Information-theoretic aspects of inseparability of mixed states

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Information-theoretic aspects of quantum inseparability of mixed states are investigated in terms of the α entropy inequalities and teleportation fidelity. Inseparability of mixed states is defined and a complete characterization of the inseparable 2×2 systems with maximally disordered subsystems is presented within the Hilbert-Schmidt space formalism. A connection between teleportation and negative conditional α entropy is also emphasized. [S1050-2947(96)06908-9]

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I. INTRODUCTION

Quantum inseparability is one of the most striking features of quantum formalism. It can be expressed as follows: *If two systems interacted in the past it is, in general, not possible to assign a single state vector to either of the two subsystems* [1,2]. This is what is sometimes called the principle of inseparability. Historically it was first recognized by Einstein, Podolsky, and Rosen (EPR) [3] and by Schrödinger [4]. In their famous paper EPR suggested a description of the world (called ''local realism'') which assigns an independent and objective reality to the physical properties of the well separated subsystems of a compound system. Then EPR used the criterion of local realism to conclude that quantum mechanics is incomplete.

EPR criticism was the source of many discussions concerning fundamental differences between quantum and classical descriptions of nature. The most significant progress toward the resolution of the EPR problem was made by Bell [5] who proved that the local realism implies constraints on the predictions of spin correlations in the form of inequalities (called Bell's inequalities) which can be violated by some quantum-mechanical predictions. The latter feature of quantum mechanics called usually ''nonlocality'' [6] is one of the most apparent manifestations of quantum inseparability.

The Bell's inequalities involve correlations between the outcomes of measurements performed on the well separated systems which have interacted in the past. It emphasizes the correlation aspect of inseparability. There is another aspect which cannot be directly related to correlations but rather to the amount of information carried by quantum states. It was first considered by Schrödinger who wrote in the context of the EPR problem: ''Thus one disposes provisionally (until the entanglement is resolved by actual observation) of only a *common* description of the two in that space of higher dimension. This is the reason that knowledge of the individual systems can *decline* to the scantiest, even to zero, while that of the combined system remains continually maximal. Best possible knowledge of a whole does *not* include best possible knowledge of its parts—and that is what keeps coming back to haunt us'' [4]. In this way Schrödinger recognized a profoundly nonclassical relation between the information which an entangled state gives us about the whole system and the information which it gives us about the subsystems. It involves an information-theoretic aspect of quantum inseparability which has attracted much attention recently [7–12]. Braunstein and Caves first considered information-theoretic Bell's inequalities, and have shown that they can be violated in the region of the violation of the usual Bell's inequalities [7] (see also Ref. [8] in this context). There is another approach which is based on the notion of the index of correlation [9,10] or, more generally, quantum redundancies [11]. In particular, it has been shown [11] that for all known states admitting the local hidden variable (LHV) model the normalized index of correlation is bounded by 1/2 [11]. More general analysis in terms of the so-called α entropy inequalities [12] shows that there is a connection between the correlation aspect and the information-theoretic one involving quantum entropies.

Recently Bennett *et al.* [13] have discovered an aspect of quantum inseparability—teleportation. It involves a separation of an input state into classical and quantum parts from which the state can be reconstructed with perfect fidelity $\mathcal{F}=1$. The basic idea is to use a pair of particles in the singlet state shared by the sender (Alice) and the receiver (Bob). Popescu [14] noticed that the pairs in a mixed state could be still useful for (imperfect) teleportation. There was a question of what value of the fidelity of transmission of an unknown state can ensure us about the nonclassical character of the state forming the quantum channel. It has been shown [14,15] that a purely classical channel can give at most $\mathcal{F}=2/3$. Subsequently, Popescu showed that there exist mixtures which are useful for teleportation although they admit the local hidden variable (LHV) model. Then basic questions concerning the possible relations between teleportation, Bell's inequalities and inseparability were addressed [14]. Quite recently the maximal fidelity for the standard teleportation scheme [16] with the quantum channel formed by any mixed two spin-1/2 state has been obtained [17].

