# Radiation reaction in the stimulated emission from a gain sheet and its similarity to superradiance

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The radiation reaction force due to individual emitters of a homogeneously broadened gain sheet is calculated from the multimode quantized electromagnetic field. This force can be placed in the dipole moment operator equation of motion, and takes a form similar to superradiance. A small correction factor in the laser rate equations for an optical gain sheet placed in a large dimension cavity is derived. For small cavity systems, such as the Fabry-Pérot microcavity, the correction factor may be increased. [S1050-2947(96)05709-5]

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The topic of superradiance from an emitter sheet, and in particular when the sheet is contained in a Fabry-Pérot microcavity, has gained recent interest [1–7]. In this problem of spontaneous emission a macroscopic polarization leads to superradiance [1–4], due in part to radiation reaction between the separate emitters, and a similar effect might be expected for stimulated emission. Typically for laser problems or in the analysis of stimulated emission, however, the individual emitters are treated as coupled independently to the driving field even though the gain region is at the same time treated as deriving a macroscopic polarization due to the driving field [8,9]. For example, Lax [8] analyzes the coupled equations of motion for a single-mode lasing field and a resonant gain polarization using

$$\frac{d\hat{a}}{dt}=-(i\omega_0+\gamma_c)\hat{a}(t)+N\mu\hat{\sigma}(t)+\hat{F}_{\hat{a}} \tag{1}$$

and

$$\frac{d\hat{\sigma}}{dt}=-(i\omega_0+\gamma_a)\hat{\sigma}(t)+\mu(\hat{\sigma}_a-\hat{\sigma}_b)\hat{a}(t)+\hat{F}_{a,b}, \tag{2}$$

where $\hat{a}(t)$ and $\hat{\sigma}(t)$ are the annihilation operators of the field and gain polarization, $\mu$ is the dipole coupling strength to the lasing mode [8], $\gamma_c$ is the cavity decay rate, $\gamma_a$ is the polarization decay rate due to emitter dephasing, $N$ is the emitter number, $(\hat{\sigma}_a-\hat{\sigma}_b)$ is the population inversion operator, and $\hat{F}_{\hat{a}}$ and $\hat{F}_{a,b}$ are noise operators, which preserve the commutator relations of the single-mode field and dipole moment operators, respectively. The collective polarization state in terms of the separate emitters is defined as

$$\hat{\sigma}(t)=\frac{1}{N}\sum_{n=1}^{N}\hat{\sigma}_n(t), \tag{3}$$

where the subscript $n$ labels the individual emitter polarizations, with the summation resulting in the macroscopic polarization. The similar approach is presented in [9]. We note that if Eq. (3) is used to describe spontaneous emission and does not average to zero, superradiance results.

In a large optical cavity a plane-wave mode expansion can be used to describe the short-time propagation of a single cavity quasimode. It is shown below that in such a case radiation reaction forces lead to a small correction factor that contains the square of the emitter density in the laser gain, and is similar to superradiance in spontaneous emission. For a microcavity system we believe that the same radiation reaction force will be even larger due to the increased mode coupling, as is also found for superradiance [5–7]. In this regard we note that a similar treatment to that presented below for open space should also be possible for the planar Fabry-Pérot microcavity, since the complete and orthogonal set of mode functions that satisfies the passive cavity boundary conditions is known [7,10].

Within the large cavity or open space, we consider an optically thin emitter sheet that is excited by an incident Gaussian mode (or standing wave). For a steady-state driving field yielding stimulated emission, incoherent pumping to level $|a\rangle$ leads to an average value of population inversion, and a macroscopic polarization evolves described by the equations of motion of the individual dipole moments driven by the same stimulating field. The Hamiltonian of the system of $N$ homogeneously broadened two-level emitters coupled to the multimode quantized electromagnetic field, and a reservoir leading to level dephasing, can be described in the notation of [7] by

