discrete action integral) at its stationary point times the typical square root prefactor. The stationarity condition determines the Legendre relation between classical velocity and canonical momentum. Applied to Eq. (19) with Eq. (31) we find the complex *Itô Legendre transformation*

$$dq=\frac{p-iM\overline{\omega}q}{M}dt+i\sqrt{\hbar}\sum_\mu \gamma_\mu d\xi_\mu, \tag{34}$$

which replaces the classical Legendre relation between canonical momentum and velocity.

We now evaluate the Gaussian momentum integrals in Eq. (19), which leads to the final expression for the propagator in Lagrangian language,

$$\begin{aligned} G_\xi(q,t;q_0,0)=&\lim_{N\to\infty}\left(\frac{M}{2\pi i\hbar\Delta t}\right)^{N/2} e^{\{-(i/2)M\sum_{k=1}^N\sum_{\mu\nu}\gamma_\mu\gamma_\nu\Delta\xi_\mu(k)\Delta\xi_\nu(k)/\Delta t\}}\int dq_1\cdots\int dq_{N-1}\\ &\times\exp\Bigg(\frac{i}{\hbar}\sum_{k=1}^N\left[\frac{1}{2}M\frac{(q_k-q_{k-1})^2}{\Delta t}+iM\overline{\omega}(q_k-q_{k-1})(q_k+q_{k-1})/2-V_{\rm eff}\left(\frac{q_k+q_{k-1}}{2}\right)\Delta t\right]\\ &+\frac{1}{\sqrt{\hbar}}\sum_{k=1}^N\sum_\mu\left[\left\{M\gamma_\mu\frac{q_k-q_{k-1}}{\Delta t}+(\beta_\mu+iM\overline{\omega}\gamma_\mu)\frac{q_k+q_{k-1}}{2}\right\}\Delta\xi_\mu(k)\right]\Bigg). \end{aligned} \tag{35}$$

In more formal terms the propagator is given by the stochastic Lagrangian path integral

$$G_\xi(q,t;q_0,0)=\exp\left\{-\frac{iM}{2}\sum_{\mu\nu}\gamma_\mu\gamma_\nu\int_0^t d\xi_\mu(\tau)\frac{d\xi_\nu(\tau)}{d\tau}\right\}\int_{(q_0,0)}^{(q,t)}\mathcal{D}[q]\exp\left\{\frac{i}{\hbar}S_\xi[q,\dot{q}]\right\}. \tag{36}$$

The stochastic Lagrangian action functional is

$$S_\xi[q,\dot{q}]=\int_0^t d\tau\mathcal{L}_{\rm eff}(q_\tau,\dot{q}_\tau)-i\sqrt{\hbar}\sum_\mu\int_0^t d\xi_\mu(\tau)\mathcal{E}_\mu(q_\tau,\dot{q}_\tau) \tag{37}$$

with the complex *effective Lagrangian*

$$\mathcal{L}_{\rm eff}(q,\dot{q})=\frac{1}{2}M\dot{q}^2+iM\overline{\omega}\dot{q}q-V_{\rm eff}(q) \tag{38}$$

and the *Lagrangian environment functions*

$$\mathcal{E}_\mu(q,\dot{q})=M\gamma_\mu\dot{q}+(\beta_\mu+iM\overline{\omega}\gamma_\mu)q. \tag{39}$$

We will now address the prefactor of the Lagrangian path integral in Eqs. (35) and (36). The singular prefactor $\sqrt{M/2\pi i\hbar\Delta t}$ is hidden in the notation $\int\mathcal{D}[q]$ of the path integral. It is unimportant when working with the formal path integral because in applications like Feynman' s perturbation theory this term drops out. However, in our case there is the additional path-independent but *stochastic* prefactor

$$\exp\left\{-\frac{iM}{2}\sum_{\mu\nu}\gamma_\mu\gamma_\nu\int_0^t d\xi_\mu(\tau)\frac{d\xi_\nu(\tau)}{d\tau}\right\} \tag{40}$$

whenever the environment operators are momentum dependent. Since the ensemble average relies on the unnormalized single realizations, it is essential to keep this prefactor. This complication is a hint that the Lagrangian version of the stochastic path integral theory complicates matters unnecessarily compared to the Hamiltonian version. More about the meaning of this prefactor becomes apparent in the following section.