The main purpose of the present paper is to investigate a relation between inseparability of mixed states and their nonclassical information-theoretic features. In particular, we provide a complete characterization of the inseparable 2×2 sys-

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tems with maximally disordered subsystems. This paper is organized in the following way. In Sec. II we discuss the inseparability principle for mixed states. In Sec. III we present the quantum α entropy (α-E) inequalities and discuss them in the context of inseparability. In particular, we point out that separable states satisfy 1-and 2-entropy inequalities. In Sec. IV we consider in detail 2×2 systems using the Hilbert-Schmidt space formalism. In particular, we provide the necessary conditions for separability of the mixed two spin-1/2 states. In Sec. V we provide a complete characterization of the mixed two spin-1/2 states with maximal entropies of subsystems, and we single out the separable states belonging to the above class. This allows us to obtain information-theoretic characterizations of the latter in terms of the α entropy inequalities and teleportation presented in Sec. VI. The characterization is also obtained in terms of purification of noisy teleportation channels [28]. Finally we discuss the idea of purification in the context of α-E inequalities.

II. INSEPARABILITY PRINCIPLE

To make our consideration clearer it is necessary to extend the notion of inseparability. Note that the above principle of inseparability when applied to mixed states becomes inadequate. Indeed, there are nonproduct mixtures that can be written as mixtures of pure product states thus separable according to the principle. Then for clarity it is convenient to introduce the following natural generalization of the latter: *If two systems interacted in the past it is possible to find the whole system in the state that cannot be written as a mixture of product states*. It involves the existence of inseparable mixed states which may be viewed as a counterpart of pure entangled states. They correspond to the Werner's EPR correlated states, i.e., the ones which cannot be written as mixtures of direct products, while the separable states (mixtures of product states) correspond to the classically correlated ones [22]. It is natural to interpret the principle as follows. As one knows, a mixed state can in general come from a reduction of some pure state or from a source producing randomly pure states. If a mixed state is separable, then it produces the statistics equivalent to the one generated by an ensemble of product states. In the latter case, the nonfactorability is due to the lack of knowledge of the observer only. However, if a mixed state is inseparable, then there is certainly no way to ascribe to the subsystems, even in principle, their state vectors.

Now, we are interested in information-theoretic aspects of inseparability as well as in the range of their manifestations. The question is also, to what degree this range can cover the whole set of inseparable states. However it is very difficult to check whether some given state can be written as a mixture of product states or not. Then it follows that more ''operational'' characterization of inseparable (separable) states is more than desirable.

It should be emphasized here that, although the above inseparability principle says about the existence of the dynamics that can convert product state into inseparable one, we are interested in the final effect of the action of the dynamics. In other words, we assume that the system is found in an inseparable state, and the task is to investigate the effects that manifest its inseparability.

III. α ENTROPY INEQUALITIES

In this section we will outline the concept of the α entropy inequalities in the context of inseparability [12]. Let us consider the quantum counterpart of the Rényi α entropy [18,19]

$$S_\alpha(\varrho)=\frac{1}{1-\alpha}\ln \mathrm{Tr}\varrho^\alpha, \quad \alpha>1. \tag{1}$$

If α tends to 1 decreasingly, one obtains the von Neumann entropy $S_1(\varrho)$ as a limiting case

$$S_1(\varrho)=-\mathrm{Tr}\varrho\ln\varrho. \tag{2}$$

One can replace the standard information measure which is the von Neumann entropy by the whole family of α entropies [20]. Then given the compound system, one can consider the relationships between the entropy of the whole system and the entropies of the subsystems. For this purpose, consider the following inequalities:

$$S_\alpha(\varrho)\geqslant \max_{i=1,2}S_\alpha(\varrho_i), \tag{3}$$

where $\alpha\geqslant 1$, $S_\alpha(\varrho)$ denotes the entropy of the system and $S_\alpha(\varrho_i)$, $i=1,2$ are the entropies of the subsystems. The above inequalities can be interpreted as the constraints imposed on the system by positivity of the conditional α entropies if the latter are defined by

$$S_\alpha(1|2)=S_\alpha(\varrho)-S_\alpha(\varrho_2), \quad S_\alpha(2|1)=S_\alpha(\varrho)-S_\alpha(\varrho_1). \tag{4}$$

Now one can expect that violation of the α-E inequalities is a manifestation of some nonclassical features of a compound system resulting from its inseparability. Indeed, one can easily see that for discrete classical systems [21] the corresponding inequalities are always satisfied, i.e., the classical conditional α entropies are positive.

