Crossed-field hydrogen atom and the three-body Sun-Earth-Moon problem

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We report results of a systematic study of the energy-field dependence of the four elementary periodic orbits of the electron motion of the hydrogen atom in crossed magnetic-electric fields from weak to strong perturbation, up into the continuum regime. We find the classical dynamics of the crossed-field atom to be intimately connected to that of the Sun-Earth-Moon three-body problem in Hill's approximation, exhibited by striking similarities of the four elementary orbits of the atomic and celestial systems in their energy-field dependence. [S1050-2947(96)04007-3]

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Periodic orbits have, since Kepler, evolved as the key concept for describing and understanding classical dynamics and, since Bohr, as the basic link in the transition to quantum mechanics. This is particularly true when atomic systems turn chaotic in the classical limit, as has been elaborated extensively by Gutzwiller [1]. In this context highly excited Rydberg hydrogen atoms, strongly perturbed by homogeneous external fields, have lately attracted a large amount of interest, because they show classically chaotic behavior and simultaneously are dynamically sufficiently simple to allow detailed experimental and theoretical studies [2]. Particularly the hydrogen atom in magnetic fields, the most simple chaotic system, has been a subject of intense research [3] and is now reasonably well understood [4]. The atom in crossed electric-magnetic fields is, however, still an open problem despite significant advances in recent years [5]. With its three nonintegrable degrees of freedom and additionally because of the possibility of the field ionization, the crossed-field atom dynamics is essentially different from that of the magnetized atom, and is naturally also much more complex. Searching for a systematics and organizational order of the electron motion in crossed fields, we discovered a set of three elementary Kepler-like periodic orbits, originating in the weak-perturbation regime [6], one of them found before by Delande and Gay [7]. An additional elementary periodic orbit, known as the quasi-Penning orbit, was earlier reported by Clark, Korevaar, and Littman [8]. It occurs in the crossed-field system only and is centered around the field-ionization saddle point. As in pure fields, these four orbits represent the roots out of which evolves the dynamics of the crossed-field atom from weak to strong perturbation, a most fundamental property not yet investigated to our knowledge.

The significance of periodic orbits in classical dynamics became apparent at the end of the last century, particularly in celestial mechanics. In this context the problem of the lunar dynamics in the Sun-Earth-Moon system played a prominent role. While the observational motion of the Moon was known with impressive precision since the ancient Greeks' time [9], the theoretical quantitative solution remained unsatisfactory until 1878, when a breakthrough was achieved by Hill [10], who reduced the three-body problem by an ingenious approximation to an effective two-body problem. As a result, the observed lunar motion is described by a Kepler-like orbit centered in the ecliptic plane with small-amplitude oscillatory deviations superimposed. Following Hill's approach later work extended to the global dynamics and the general periodic orbit structure [11] showed the existence of three further elementary periodic orbits governing the overall dynamics of this system. A more recent comprehensive study has been performed by Hénon [12].

The objective of this work is to investigate the energy dependence of the four elementary orbits of the crossed-field atom. These trajectories turned out to exhibit features resembling the orbits in Hill's lunar theory, resulting from similarities of the Hamiltonians of these apparently different systems. Exploiting the knowledge about the Moon dynamics thus leads to an understanding of properties and structures that are observed in the crossed-field system.

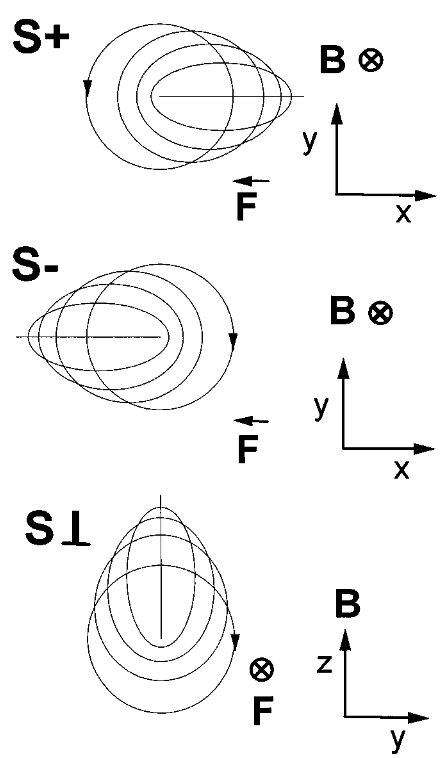

FIG. 1. The three elementary Kepler ellipses of the crossed-field hydrogen atom in the perturbative regime for different combinations of external fields.