As we can see from Eqs. (38) and (39), the functions $\mathcal{L}_{\rm eff}(q,\dot{q})$ and $\mathcal{E}_\mu(q,\dot{q})$ are equal to their corresponding phase space expressions with the deterministic replacement $p \to M(\dot{q}+i\overline{\omega q})$.

$$\mathcal{L}_{\rm eff}(q,\dot{q}) = M\dot{q}(\dot{q}+i\overline{\omega q}) - H_{\rm eff}(q,M(\dot{q}+i\overline{\omega q})) \quad (41)$$

and

$$\mathcal{E}_\mu(q,\dot{q}) = \sqrt{\hbar}E_\mu(q,M(\dot{q}+i\overline{\omega})q). \quad (42)$$

Again, we see how the Lagrangian environment functions $\mathcal{E}_\mu$ play the role of the environment operators in the LQSD equation in determining the diffusion, while the complex Lagrangian includes both the dynamics of the isolated system in its real part, and contributions to the drift arising from the trace condition (4) in its imaginary part.

As in the Hamiltonian case it is worth separating the stochastic action functional (36) into its real and imaginary part in order to clarify the meaning of expression (36). From Eqs. (38) and (39) we find

$$\mathrm{Im}\{\mathcal{L}_{\rm eff}(q,\dot{q})\} = \frac{1}{2}\sum_\mu |\mathcal{E}_\mu(q,\dot{q})|^2 + \frac{\Gamma}{2}\left\{ \mathrm{Im}\{M\}\frac{d}{dt}(q^2) + 2\overline{\omega}\mathrm{Re}\{M\}q^2 - \hbar \right\}. \quad (43)$$

We see that in the Lagrangian case the imaginary part of the Lagrangian $\mathcal{L}_{\rm eff}$ is the sum of the squared modulus of the corresponding Lagrangian environment functions $\mathcal{E}_\mu$ up to additional, path-dependent terms including a total time derivative. This is in contrast to the Hamiltonian case, where the imaginary part of $H_{\rm eff}$ is actually equal to the squared modulus of the Hamiltonian environment functions E_μ up to a constant [Eqs. (6) and (26), for linear $\hat{E}_\mu$].

Taking all these results together, we can express the propagator for the LQSD equation with standard Hamiltonian (28) and linear environment operators (29) as

$$\begin{aligned} G_\xi(q,t;q_0,0) = {}& \exp\left\{ -\frac{iM}{2}\sum_{\mu\nu} \gamma_\mu\gamma_\nu \int_0^t d\xi_\mu(\tau)\frac{d\xi_\nu(\tau)}{d\tau} \right\} \\ & \times \exp\left\{ \frac{\Gamma t}{2} - \frac{\Gamma + i\overline{\omega}}{2\hbar}\mathrm{Im}\{M\}(q^2-q_0^2) \right\} \\ & \times \int_{(q_0,0)}^{(q,t)} \mathcal{D}[q]\exp\left\{ \frac{i}{\hbar}(S_{\rm cl}[q,\dot{q}] + \sqrt{\hbar}\Phi_\xi[q,\dot{q}]) \right\} \exp\left\{ -\frac{1}{2}D_\xi[q,\dot{q}] \right\}, \end{aligned} \quad (44)$$

where

$$S_{\rm cl}[q,\dot{q}] = \int_0^t d\tau \left\{ \frac{1}{2}\mathrm{Re}\{M\}\dot{q}_\tau^2 - \left(V(q_\tau) + \frac{1}{2}\mathrm{Re}\{M\}\overline{\omega^2}q_\tau^2 \right) \right\}. \quad (45)$$