$$\begin{aligned}\hat{H}=\hat{H}_A(t)&=\hat{H}_F(t)+\hat{H}_R(t)+\hat{H}_{IAF}(t)+\hat{H}_{IRA}(t)\\&=\sum_{n=1}^{N}\frac{\hbar\omega_0}{2}[\hat{\sigma}_{a,n}(t)-\hat{\sigma}_{b,n}(t)]+\sum_{\mathbf{m}}\frac{\hbar\omega_{\mathbf{m}}}{2}[\hat{a}_{\mathbf{m}}^{\dagger}(t)\hat{a}_{\mathbf{m}}(t)\\&\quad+\hat{a}_{\mathbf{m}}(t)\hat{a}_{\mathbf{m}}^{\dagger}(t)]+\hat{H}_R(t)+q\sum_{n=1}^{N}\sum_{\mathbf{m}}[\mathbf{E}_{\mathbf{m}}(\mathbf{r}_n)\hat{a}_{\mathbf{m}}(t)\\&\quad+\mathbf{E}_{\mathbf{m}}^{*}(\mathbf{r}_n)\hat{a}_{\mathbf{m}}^{\dagger}(t)]\cdot[\mathbf{d}_n\hat{\sigma}_n(t)+\mathbf{d}_n^{*}\hat{\sigma}_n^{\dagger}(t)]+\hat{H}_{IRA}(t),\end{aligned} \tag{4}$$

where the emitter and field commutator relations are well known for a two-level system coupled to a multimode field with the terms defined in [7], and the reservoir $\hat{H}_R(t)$ that leads to emitter dephasing through $\hat{H}_{IRA}(t)$ need not yet be specified, except that it has finite bandwidth. The equations of motion for each optical field operator are derived from its commutation relation with the Hamiltonian, and given by

$$\frac{d\hat{a}_{\mathbf{m}}}{dt}=-i\omega_{\mathbf{m}}\hat{a}_{\mathbf{m}}(t)-\frac{iq}{\hbar}\sum_{n}\left[\mathbf{d}_{n}\cdot\mathbf{E}_{\mathbf{m}}^{*}(\mathbf{r}_{n})\hat{\sigma}_{n}(t)+\mathbf{d}_{n}^{*}\cdot\mathbf{E}_{\mathbf{m}}^{*}(\mathbf{r}_{n})\hat{\sigma}_{n}^{\dagger}(t)\right], \quad (5)$$

while the equation of motion for each dipole moment operator is given by

$$\frac{d\hat{\sigma}_{n}}{dt}=-i\omega_{0}\hat{\sigma}_{n}(t)+\frac{iq}{\hbar}[\hat{\sigma}_{a,n}(t)-\hat{\sigma}_{b,n}(t)]\times\sum_{\mathbf{m}}\left[\mathbf{d}_{n}^{*}\cdot\mathbf{E}_{\mathbf{m}}(\mathbf{r}_{n})\hat{a}_{\mathbf{m}}(t)+\mathbf{d}_{n}^{*}\cdot\mathbf{E}_{\mathbf{m}}^{*}(\mathbf{r}_{n})\hat{a}_{\mathbf{m}}^{\dagger}(t)\right]+\frac{i}{\hbar}[\hat{H}_{IRA'}(t),\hat{\sigma}_{n}(t)]. \quad (6)$$

The reservoir interaction with the dipole moment leads to an exponential decay of the emitter dephasing on a time scale that is long compared to the actual emitter dephasing time [9]. Equation (5) can be integrated over a short-time interval $\Delta t$ to find the multimode influence of any emitter on any other emitter making up the gain sheet through the coupled equation (6). Using the normalized, traveling plane-wave mode basis of open space satisfying orthogonality over large periodic boundary conditions, the result becomes