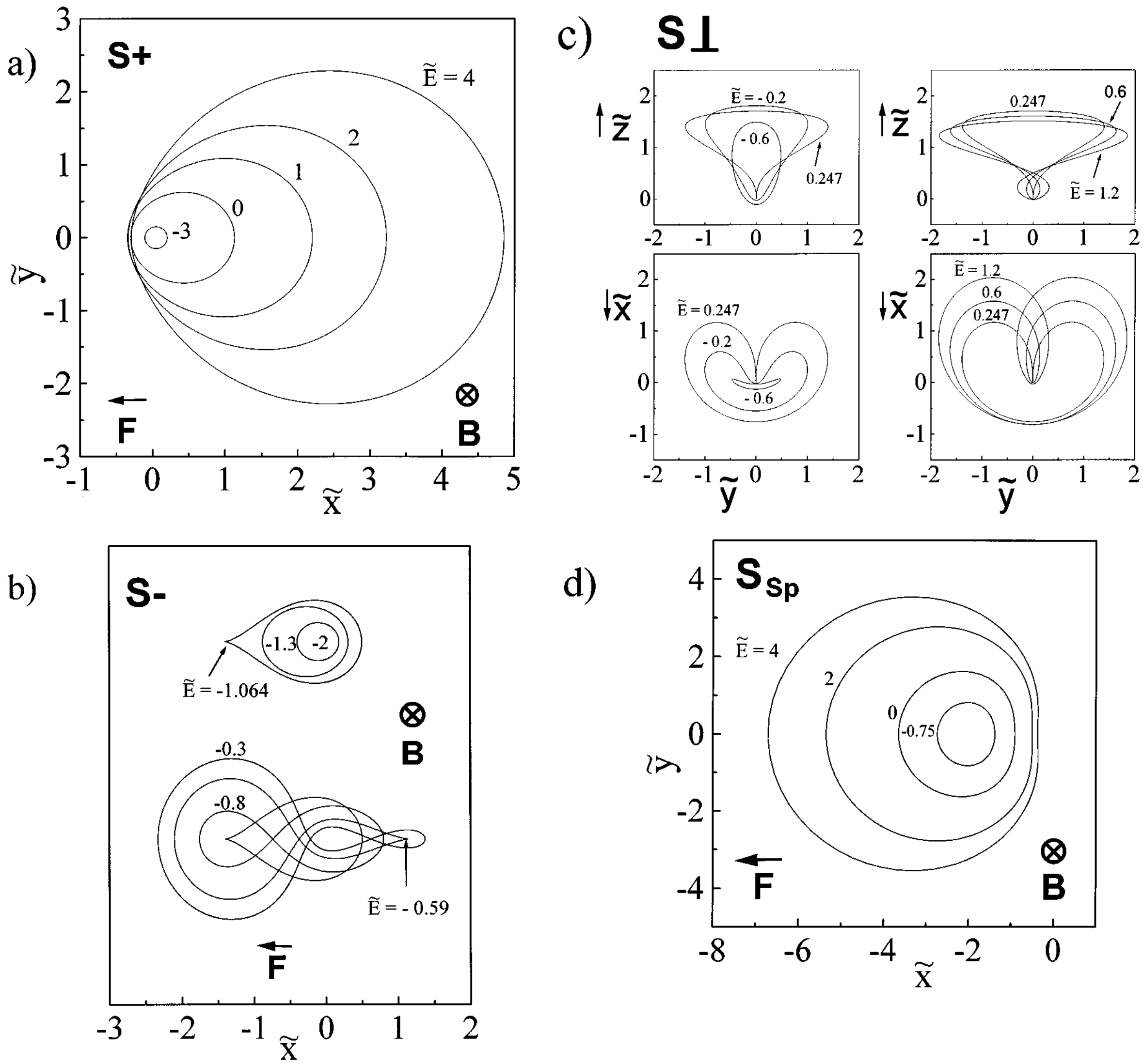

FIG. 2. Energy dependence of the elementary periodic orbits of the crossed-field hydrogen atom at $\widetilde{F}=0.25$: (a) $S+$, (b) $S-$, (c) projections of the $S\perp$ orbit onto the y-x and y-z planes, on the left side for energies below and on the right side for energies above the collision. (d) Quasi-Penning orbit, S_{Sp}, centered on the Stark saddle point $\widetilde{x}_{Sp}=1/\sqrt{\widetilde{F}}=-2$.