Notice that the Lagrangian classical action requires renormalized mass and potential when compared to the original Hamiltonian (28). The stochastic phase functional, similar to Eq. (24), is

$$\Phi_\xi[q,\dot{q}] = \sum_\mu \mathrm{Im}\left\{ \int_0^t d\xi_\mu(\tau)\mathcal{E}_\mu(q_\tau,\dot{q}_\tau) \right\}. \quad (46)$$

In contrast to the Hamiltonian case we find additional and more complicated (since position dependent) prefactors in front of the path integral. Again the paths are weighted by a weight functional $\exp(-D_\xi[q,\dot{q}]/2)$, where

$$\begin{aligned} D_\xi[q,\dot{q}] = {}& \sum_\mu \int_0^t d\tau \left| \frac{\mathcal{E}_\mu(q_\tau,\dot{q}_\tau)}{\sqrt{\hbar}} \right|^2 - \frac{2}{\sqrt{\hbar}}\mathrm{Re}\left\{ \int_0^t d\xi_\mu(\tau)\mathcal{E}_\mu(q_\tau,\dot{q}_\tau) \right\} + \frac{2\overline{\omega}\Gamma\,\mathrm{Re}\{M\}}{\hbar}\int_0^t d\tau q_\tau^2 \\ = {}& \sum_\mu \int_0^t d\tau \left| \frac{\mathcal{E}_\mu(q_\tau,\dot{q}_\tau)}{\sqrt{\hbar}} - \frac{d\xi_\mu^*(\tau)}{d\tau} \right|^2 - \left| \frac{d\xi_\mu(\tau)}{d\tau} \right|^2 + \frac{2\overline{\omega}\Gamma\,\mathrm{Re}\{M\}}{\hbar}q_\tau^2. \end{aligned} \quad (47)$$

We find an unexpected additional term

$$\frac{2\overline{\omega}\Gamma\,\mathrm{Re}\{M\}}{\hbar}\int_0^t d\tau q_\tau^2 \quad (48)$$

contributing to the weight functional. Notice that this term vanishes for $\overline{\omega}=0$. This is fulfilled when the environment operators are creation or annihilation operators or when they are purely position or purely momentum dependent [see Eq. (30)]. Apart from this term, the interpretation of the weight functional (47) of paths is similar to the Hamiltonian case. We see how the stochastic processes $\xi_\mu(\tau)$ determine the center of a Gaussian-type weight functional of paths.

V. LQSD DERIVED FROM THE LAGRANGIAN PATH INTEGRAL PROPAGATOR

Conceptually, Lagrangian path integrals appear to be more desirable, particularly for relativity. One might choose to found a generalized LQSD theory on a stochastic Lagrangian path integral. In order to get some experience of how this program might work, it is instructive to see the last two sections reversed and find out how we can *derive* the LQSD equation (7) *starting* from a stochastic Lagrangian path integral.

Following Feynman's original approach [3,2], we find the propagator for finite times by the composition

$$G_\xi(t;0)=\lim_{N\to\infty}\prod_{k=1}^{N} G_\xi(k\Delta t;(k-1)\Delta t), \tag{49}$$

of short time propagators. It seems natural to base a stochastic linear quantum theory on a short time propagator of the form

$$\langle x|G_\xi(t+\Delta t,t)|y\rangle=\mathcal{P}\exp\left\{\frac{i}{\hbar}\mathcal{L}_{\rm eff}\left(\frac{x+y}{2},\frac{x-y}{\Delta t},t\right)\Delta t + \frac{1}{\sqrt{\hbar}}\sum_\mu \mathcal{E}_\mu\left(\frac{x+y}{2},\frac{x-y}{\Delta t},t\right)\Delta\xi_\mu(t)\right\} \tag{50}$$

with some generalized Lagrangian $\mathcal{L}_{\rm eff}(q,\dot q,t)$ responsible for the drift and Lagrangian environment functions $\mathcal{E}_\mu(q,\dot q,t)$ responsible for the diffusion. The Lagrangian path integral expression for the propagator then follows immediately from the composition rule (49) and the inclusion of $N-1$ identities $\int dq|q\rangle\langle q|$. Notice again the use of the midpoint rule for the position in the exponent of the short time propagator Eq. (50).