$$\sum_{\mathbf{m}}\mathbf{E}_{\mathbf{m}}(\mathbf{r}_{n})\hat{a}_{\mathbf{m}}(t+\Delta t)\approx\sum_{\mathbf{m}}\mathbf{E}_{\mathbf{m}}(\mathbf{r}_{n})\hat{a}_{\mathbf{m}}(t)e^{-i\omega_{m}\Delta t}-\frac{iq}{16\pi^{3}\epsilon_{0}}\sum_{s=1}^{2}\int d^{3}k\,\omega\,\mathbf{a}_{s}(\mathbf{k})\sum_{n'}e^{i\mathbf{k}\cdot(\mathbf{r}_{n}-\mathbf{r}_{n'})}\times\left[\mathbf{d}_{n'}\cdot\mathbf{a}_{s}(\mathbf{k})\hat{\sigma}_{n'}(t)e^{-i\omega_{0}\Delta t}\frac{1-e^{-i(\omega-\omega_{0})\Delta t}}{i(\omega-\omega_{0})}+\mathbf{d}_{n'}^{*}\cdot\mathbf{a}_{s}(\mathbf{k})\hat{\sigma}_{n'}^{\dagger}(t)e^{i\omega_{0}\Delta t}\frac{1-e^{-i(\omega+\omega_{0})\Delta t}}{i(\omega+\omega_{0})}\right], \quad (7)$$

where $\mathbf{a}_s(\mathbf{k})$ is the field polarization of mode $\mathbf{k}$. The summation over the emitter positions given by $n'$ restricts the angular range of modes included in the radiated field. In open space the mode interference is such that constructive interference occurs only over a resonant wavelength squared area. Therefore, when the emitter sheet has a much larger area than this, it is a very good approximation in changing the summation over $n'$ to a two-dimensional integral to normalize by the emitter density $n_0$, so that the emitter area can be extended to infinity. The summation over the angular range of modes then becomes a single integration over frequency. We take the position at the center of the gain sheet as (0,0,0), and calculate the radiation field at that point. Adding Eq. (7) to its adjoint to find the total field at time $t+\Delta t$ and taking the dipole moments as aligned in the $x$ direction gives

$$\hat{\mathbf{E}}(0,0,0,t+\Delta t)\approx\sum_{\mathbf{m}}\left[\mathbf{E}_{\mathbf{m}}(0,0,0)\hat{a}_{\mathbf{m}}(t)e^{-i\omega_{m}\Delta t}+\mathbf{E}_{\mathbf{m}}^{*}(0,0,0)\hat{a}_{\mathbf{m}}^{\dagger}(t)e^{i\omega_{\mathbf{m}}\Delta t}\right]-\frac{i\mathbf{a}_{x}qn_{0}}{2\pi\epsilon_{0}c}\left[d\hat{\sigma}(t)e^{-i\omega_{0}\Delta t}\times\int_{-\infty}^{\infty}d\omega(\omega'+\omega_{0})\frac{1-e^{-i\omega'\Delta t}}{i\omega'}-d^{*}\hat{\sigma}^{\dagger}(t)e^{i\omega_{0}\Delta t}\times\int_{-\infty}^{\infty}d\omega(\omega'+\omega_{0})\frac{1-e^{i\omega'\Delta t}}{-i\omega'}\right], \quad (8)$$

where the dipole operators refer to an emitter at the center of the gain sheet. Now we make an important point regarding the mathematical form of Eq. (8). The real part of the integrals are independent of $\Delta t$ and yield $\pi$. The imaginary part of the integrals contain leading terms that follow $(\Delta t)^2$. If we rewrite a differential equation for the total field from the emitter sheet from Eq. (8) by dividing by $\Delta t$ in the limit of $\Delta t\to 0$, we find that the real parts of the integrals diverge due to division by zero, while the imaginary parts of the integrals go to zero. The radiation reaction field between the emitters, therefore, acts instantaneously with a finite force, and should properly be placed within the dipole moment equation of motion. The instantaneous action is due to the transverse extent of the approximate plane wave radiated mode selected by the emitter geometry. For a single-point source emitter, the same analysis shows that the radiation reaction force appears instantaneously only at the position of the emitter. We emphasize that the result is independent of the dipole approximation, as we can as well work within the electronic wave functions of the emitters that will exhibit the collective motion of the dipole sheet.