Neglecting relativistic and center-of-mass effects the Hamiltonian for the electron of the hydrogen atom in crossed fields is given by

$$H=E=\frac{p^2}{2}-\frac{1}{r}+\tfrac{1}{2}BL_z+\tfrac{1}{8}B^2(x^2+y^2)+Fx, \qquad (1)$$

with the electric and magnetic fields F and B in the x and z directions, respectively. The angular momentum component L_z is not conserved and the potential is velocity dependent. Symmetries are the z parity, with $z=0$ as symmetry plane, and the y parity connected with time reversal. At the Stark saddle energy $E_{Sp}=-2\sqrt{F}$, the equations of motion have a fix point at $x_{Sp}=-1/\sqrt{F}$. The dynamical properties are similar for all values in E, F, and B, leading to the same scaled energy $\widetilde{E}=EB^{-2/3}$ and scaled electric field $\widetilde{F}=FB^{-4/3}$. The scaled Hamiltonian reads

$$\widetilde{E}=EB^{-2/3}=\frac{\widetilde{p}^{\,2}}{2}-\frac{1}{\widetilde{r}}+\tfrac{1}{2}\widetilde{L}_{\widetilde{z}}+\tfrac{1}{8}(\widetilde{x}^{\,2}+\widetilde{y}^{\,2})+\widetilde{F}\widetilde{x} \qquad (2)$$

with the scaled coordinate $\widetilde{r}=rB^{2/3}$ and scaled momentum $\widetilde{p}=pB^{-1/3}$.

As shown previously [6], there exist three elementary Kepler orbits of the crossed-field atom in the perturbation regime shown in Fig. 1: two ellipses, $S+$ and $S-$, located in the $z=0$ plane and the third one, $S\perp$, in the z-y plane perpendicular to the electric field. $S-$ and $S+$ rotate clockwise and anticlockwise with respect to the B direction, changing from circular in the magnetic to linear in the electric field (parallel or antiparallel). $S\perp$ varies from linear in the magnetic to circular in the electric field.

As the energy is increased from weak to strong perturbation the three orbits change shape significantly, as shown in Fig. 2 for the field strength $\widetilde{F}=0.25$, arbitrarily chosen simply because it corresponds to a scaled saddle point energy

$\tilde{E}_{Sp}=-2\sqrt{\tilde{F}}=-1$. The important point is that the orbits show qualitatively the same energy dependence, essentially independent of $\tilde{F}$. Characteristic features are as follows.

$S+$ remains quasielliptical, largly orientated towards the electric field with increasing polarization [Fig. 2(a)]. In contrast, $S-$ changes its shape [Fig. 2(b)]: The initially elliptic trajectory develops a cusp turning to a loop in the electric-field direction, then stretching in the opposite direction, developing another cusp and a second loop. Both $S+$ and $S-$ are located in the $z=0$ plane at all energies. Their shape variation is qualitatively the same for all external field combinations, except in pure fields where they remain circular or linear. $S\perp$, while in the perturbation approximation planar in the y-z plane, evolves with increasing energy in three dimensions, more and more polarized by the electric field. Figure 2(c) shows examples of orbits in z-y and x-y projections. Notably, this orbit exhibits a cusp at the origin, corresponding to a collision of the electron with the proton. In addition to these three orbits the quasi-Penning orbit S_{Sp} occurs for energies above the field-ionization threshold, centered on the saddle point and with energy increasingly polarized in the electric-field direction [Fig. 2(d)]. An unexpected property, common to all four orbits, is that they still exist in the continuum regime.

In Hill's theory the Moon is assumed to be a massless particle, the Earth is considered to evolve on a circular Kepler orbit around the Sun, and the variation of the gravitational potential of the Sun on the Moon is approximated by a quadrupole field. In the rotating frame of the Earth around the Sun the Moon's motion is described by the Hamiltonian

$$H=E=\frac{p^2}{2}-\frac{1}{r}-\underbrace{nL_z}_{a}-\underbrace{n^2\tfrac{1}{2}(3x^2-r^2)}_{b} \tag{3}$$

with the coordinate system centered in the Earth, x and y spanning the ecliptic plane, x orientated in the Sun's direction, r the Earth-Moon distance, and n the rotation frequency of the Earth around the Sun. The terms a and b describe the interaction of the two-body system Earth and Moon with the Sun.

This Hamiltonian satisfies a scaling property with respect to n, in direct analogy to the B scaling of the atomic system. The dynamics is governed by the scaled Hamiltonian

$$\tilde{E}=En^{-2/3}=\frac{\tilde{p}^{\,2}}{2}-\frac{1}{\tilde{r}}-\tilde{L}_{\tilde{z}}-\tfrac{1}{2}(3\tilde{x}^{\,2}-\tilde{r}^{\,2}) \tag{4}$$

with the scaled coordinate $\tilde{r}=rn^{2/3}$ and momentum $\tilde{p}=pn^{-1/3}$.