Note that the classical systems are always separable: joint distributions can always be written as convex combinations of product distributions [22]. Then it is natural to ask about the connection between the violation of the α-E inequalities and inseparability of quantum states. In fact one can prove [12]

Theorem 1. For any separable state ϱ on the finite-dimensional Hilbert space the inequality (3) is satisfied for $\alpha=1,2$.

The above theorem provides necessary conditions for separability. In particular, it turns out that the 2-E inequality is essentially stronger than the Bell-CHSH inequality for the two spin-$\frac{1}{2}$ cases [12]. Then it constitutes a nontrivial and extremely simple computationally necessary condition for separability. This is a useful result as there is still no operational criterion of inseparability, in general. As we will see further, for a class of two spin-1/2 states, it is possible to provide the criterion in terms of the α-E inequalities.

IV. POSITIVITY AND SEPARABILITY CONDITIONS IN THE HILBERT-SCHMIDT SPACE

In this section and further we will restrict our consideration to the 2×2 systems. Consequently, consider the Hilbert space $\mathcal{H}=C^2\otimes C^2$. All Hermitian operators acting on $\mathcal{H}$ constitute a Hilbert-Schmidt (HS) space $\mathcal{H}_{\rm HS}$ with a scalar product $\langle A,B\rangle=\mathrm{Tr}(A^\dagger B)$. An arbitrary state of the system can be represented in $\mathcal{H}_{\rm HS}$ as follows:

$$\varrho=\frac{1}{4}\left(I\otimes I+\boldsymbol{r}\cdot\boldsymbol{\sigma}\otimes I+I\otimes\boldsymbol{s}\cdot\boldsymbol{\sigma}+\sum_{m,n=1}^{3}t_{nm}\sigma_n\otimes\sigma_m\right). \quad (5)$$

Here I stands for identity operator, $\boldsymbol{r}$, $\boldsymbol{s}$ belong to R^3, $\{\sigma_n\}_{n=1}^3$ are the standard Pauli matrices $\boldsymbol{r}\cdot\boldsymbol{\sigma}=\Sigma_{i=1}{}^3 r_i\sigma_i$. The coefficients $t_{mn}=\mathrm{Tr}(\rho\sigma_n\otimes\sigma_m)$ form a real matrix denoted by T. Note that $\boldsymbol{r}$ and $\boldsymbol{s}$ are local parameters as they determine the reductions of the state ϱ

$$\varrho_1\equiv\mathrm{Tr}_{\mathcal{H}_2}\varrho=\tfrac{1}{2}(I+\boldsymbol{r}\cdot\boldsymbol{\sigma}),\quad \varrho_2\equiv\mathrm{Tr}_{\mathcal{H}_1}\varrho=\tfrac{1}{2}(I+\boldsymbol{s}\cdot\boldsymbol{\sigma}), \quad (6)$$

while the T matrix is responsible for correlations

$$E(\boldsymbol{a},\boldsymbol{b})\equiv\mathrm{Tr}(\varrho\boldsymbol{a}\cdot\boldsymbol{\sigma}\otimes\boldsymbol{b}\cdot\boldsymbol{\sigma})=(\boldsymbol{a},T\boldsymbol{b}). \quad (7)$$

Now we will reduce the number of the parameters that are essential for the problem we discuss in this paper. Note that inseparability is invariant under product unitary transformations, i.e., if a state ϱ is inseparable (separable) then any state of the form $U_1\otimes U_2\varrho U_1^\dagger\otimes U_2^\dagger$ also has this property. Then, without loss of generality we can restrict our considerations to some representative class of the states described by less number of parameters. The class should be representative in the sense that any state ϱ should be of the form $\varrho=U_1\otimes U_2\tilde{\varrho}U_1^\dagger\otimes U_2^\dagger$ where $\tilde{\varrho}$ belongs to the class. Consequently, denote by $\mathcal{D}$ the set of all states with diagonal T. This set is a convex subset of the set of all states. To show that it is representative, we will use the fact, that for any unitary transformation U there is a unique rotation O such that [23]

$$U\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}U^\dagger=(O\hat{\boldsymbol{n}})\cdot\boldsymbol{\sigma}. \quad (8)$$