The prefactor $\mathcal{P}$ in (50) is determined by the lowest order in Δt, for which we require

$$\langle x|G_\xi(t+\Delta t,t)|y\rangle\to\delta(x-y)\quad \text{for}\quad \Delta t\to 0. \tag{51}$$

Given the short time propagator (50) we can now derive the corresponding stochastic linear Schrödinger equation for the state vector $|\psi\rangle$. As with Feynman's original derivation of the Schrödinger equation from the path integral, we assume at most quadratic dependence of the Lagrangian on the velocities,

$$\mathcal{L}_{\rm eff}(q,\dot q)=\frac{1}{2}m\dot q^2+\mathcal{A}(q)\dot q-\mathcal{V}(q). \tag{52}$$

We restrict ourselves to Lagrangian environment functions linear in the velocities,

$$\mathcal{E}_\mu(q,\dot q)=\mathcal{B}_\mu(q)+\mathcal{C}_\mu\dot q, \tag{53}$$

allowing the constants $m,\mathcal{C}_\mu$ and the functions $\mathcal{A}(q)$, $\mathcal{V}(q)$, and $\mathcal{B}_\mu(q)$ to be complex. Keep in mind that the trace condition for the density operator determined the non-Hermitian part of the drift term according to Eqs. (4) and (5). We expect, therefore, that given the Lagrangian environment functions (53), the trace condition determines the imaginary part of the Lagrangian (52).

We propagate the wave function $\langle x|\psi(t)\rangle$ a short time step Δt to get

$$\langle x|\psi(t+\Delta t)\rangle=\mathcal{P}\int dy\exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}m\left(\frac{x-y}{\Delta t}\right)^2 + \mathcal{A}\left(\frac{x+y}{2}\right)\frac{x-y}{\Delta t}-\mathcal{V}\left(\frac{x+y}{2}\right)\right]\Delta t + \frac{1}{\sqrt{\hbar}}\sum_\mu\left[\mathcal{B}_\mu\left(\frac{x+y}{2}\right) + \mathcal{C}_\mu\left(\frac{x-y}{\Delta t}\right)\right]\Delta\xi_\mu(t)\right\}\langle y|\psi(t)\rangle \tag{54}$$

and evaluate the y integral to first order in Δt. The lowest order determines the prefactor $\mathcal{P}$ [condition (51)] and the next order the linear Itô-Schrödinger equation for the wave function. We find

$$\mathcal{P}=\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left\{-\frac{i}{2M}\sum_{\mu\nu}\mathcal{C}_\mu\mathcal{C}_\nu\frac{\Delta\xi_\mu(t)\Delta\xi_\nu(t)}{\Delta t}\right\} \tag{55}$$

in agreement with the preceding section [Eq. (35)]. This shows that the complicated stochastic prefactor in the Lagrangian version of stochastic path integrals has its origin in preserving the correct short time behavior (51).

It is more elegant to express the resulting linear stochastic Itô-Schrödinger equation in operator language by replacing spatial derivatives by the momentum operator $\hat p=-i\hbar\partial/\partial q$. The stochastic Schrödinger equation corresponding to the short time propagator (50) with the choices (52) and (53) then reads

$$|d\psi\rangle=-\frac{i}{\hbar}\left\{\frac{(\hat p-\mathcal{A}(\hat q))^2}{2m}+\mathcal{V}(\hat q)\right\}|\psi\rangle dt + \frac{1}{\sqrt{\hbar}}\sum_\mu\left\{\mathcal{B}_\mu(\hat q)+\frac{\mathcal{C}_\mu}{m}[\hat p-\mathcal{A}(\hat q)]\right\}|\psi\rangle d\xi_\mu(t). \tag{56}$$