As in the crossed-field atom there are three nonintegrable degrees of freedom. Symmetries are the z parity and the x and y parity connected with time reversal. Also L_z is not conserved, rendering the potential velocity dependent. Two symmetric fix points at $\tilde{x}_{Lp}=\pm 3^{-1/3}$ exist, known as Lagrange points, at the escape-threshold energy $\tilde{E}_{Lp}>-3^{4/3}/2$, in analogy to the field-ionization saddle point of the atomic system.

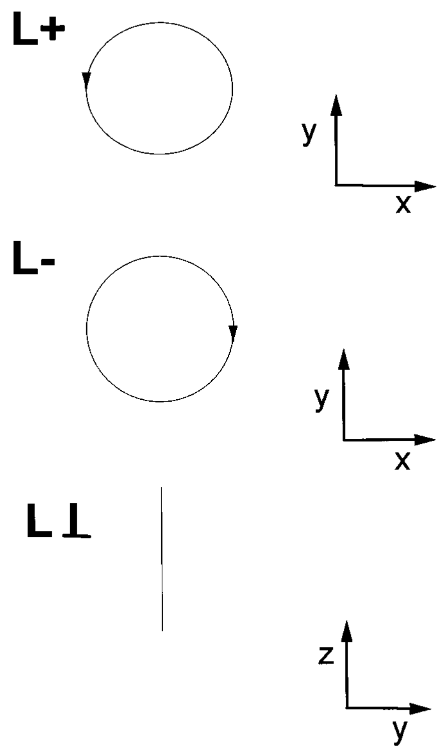

FIG. 3. The three elementary Kepler ellipses of the three-body problem Sun-Earth-Moon in Hill's approximation in the perturbative regime.

As previously pointed out by Gutzwiller [1], the terms a and b in the Hamiltonian can be taken as a paramagnetic term a, resulting from a magnetic field with $-z$ orientation and an electric quadrupole field b destabilizing in the $\pm x$ direction. In this respect the situation resembles that of the crossed-field atomic system. Differences are the missing diamagnetic term and the electric-field interaction being of a quadrupole instead of a dipole nature.

If, in a perturbative approximation, the $-(3\tilde{x}^{\,2}-\tilde{r}^{\,2})/2$ term is omitted one naturally obtains the same elementary Kepler ellipses as for the magnetized atom as shown in Fig. 3: two circular ones $L+$ and $L-$ with the angular momentum L_z parallel and antiparallel to the magnetic field and the linear one $L\perp$ along the z axis. If the electric field is taken into account, $L\perp$ remains linear, since it does not exert a force vertically to the z axis, while the orbits $L+$ and $L-$ obtain an eccentricity in the y direction. While $L+$ stays elliptic at all energies [Fig. 4(a)], $L-$ develops two symmetric cusps, which then evolve further to loops [Fig. 4(b)]. The fourth elementary orbit, L_{Lp}, consists of two symmetric orbits, existing only above the Moon escape threshold and surrounding the two Lagrange points [Fig. 4(c)].

The properties of the lunar and electron orbits naturally mirror the symmetries of the respective fields: Both the orbits $S+$, $S-$ and $L+$, $L-$ are bound to the respective symmetry planes and also show similar energy dependences. While $S+$ and $L+$ keep their elliptical or quasielliptical shape, $S-$ and $L-$ exhibit cusps and loops as characteristic and significant features. However, the cusps of $L-$ appear on the y axis symmetrically at a given energy because of the orbits' parity symmetry. On the other hand the cusps of