Letting $\Delta t\to 0$, the field within the gain sheet becomes

$$\hat{\mathbf{E}}(0,0,0,t)=\hat{\mathbf{E}}(0,0,0,t)-\frac{i\mathbf{a}_{x}q\omega_{0}n_{0}d\hat{\sigma}(t)}{2\epsilon_{0}c}+\frac{i\mathbf{a}_{x}q\omega_{0}n_{0}d^{*}\hat{\sigma}^{\dagger}(t)}{2\epsilon_{0}c}, \quad (9)$$

but where the second term requires the presence of the emitter sheet in the superposition state. If we consider again the case of superradiance, an emitter at the center of the sheet has a level decay given from the Hamiltonian by $d\langle\hat{H}_A\rangle/dt=iq\omega_0\langle\hat{\mathbf{E}}(0,0,0,t)[\mathbf{d}\hat{\sigma}(t)-\mathbf{d}^{*}\hat{\sigma}^{\dagger}(t)]\rangle$, and inserting the dipole part of the field of Eq. (9) leads to a spontaneous decay of $(d\langle\hat{H}_A\rangle/dt)_{\mathrm{RR}}=-(q^2\omega_0^2 n_0|d|^2/2\epsilon_0 c)\langle\hat{\sigma}_a(t)+\hat{\sigma}_b(t)\rangle$, where the RR subscript designates radiation reaction. Analysis of the broadband vacuum fluctuations contained in the first field term of Eq. (9) leads to an additional decay rate of $(d\langle\hat{H}_A\rangle/dt)_{\mathrm{VF}}=-(q^2\omega_0^2 n_0|d|^2/2\epsilon_0 c)\langle\hat{\sigma}_a(t)-\hat{\sigma}_b(t)\rangle$, or a superradiance decay rate of $(3\lambda_0^2 n_0)/(4\pi^2\tau_{\mathrm{sp}})$ where $\lambda_0$ is the emission wavelength and $\tau_{sp}$ is the single isolated emitter spontaneous decay rate. This is the typical result for a perturbation analysis of superradiance from an emitter sheet [1], and note that the analysis yields the result that 50% of the emission rate arises from radiation reaction while 50% arises from vacuum fluctuations, and where the vacuum fluctuations yield a stable ground state for the emitters coupled to the radiation field [11,12]. Inserting the radiation reaction field of Eq. (9) into Eq. (6), the new equation of motion for the dipole moment operator, including the radiation reaction force, becomes

$$\frac{d\hat{\sigma}}{dt}=-\left\{i\omega_0-\frac{\omega_0 q^2|d|^2 n_0[\hat{\sigma}_a(t)-\hat{\sigma}_b(t)]}{2\hbar\epsilon_0 c}\right\}\hat{\sigma}(t)+\frac{iq}{\hbar}[\hat{\sigma}_{a,n}(t)-\hat{\sigma}_{b,n}(t)]\sum_{\mathbf{m}}[\mathbf{d}_n^*\cdot\mathbf{E}_{\mathbf{m}}(\mathbf{r}_n)\hat{a}_{\mathbf{m}}(t)+\mathbf{d}_n^*\cdot\mathbf{E}_{\mathbf{m}}^*(\mathbf{r}_n)\hat{a}_{\mathrm{m}}^\dagger(t)]+\frac{i}{\hbar}[\hat{H}_{IRA}(t),\hat{\sigma}(t)], \quad (10)$$

where the coupling to the dipole adjoint operator is neglected based on the rotating wave approximation.