Then if a state is subjected to the $U_1\otimes U_2$ transformation, the parameters $\boldsymbol{r},\boldsymbol{s}$ and T transform themselves as follows

$$\boldsymbol{r}'=O_1\boldsymbol{r},\quad \boldsymbol{s}'=O_2\boldsymbol{s},\quad T'=O_1TO_2^\dagger,$$

where O_i's correspond to U_i's via formula (8). Thus given an arbitrary state, we can always choose such U_1,U_2 that the corresponding rotations will diagonalize its matrix T [24], and the transformed state will belong to $\mathcal{D}$.

Now we are in position to present the conditions imposed on the parameters contained in the T matrix by positivity of ϱ and by its separability [17]. As we consider the states with diagonal T we can identify the latter with the vector $\boldsymbol{t}\in R^3$ given by $\boldsymbol{t}=(t_{11},t_{22},t_{33})$. Henceforth we will identify diagonal matrices with corresponding vectors. By the notation $T\in\mathcal{A}$, where T is a matrix and $\mathcal{A}$ is a subset of R^3, we mean that T is diagonal and the corresponding vector belongs to $\mathcal{A}$. We have

Prop. 1. For any $\varrho\in\mathcal{D}$ the T matrix given by (5) belongs to the tetrahedron $\mathcal{T}$ with vertices $\boldsymbol{t}_0=(-1,-1,-1)$, $\boldsymbol{t}_1=(-1,1,1)$, $\boldsymbol{t}_2=(1,-1,1)$, $\boldsymbol{t}_3=(1,1,-1)$.

Proof. ϱ is positive if and only if the following inequalities are satisfied:

$$\mathrm{Tr}(\varrho P)\geqslant 0, \quad (9)$$

for any projector P. Consider four projectors given by the Bell basis

$$\psi_{(2)}^1=\frac{1}{\sqrt{2}}(e_1\otimes e_1\pm e_2\otimes e_2), \quad (10)$$

$$\psi_{(4)}^3=\frac{1}{\sqrt{2}}(e_1\otimes e_2\pm e_2\otimes e_1), \quad (11)$$

where $\{e_i\}$ is the standard basis in C^2. The parameters of the above projectors in the Hilbert-Schmidt space are

$$\boldsymbol{r}_i=0,\quad \boldsymbol{s}_i=0,\quad i=0,1,2,3$$

$$T_0=\mathrm{diag}(-1,-1,-1)$$

$$T_1=\mathrm{diag}(-1,1,1)$$

$$T_2=\mathrm{diag}(1,-1,1)$$

$$T_3=\mathrm{diag}(1,1,-1). \quad (12)$$

Now as for two states ϱ and ϱ' given by $(\boldsymbol{r},\boldsymbol{s},T)$ and $(\boldsymbol{r}',\boldsymbol{s}',T')$, respectively, one has

$$\mathrm{Tr}\varrho\varrho'=\tfrac{1}{4}[1+(\boldsymbol{r},\boldsymbol{r}')+(\boldsymbol{s},\boldsymbol{s}')+\mathrm{Tr}(TT'^\dagger)]. \quad (13)$$

Then one obtains that the inequalities $\mathrm{Tr}\varrho P_i\geqslant 0$, $i=0,1,2,3$ are equivalent to the following ones:

$$1-t_{11}-t_{22}-t_{33}\geqslant 0,\quad 1-t_{11}+t_{22}+t_{33}\geqslant 0,$$

$$1+t_{11}-t_{22}+t_{33}\geqslant 0,\quad 1+t_{11}+t_{22}-t_{33}\geqslant 0.$$

Clearly, the above conditions mean that T belongs to the tetrahedron $\mathcal{T}$.

Now one can establish conditions implied by separability of a given state ϱ. Consequently, we have

Prop. 2. For any separable state $\varrho\in\mathcal{D}$ the T matrix given by (5) belongs to the octahedron $\mathcal{L}$ with vertices $\boldsymbol{o}_1^\pm=(0,0,\pm 1)$, $\boldsymbol{o}_2^\pm=(0,\pm 1,0)$, $\boldsymbol{o}_3^\pm=(0,0,\pm 1)$.