In order to be a sensible equation, we require the resulting density operator to be trace preserving (4),

$$d\mathcal{M}[\langle\psi_t|\psi_t\rangle]=0. \tag{57}$$

This condition leads to the following restrictions on the imaginary part of the Lagrangian (52):

$$\mathrm{Im}\{m\}=\sum_\mu|\mathcal{C}_\mu|^2, \tag{58}$$

$$\mathrm{Im}\{\mathcal{A}(q)\}=\frac{\sum_\mu\mathrm{Re}\{\mathcal{B}_\mu(q)\mathcal{C}_\mu^* m\}}{\mathrm{Re}\{m\}},$$

$$\mathrm{Im}\{\mathcal{V}(q)\}=-\frac{1}{2}\mathrm{Im}\left\{\frac{\mathcal{A}(q)^2}{m}\right\}-\frac{1}{2}\sum_\mu\left\{\left|\frac{\mathcal{C}_\mu\mathcal{A}(q)}{m}\right|^2+|\mathcal{B}(q)|^2-2\mathrm{Re}\left\{\frac{\mathcal{C}_\mu\mathcal{B}_\mu^*(q)\mathcal{A}(q)}{m}\right\}+\hbar\,\mathrm{Im}\left\{\frac{\mathcal{C}_\mu^*\mathcal{B}_\mu'(q)}{m^*}\right\}-\hbar\left|\frac{\mathcal{C}_\mu}{m}\right|^2\mathrm{Im}\{\mathcal{A}'(q)\}\right\}.$$

With these restrictions, the stochastic Schrödinger equation (56) is indeed an LQSD equation of the form (7) with operators

$$\hat{H}=\frac{\mathrm{Re}\{m\}}{2|m|^2}\left\{\hat{p}-\frac{\mathrm{Re}\{m\mathcal{A}(\hat{q})^*\}}{\mathrm{Re}\{m\}}\right\}^2+\mathrm{Re}\{\mathcal{V}(\hat{q})\}-\frac{[\mathrm{Im}\{\mathcal{A}(\hat{q})\}]^2}{2\mathrm{Re}\{m\}}$$

and

$$\hat{E}_\mu=\frac{1}{\sqrt{\hbar}}\left\{\mathcal{B}_\mu(\hat{q})+\frac{\mathcal{C}_\mu}{m}[\hat{p}-\mathcal{A}(\hat{q})]\right\}. \tag{59}$$

We recover the results of the preceding section with the choices

$$m=M,\quad \mathcal{A}(q)=iM\overline{\omega q},\quad \mathcal{V}_{\mathrm{eff}}(q)=V_{\mathrm{eff}}(q) \tag{60}$$

and

$$\mathcal{B}_\mu(q)=(\beta_\mu+iM\overline{\omega\gamma_\mu})q,\quad \mathcal{C}_\mu=M\gamma_\mu. \tag{61}$$

This completes the relations between the LQSD equation (7), the Hamiltonian (20), Lagrangian (36) path integral propagator, and the short time Lagrangian propagator (50). Further insight into the solutions of the LQSD equation is gained by investigating soluble models as in the next two sections.

VI. PROPAGATOR FOR THE HARMONIC OSCILLATOR WITH LINEAR ENVIRONMENT OPERATORS

Soluble models give us further insight into the meaning of the general results of the last three sections. In ordinary quantum mechanics the path integral for the harmonic oscillator with Hamiltonian

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{q}^2 \tag{62}$$

can be evaluated in closed form, due to the Gaussian nature of the momentum and position path integrals. Gaussian integrals are given by the value of their integrand at the stationary point times a constant factor. For path integrals this implies that only the *classical* path (the stationary path of the action functional) has to be taken into account [2,32].

Similarly, in our stochastic theory, we can determine the propagator for the harmonic oscillator (62) with linear environment operators like Eq. (29),

$$\hat{E}_\mu=\frac{\beta_\mu\hat{q}+\gamma_\mu\hat{p}}{\sqrt{\hbar}}, \tag{63}$$

in closed form by calculating the action integral (21) along the stationary path analytically.