We can estimate the size of the radiation reaction force on the stimulated emission rate in a large cavity laser system if the gain in fact arises from such an emitter sheet. Assuming that Eq. (10) holds in the long-time approximation leading to an exponential dipole dephasing rate [9], the dipole moment operator can be adiabatically eliminated from Eq. (1) to yield the standard form for the lasing quasimode field operator. Taking the commutator between the dipole moment operator and the dephasing reservoir as yielding a dipole dephasing rate of $\gamma_a$ plus a noise operator, the result is

$$\frac{d\hat{a}}{dt}=-(i\omega_0+\gamma_c)\hat{a}(t)+\frac{\omega_0 q^2|d|^2 n_0[\hat{\sigma}_a(t)-\hat{\sigma}_b(t)]}{\hbar\epsilon_0\gamma_a' L}\hat{a}(t)+\hat{F}'_{a,b}, \quad (11)$$

where $L$ is the cavity length, the new dipole noise operator $\hat{F}'_{a,b}$ has been placed in the lasing field equation of motion, and now

$$\frac{1}{\gamma_a'}=\frac{1}{\gamma_a-\omega_0 q^2|d|^2 n_0[\hat{\sigma}_a(t)-\hat{\sigma}_b(t)]/2\hbar\epsilon_0 c}\approx\frac{1}{\gamma_a}\left\{1+\frac{\omega_0 q^2|d|^2 n_0[\hat{\sigma}_a(t)-\hat{\sigma}_b(t)]}{2\hbar\epsilon_0 c\gamma_a}\right\}, \quad (12)$$

where the approximation holds for the typical case of dipole dephasing dominating any radiation reaction effects. Inserting Eq. (12) into Eq. (11) then gives

$$\frac{d\hat{a}}{dt}=-(i\omega_0+\gamma_c)\hat{a}(t)+\left\{\frac{\omega_0 q^2|d|^2 n_0[\hat{\sigma}_a(t)-\hat{\sigma}_b(t)]}{\hbar\epsilon_0\gamma_a L}+\frac{\omega_0^2 q^4|d|^4 n_0^2[\hat{\sigma}_a(t)-\hat{\sigma}_b(t)]^2}{2\hbar^2\epsilon_0^2 c\gamma_a^2 L}\right\}\hat{a}(t)+\hat{F}'_{a,b}. \quad (13)$$

The size of the term depending on the square of the emitter density can be estimated from the threshold condition in the large cavity Fabry-Pérot laser, and for steady-state lasing we have

$$\frac{\omega_0 q^2|d|^2 n_0[\hat{\sigma}_a(t)-\hat{\sigma}_b(t)]}{2\hbar\epsilon_0 c\gamma_a}\approx\frac{L\gamma_c}{2c}=\frac{1}{2}\ln\left(\frac{1}{R}\right), \quad (14)$$

where $R$ is the intensity reflectivity of the mirrors. For high reflectivity the radiation reaction force therefore yields an increase in the stimulated emission rate of at most a few percent (1% for $R=0.98$). For transient effects, such as gain switching, the emitter density can exceed its threshold value and increase the radiation reaction force.

The strong similarity between the square dependence term for stimulated emission in Eq. (13) and that expected for superradiance from the same sheet also has an analogous description given from classical field interference. The gain sheet is driven by a stimulating field that can be given at an emission time by $\mathbf{E}_{St}(\mathbf{r},t)$, and results in an induced field from each dipole emitter given by $\mathbf{E}_n(\mathbf{r},t)$. The resulting field energy is then found in the standard way by a volume integration of the square of the summed fields, and can be expressed by

$$\int d^3r\left|\mathbf{E}_{St}(\mathbf{r},t)+\sum_n\mathbf{E}_n(\mathbf{r},t)\right|^2=\int d^3r|\mathbf{E}_{St}(\mathbf{r},t)|^2+2\,\mathrm{Re}\int d^3r\,\mathbf{E}_{St}^*(\mathbf{r},t)\cdot\sum_n\mathbf{E}_n(\mathbf{r},t)+\int d^3r\sum_n\sum_{n'}\mathbf{E}_n^*(\mathbf{r},t)\cdot\mathbf{E}_n{}'(\mathbf{r},t). \quad (15)$$

When the overlap integrals between the dipole fields and stimulating field are zero, due to a temporally short stimulating pulse, the field interference between the dipole fields in Eq. (15) yields a classical description of superradiance. For steady-state excitation, if the emitter distribution closely matches the field distribution, as for a well-designed laser, the extra stimulated emission is radiated into the lasing mode.

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