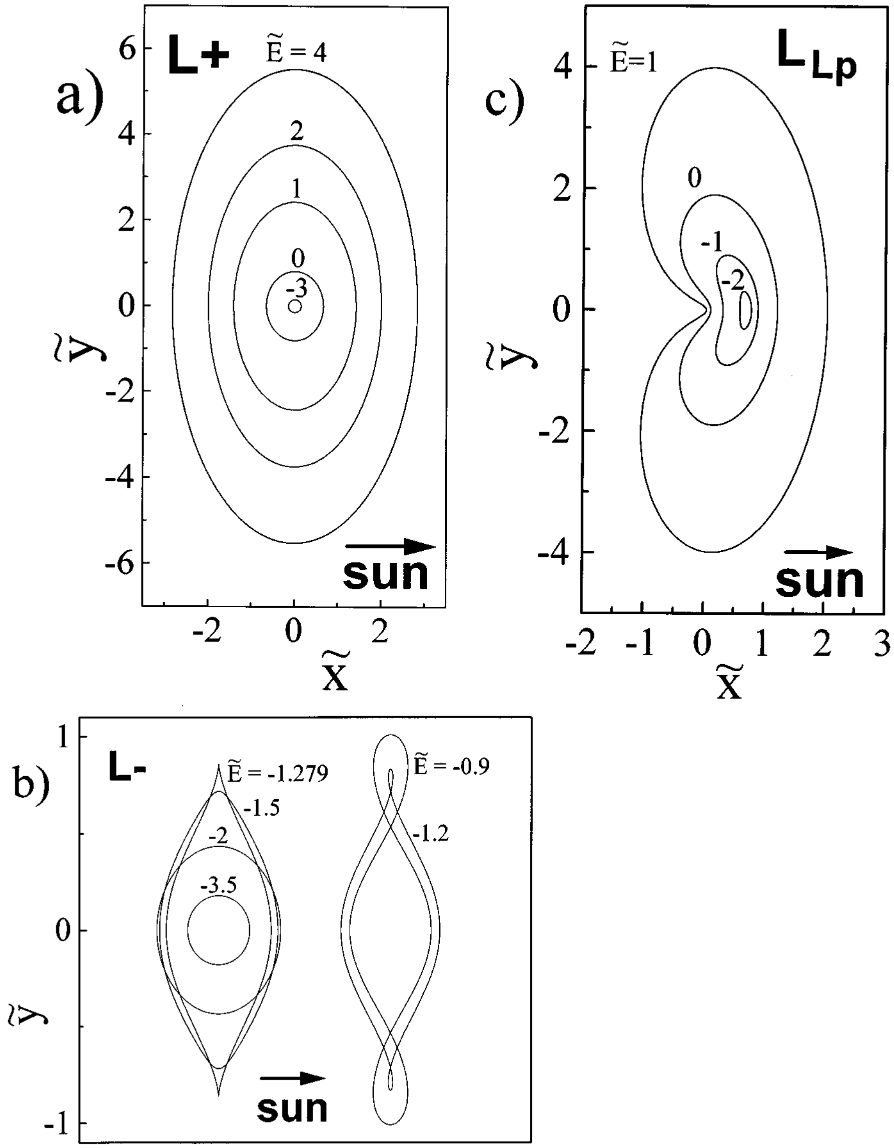

FIG. 4. Energy dependence of the elementary periodic lunar orbits: (a) $L+$, (b) $L-$, c) L_{Lp}, surrounding the Lagrange point $\tilde{x}_L = 3^{-1/3}$.

$S-$ appear on the x axis asymmetrically at different energies, because the atomic orbit is nonsymmetric with respect to the electric-field direction. As found by Strömgren in 1907 [13], such cusps and loops are a general consequence of the velocity dependence of the potential, which thus also explains their occurrence in the crossed-field atom. A substantial difference is observed for $S\perp$ and $L\perp$ resulting from the difference in the ''electric field'' forces of both systems. While no force is acting on the Moon in the x or y direction, the electron is pulled out of the y-z plane by the electric field in the x direction, moreso with higher energy. S_{Sp} and L_{Lp}, though different in shape have their common source in and are a general consequence of the saddle-point structure of the potentials and the Coriolis force caused by the nonconserved L_z angular momentum. Finally, another common property is that all elementary orbits exist up into the continuum; i.e., they are stable against ionization. However, this result gives no information about their dynamical stability and their signature in the quantum spectrum. Investigations on the effects of these orbits on the experimental photoabsorption spectrum particularly in the continuum regime are in progress.

Somewhat surprisingly the two systems show these similarities although the quadratic diamagnetic term of the atomic system is not contained in the Moon Hamiltonian. If the diamagnetic term is neglected, describing the hydrogen atom in a circular polarized microwave field [14], the four periodic orbits show the same basic qualitative structures—they are affected only quantitatively by the quadratic field term.

In summary, we have provided a first systematic study of the energy dependence of the four elementary periodic orbits governing the classical dynamics of the crossed-field hydrogen atom. We have found that the dynamics of this system is intimately connected to the Sun-Earth-Moon three-body problem in Hill's approximation. Due to analogies in the equations of motion with the characteristic velocity dependence and the saddle-point structures of the potentials, striking similarities in the energy dependence of the orbits in both systems exist.

The work supplies a starting point for further investigations of the crossed-field atom problem. Remaining basic-questions are, for instance, (a) the classical dynamical stability properties of the four elementary orbits, likely revealing further connections to Hill's theory, (b) the evolution of the global phase space dynamics involving the bifurcation-schemes of the periodic orbits, and (c) the semiclassical quantization of the system in the strong perturbation regime.

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