The equations of motion are obtained by variation of the action (21) with respect to **q** and **p**. We find the complex Langevin-Itô equations

$$d\mathbf{q}=\frac{\partial H_{\mathrm{eff}}(\mathbf{q},\mathbf{p})}{\partial\mathbf{p}}dt+i\hbar\sum_\mu\frac{\partial E_\mu(\mathbf{q},\mathbf{p})}{\partial\mathbf{p}}d\xi_\mu \tag{64}$$

and

$$d\mathbf{p}=-\frac{\partial H_{\mathrm{eff}}(\mathbf{q},\mathbf{p})}{\partial\mathbf{q}}dt-i\hbar\sum_\mu\frac{\partial E_\mu(\mathbf{q},\mathbf{p})}{\partial\mathbf{q}}d\xi_\mu, \tag{65}$$

which in our special case reduce to the linear equations for a (complex) classical phase space path

$$\begin{pmatrix}dq\\dp\end{pmatrix}=\begin{pmatrix}-i\overline{\omega} & \frac{1}{M}\\ -(m\omega^2-i|\beta|^2) & i\overline{\omega}\end{pmatrix}\begin{pmatrix}q\\p\end{pmatrix}dt+i\sqrt{\hbar}\sum_\mu\begin{pmatrix}\gamma_\mu\\-\beta_\mu\end{pmatrix}d\xi_\mu. \tag{66}$$

Evaluating the stochastic action (21) along the solution of Eq. (66) with boundary conditions $q(0)=q_0$ and $q(t)=q$ is quite cumbersome, which is why we simply state the result. We introduce an effective, complex frequency Ω through

$$M\Omega^2=m\omega^2-i|\beta|^2+M\overline{\omega^2}, \tag{67}$$

where again we used the abbreviations (30) for $\overline{\omega}$ and Eq. (32) for the effective mass M. Moreover, we introduce the functions

$$\varepsilon_\mu(t)=(\beta_\mu+iM\overline{\omega\gamma_\mu})\sin(\Omega t)+M\Omega\gamma_\mu\cos(\Omega t) \tag{68}$$

and find that the stochastic action (21) for the harmonic Hamiltonian (62) and linear environment operators (63) along the classical path (66) is

$$\begin{aligned}S_\xi^{\mathrm{cl}}(q,t;q_0,0)=&-\frac{i\hbar t}{2}\Gamma+\frac{i}{2}M\overline{\omega}(q^2-q_0^2)+\frac{M\Omega}{2\sin(\Omega t)}\\&\times[(q^2+q_0^2)\cos(\Omega t)-2qq_0]\\&-\frac{i\sqrt{\hbar}}{\sin(\Omega t)}\sum_\mu\int_0^t d\xi_\mu(s)\\&\times[q\varepsilon_\mu(s)-q_0\varepsilon_\mu(s-t)]\\&-\frac{\hbar}{M\Omega\sin(\Omega t)}\sum_{\mu\nu}\int_0^t d\xi_\mu(s)\varepsilon_\mu(s-t)\\&\times\int_0^s d\xi_\nu(s')\varepsilon_\nu(s').\end{aligned} \tag{69}$$

The stochastic LQSD propagator is therefore

$$G_\xi(q,t;q_0,0)=\sqrt{\frac{i}{2\pi\hbar}\frac{\partial^2 S_\xi^{\rm cl}}{\partial q_0\partial q}}\exp\left\{\frac{i}{\hbar}S_\xi^{cl}(q,t;q_0,0)\right\}$$
$$=\sqrt{\frac{M\Omega}{2\pi i\hbar\sin(\Omega t)}}\exp\left\{\frac{i}{\hbar}S_\xi^{cl}(q,t;q_0,0)\right\}. \tag{70}$$

The prefactor is determined according to the general theory of semiclassical propagators [32]. Clearly, this result can also be checked directly with the help of the corresponding LQSD equation (7).

It is more instructive to look at the time evolution of wave packets under the propagator (70). Due to its quadratic dependence of the exponent on the initial and final positions, Gaussian wave packets remain Gaussian under time evolution. A special class of them will be further investigated in the following section. Similar investigations on the time evolution of Gaussian wave packets with LQSD can be found in [27,28].