Proof. Consider the operator V given by $V\phi\otimes\tilde{\phi}=\tilde{\phi}\otimes\phi$. Note that [22] one has

$$\mathrm{Tr}VA\otimes B=\mathrm{Tr}AB. \quad (14)$$

Then it follows that for any separable state ϱ we have

$$\mathrm{Tr}V\varrho\geqslant 0. \quad (15)$$

Consider now four operators V_i, $i=0,1,2,3$ given by

$$V_i=\sigma_iV\sigma_i, \quad (16)$$

where $\sigma_0\equiv I$. As the set of separable states is $U_1\otimes U_2$ invariant, it follows from (15) that for separable state ϱ we have

$$\mathrm{Tr}V_i\varrho \geqslant 0, \quad i=0,1,2,3. \tag{17}$$

The operators V_i can be written in terms of the HS space

$$V_i=\frac{1}{4}\left(2I\otimes I-\sum_{j=1}^{3} t_{jj}^{i}\sigma_j\otimes\sigma_j\right), \tag{18}$$

where $T^i\equiv\mathrm{diag}(t_{11}^i,t_{22}^i,t_{33}^i)=-2T_i$. Then the inequalities (17) imply $T\in-\mathcal{T}$ ($\boldsymbol{t}\in-\mathcal{T}$ if and only if $-\boldsymbol{t}\in\mathcal{T}$). Combining this with Prop. 1 we obtain that for separable states with diagonal T, the latter belongs to the octahedron $\mathcal{L}=\mathcal{T}\cap-\mathcal{T}$.

V. 2×2 SYSTEMS WITH MAXIMALLY DISORDERED SUBSYSTEMS

In this section we shall deal with the states whose reduced density matrices are maximally disordered, i.e., are normalized identities. The only pure states with this property are maximally entangled ones, i.e., $U_1\otimes U_2$ transformations of the singlet state. The latter appears to be the most nonclassical of all pure states. Many of the mixtures belonging to the class of the states with maximally disordered subsystems should exhibit nonclassical properties as, if the entropies of the subsystems are maximal, we expect that the inequalities (3) should often be violated.

The states with maximally disordered subsystems are completely characterized by T matrix (we will call them further the T states). Again, we can restrict our considerations to the states with diagonal T. We prove here

Prop. 3. Any operator of the form (5) with $\boldsymbol{r},\boldsymbol{s}=0$ *and diagonal* T *is a state if and only if* T *belongs to the tetrahedron* $\mathcal{T}$.

Proof. If the operator is a state, then from Prop. 1 it follows that T belongs to the tetrahedron. Now, let T belong to the tetrahedron. Then it can be written as a convex combination of its vertices treated as matrices. Thus the operator given by T appears to be a convex combination of the projectors given by (11).

Note that the necessary condition of positivity of the operators in the HS space given by Prop. 1 is now also sufficient. The above proposition gives us complete characterization of the states with maximal entropies of the subsystems. Any such state is of the form

$$\varrho=U_1\otimes U_2\left(\sum_{i=0}^{3} p_iP_i\right)U_1^\dagger\otimes U_2^\dagger, \tag{19}$$

where U_1,U_2 are unitary transformations, P_i's are given by (11) and $\Sigma_i p_i=1$. Thus one can see that the above class appears to be a very poor one: up to the $U_1\otimes U_2$ isomorphism, the states have only one partition of unity (in the nondegenerate case).

Now it is important to know which of the considered states are inseparable. It turns out that, again, the necessary condition for separability of the states in the HS space given by Prop. 2 appears to be sufficient in the case of the considered states. Namely, we have

Prop. 4. Any two spin-1/2 state with maximally disordered subsystems and diagonal T *is separable if and only if* T *belongs to the octahedron* $\mathcal{L}$.