VII. WAVE PACKET SOLUTIONS FOR THE HARMONIC OSCILLATOR WITH LINEAR ENVIRONMENT OPERATORS

In the preceding section we found the stochastic propagator $G_\xi(t;0)$ for the LQSD equation (7) for the case of a harmonic oscillator (62) and linear environment operators (63). This propagator allows certain Gaussian wave packets to keep their shape under time evolution, generalizing the coherent states of the isolated harmonic oscillator. They are given by the family of states $|c\rangle$ with arbitrary complex c, in position representation

$$\langle x|c\rangle=\left(\frac{\mathrm{Re}\{1/\lambda^2\}}{\pi}\right)^{1/4}\exp\left\{-\frac{\mathrm{Im}\{\lambda c\}^2}{\mathrm{Re}\{\lambda^2\}}\right\}$$
$$\times\exp\left\{-\frac{(x-\sqrt{2}\lambda c)^2}{2\lambda^2}\right\} \tag{71}$$

with the constant λ given by

$$\lambda=\sqrt{\frac{\hbar}{M(\Omega+\overline{\omega})}}. \tag{72}$$

The normalization in Eq. (71) is chosen to ensure $\langle c|c\rangle=1$. These states are eigenstates of the operator

$$\hat{C}=\frac{1}{\sqrt{2}}\left(\frac{\hat{q}}{\lambda}+i\frac{\lambda}{\hbar}\hat{p}\right) \tag{73}$$

with eigenvalue c, $\hat{C}|c\rangle=c|c\rangle$. Since λ is in general complex these are squeezed states.

From $c=\langle c|\hat{C}|c\rangle$ we find the expectation values for position and momentum expressed in terms of the parameter c,

$$\langle c|\hat{q}|c\rangle=\sqrt{2}\frac{|\lambda|^2\mathrm{Re}\{\lambda^*c\}}{\mathrm{Re}\{\lambda^2\}}\quad\text{and}\quad\langle c|\hat{p}|c\rangle=\sqrt{2}\frac{\hbar\,\mathrm{Im}\{\lambda c\}}{\mathrm{Re}\{\lambda^2\}}. \tag{74}$$

We call

$$|\gamma_t\rangle=G_\xi(t;0)|c_0\rangle \tag{75}$$

the unnormalized solution of the LQSD equation with initial state $|c_0\rangle$. From the explicit expression (70) for the propagator and Eq. (71) for the state $|c_0\rangle$ we find after some manipulations

$$|\gamma_t\rangle=\exp\{F(t)\}|c_t\rangle, \tag{76}$$

which shows that the wave packet $|\gamma_t\rangle$ after time t is given by a shifted Gaussian $|c_t\rangle$ of the same shape but different amplitude $\exp\{F(t)\}$. The time evolution of the center of the wave packet c_t is determined by the linear Langevin-Itô equation

$$dc_t=-i\Omega c_t dt+\frac{1}{\sqrt{2\hbar}}\sum_\mu\left(\lambda\beta_\mu+i\frac{\hbar}{\lambda}\gamma_\mu\right)d\xi_\mu, \tag{77}$$

with initial condition $c_{t=0}=c_0$. The amplitude $\exp\{F(t)\}$ of the wave packet is given by

$$F(t)=-\frac{it}{2}(\Omega+i\Gamma)+\left(\frac{\Omega-\overline{\omega}}{2\Omega}c_\tau^2+\frac{1}{\mathrm{Re}\{\lambda^2\}}\mathrm{Im}\{\lambda c_\tau\}^2\right)\Bigg|_0^t$$
$$+\frac{1}{\sqrt{2\hbar}\Omega}\sum_\mu\left\{\left(\lambda(\Omega+\overline{\omega})\beta_\mu-\frac{i\hbar}{\lambda}(\Omega-\overline{\omega})\gamma_\mu\right)\right.$$
$$\left.\times\int_0^t d\xi_\mu(\tau)c_\tau\right\}. \tag{78}$$

This shows the complicated stochastic time dependence of the amplitude of the wave packet. If we consider the time dependence of the norm

$$N(t)=\langle\gamma_t|\gamma_t\rangle=\exp\{2\mathrm{Re}\{F(t)\}\} \tag{79}$$

explicitly, we find after a lengthy calculation involving Eqs. (74), (77), and (78) that

$$dN(t)=\frac{2}{\sqrt{\hbar}}\sum_\mu\mathrm{Re}\{(\beta_\mu\langle\gamma_t|\hat{q}|\gamma_t\rangle+\gamma_\mu\langle\gamma_t|\hat{p}|\gamma_t\rangle)d\xi_\mu\}, \tag{80}$$

which confirms that the time dependence of the norm is governed by a diffusion process only. The result (80) is of course a consequence of the general result (9).