Proof. To prove sufficiency, note that the vertices $o_k^{\pm}$, $k=1,2,3$, of the octahedron represent some separable states. Indeed, one can easily check that they represent the states

$$\varrho_k^{\overset{+}{(-)}}=\frac{1}{2}(P_k^{\overset{+}{(+)}}\otimes P_k^{\overset{+}{(-)}}+P_k^{\overset{-}{(-)}}\otimes P_k^{\overset{-}{(+)}}), \quad k=1,2,3, \tag{20}$$

where $P_k^{\pm}$ correspond to the eigenvectors of σ_k with eigenvalues ±1. Now if T belongs to the octahedron, it can be written as a convex combination of its vertices. Thus the corresponding state can be written as a mixture of the states $\varrho_k^{\pm}$, hence it is a separable state.

It is remarkable that the above result can easily be expressed in terms of spectra of the considered states. Indeed, the octahedron represents the T states with diagonal T that have all the eigenvalues less than or equal to 1/2. As the spectrum is invariant under unitary transformations (then $U_1\otimes U_2$ invariant) we obtain

Prop. 5. Any state ϱ *with maximal entropy of the subsystems is separable if and only if* $\sigma\subset[0,1/2]$ *where* σ *is the spectrum of* ϱ.

VI. INSEPARABLE 2×2 SYSTEMS WITH MAXIMALLY DISORDERED SUBSYSTEMS: INFORMATION-THEORETIC CHARACTERIZATIONS

A. α entropy characterization

There is an interesting relation between inseparability of the T states and their nonclassical information-theoretic features. It turns out that the condition imposed on the spectrum of the states can be expressed as the amount of the classical conditions for the α entropies. Namely, we have

Prop. 6. A state with maximal entropies of subsystems is separable if and only if it satisfies the α-E *inequalities for all* $\alpha\geqslant1$.

Proof. A simple proof is based on geometrical arguments. For the T states the α-E inequalities read

$$\sum_i p_i^\alpha\leqslant 2^{1-\alpha}, \quad \alpha>1; \quad -\sum_i p_i\ln p_i\geqslant\ln 2, \quad \alpha=1, \tag{21}$$

where $\{p_i\}$ is the spectrum. Thus the set of distributions $\{p_i\}$ satisfying the inequalities is convex, hence the subset $\mathcal{E}$ of the T states with diagonal T satisfying them is also convex. Now, as the states $\varrho_k^{\pm}$ given by (20) satisfy the α-E inequalities, then all the states from the octahedron must also satisfy them. To see that there are no states beyond the octahedron with this property, it suffices to consider the states represented by the line connecting one of the vertices of $\mathcal{T}$ (e.g., the one representing the singlet state ψ_0) with the origin of the frame [25] (see Fig. 1). It is straightforward to check that a state belonging to this line satisfies the α-E inequalities if it is separable [12] (i.e., if and only if it belongs to the octahedron).

Now from $U_1\otimes U_2$ invariance of the set $\mathcal{E}$ it follows that it is invariant under the group of proper rotations of a regular tetrahedron. Then from the convexity of the considered set it follows that it must be the octahedron, i.e., we have $\mathcal{E}=\mathcal{L}$.

Thus we see that the inseparability of the 2×2 states with maximal entropies of the reductions manifests itself by violation of the α-entropy inequalities for some α, i.e., by negativity of some conditional α entropy [26]. Recently the negative conditional von Neumann entropy was considered in the context of the teleportation and superdense coding [27]. As

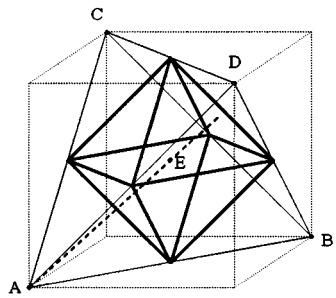

FIG. 1. Geometrical representation of the states with diagonal T and maximally disordered reduced density matrices: the bold-line-contoured octahedron represents separable states, the dashed line denotes the set of the Werner states with the singlet state A and normalized identity E. Here $A=(-1,-1,-1)$, $B=(1,1,-1)$, $C=(1,-1,1)$, $D=(-1,1,1)$.

we will see further, within the considered class of the states, the negativity of any conditional α entropy makes the state useful for teleportation.