From Eq. (74) we see how Eq. (77) implies that the time dependence of the normalized expectation values of position and momentum are governed by a linear Langevin-Itô equation. To get the correct ensemble average, however, it is essential to weight these expectation values with the norm $N(t)$. The relevant weighted expectation values

$$Q_t=\langle\gamma_t|\hat{q}|\gamma_t\rangle=N(t)\langle c_t|\hat{q}|c_t\rangle,$$
$$P_t=\langle\gamma_t|\hat{p}|\gamma_t\rangle=N(t)\langle c_t|\hat{p}|c_t\rangle, \tag{81}$$

evolve according to

$$dQ(t)=\left(-\Gamma Q(t)+\frac{P(t)}{m}\right)dt+\frac{2Q(t)}{N(t)\sqrt{\hbar}}\times\sum_{\mu}\text{Re}\{[\beta_\mu Q(t)+\gamma_\mu P(t)]d\xi_\mu\}+\frac{N(t)|\lambda|^4}{\sqrt{\hbar}\text{Re}\{\lambda^2\}}\times\sum_{\mu}\text{Re}\left\{\left(\beta_\mu+\frac{i\hbar}{\lambda^2}\gamma_\mu\right)d\xi_\mu\right\} \quad (82)$$

and

$$dP(t)=[-m\omega^2Q(t)-\Gamma P(t)]dt+\frac{2P(t)}{N(t)\sqrt{\hbar}}\times\sum_{\mu}\text{Re}\{[\beta_\mu Q(t)+\gamma_\mu P(t)]d\xi_\mu\}+\frac{N(t)\sqrt{\hbar}}{\text{Re}\{\lambda^2\}}\times\sum_{\mu}\text{Im}\{(\lambda^2\beta_\mu+i\hbar\gamma_\mu)d\xi_\mu\}. \quad (83)$$

Equations (80), (82), and (83) are a system of three coupled *nonlinear* Langevin-Itô equations. It might appear strange at first sight that the evolution equations for the relevant weighted expectation values of the harmonic oscillator with linear environment operators are nonlinear. This nonlinearity, however, appears in the noise terms only so that we find linear evolution equations for the average expectation values by taking the ensemble mean in Eqs. (82) and (83), showing how the density operator for the harmonic oscillator with linear environment operators follows a damped spiraling motion with damping rate Γ. These results highlight the peculiarities that one encounters when dealing with the linear but not norm preserving LQSD equation.

VIII. CONCLUSIONS

In this paper we introduce stochastic path integrals for the propagator of the general LQSD equation (7). These linear stochastic Schrödinger equations are uniquely connected to the most general positive trace preserving master equation (8). In this respect, we have introduced the most general, stochastic path integral propagators for the description of (Markovian) open quantum systems.

The Hamiltonian and where possible the Lagrangian version of the stochastic path integral have been investigated. The latter appears to be more complicated compared to the former, suggesting that a phase space description is the more natural approach to dissipation in quantum mechanics. The analytical evaluation of the path integral for soluble standard models shows the peculiarities of the not norm preserving LQSD equation.

The developed formalism is a starting point for many further investigations. All standard methods that have been developed for ordinary path integrals like perturbation theory and semiclassical approximations can also be applied to the stochastic path integrals. It also allows a simple derivation of the path integral theory of the corresponding master equation (8) [34]. Stochastic path integrals might also serve as the starting point for relativistic generalizations of quantum Brownian motion models. Abandoning the restriction on white Gaussian noise, these path integrals might also be useful for the generalization to non-Markovian situations.

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