B. Teleportation characterization

Some inseparable states have a very striking feature. Namely, they can be used for transmission of quantum information with better fidelity than by means of classical bits themselves. For example, two-particle system in pure singlet state shared by a sender and a receiver allows to transmit faithfully an unknown two spin-1/2 state, with additional use of two classical bits [13]. This is what is called quantum teleportation. In absence of the quantum channel, the only thing the sender can do is to measure the unknown state and then to inform the receiver about the outcome of the measurement by means of classical bits. Now, if in the presence of the quantum channel the receiver can reconstruct the state better than it is possible by using the best possible strategy basing only on classical bits, then we say that the state forming the quantum channel is useful for teleportation [14].

Then there is a basic question: are all the inseparable T states useful for teleportation? As we will see below, the answer is ''yes.'' As a measure of efficiency of teleportation we will use fidelity

$$\mathcal{F}=\int_S dM(\phi)\sum_k p_k \mathrm{Tr}(\varrho_k P_\phi). \tag{22}$$

Here P_ϕ is the input state, ϱ_k is the output state, provided the outcome k was obtained by Alice. The quantity $\mathrm{Tr}(\varrho_k P_\phi)$ which is a measure of how the resulting state is similar to the input one, is averaged over the probabilities of the outcomes, and then over all possible input states (M denotes uniform distribution on the Bloch sphere S). It has been shown, that the purely classical channel can give at most $\mathcal{F}=2/3$ [14,15]. Recently it has been proven [17] that within the standard teleportation scheme [16] the inequality $\mathcal{F}\leqslant 2/3$ is equivalent to the following one:

$$N(\varrho)\leqslant 1, \tag{23}$$

where $N(\varrho)\equiv\sqrt{T^\dagger T}$. Moreover, if a state is useful for the standard teleportation, then the maximal fidelity amounts to

$$\mathcal{F}_{\max}=\tfrac{1}{2}[1+\tfrac{1}{3}N(\varrho)]. \tag{24}$$

Now we observe that within the set $\mathcal{D}$ the inequality $N(\varrho)\leqslant 1$ holds if and only if T belongs to the octahedron. Thus for the T states with diagonal T this is equivalent to the separability condition given by Prop. 4. Obviously, if a state is separable then not only the standard teleportation procedure but any possible one cannot produce fidelity greater than 2/3. Then, under the $U_1\otimes U_2$ invariance of $N(\varrho)$ we obtain

Prop. 7. A two spin-1/2 state with maximal entropies of the subsystems is useful for teleportation if and only if it is inseparable.

Thus, again, inseparability of the T states manifests itself inherently by better fidelity of teleportation than the maximal one produced by a purely classical channel. It is remarkable that, within the considered class of states, the ability of forming an efficient teleportation channel is due to the negative conditional entropy for all large α. This generalizes earlier result concerning Werner spin-1/2 states (see Fig. 1) [12].

So far we have considered teleportation directly via mixed states. Recently, Bennett *et al.* (BBPSSW) [28] presented the idea of purification of noisy channels. The authors show how to obtain asymptotically faithful teleportation via mixed states using local operations and classical communication in order to purify them. The state can be purified by BBPSSW procedure if $\mathrm{Tr}\varrho P_0>1/2$ where $P_0=|\psi_0\rangle\langle\psi_0|$ is the singlet state. Of course, one can immediately see that it can be also purified if $\mathrm{Tr}\varrho P>1/2$, where P is a projector corresponding to any maximally entangled pure state. Thus by Prop. 5 we obtain that given a mixed 2×2 state with maximal entropies of the subsystems, one can distill a nonzero entanglement by using the BBPSSW procedure if the state is inseparable.

VII. CONCLUSION

We have investigated information-theoretic aspects of inseparability of mixed states in terms of the α entropy inequalities and teleportation. We have discussed some general properties of the α-E inequalities. Subsequently, using the Hilbert-Schmidt space formalism we have provided the separability conditions for 2×2 systems. Then the set of the T states (the two spin-1/2 states with maximal entropies of the subsystems) has been considered in detail.

It appears that, up to the $U_1\otimes U_2$ isomorphism, the set of the T states can be identified with some tetrahedron $\mathcal{T}$ in R^3, whereas the separable T states can be identified with an octahedron contained in $\mathcal{T}$. The above very illustrative geometrical representation of both sets allowed to obtain information-theoretic interpretation of inseparability of the considered states. Namely, it appears that the states lying beyond the octahedron violate α entropy inequalities for all large α. The resulting negative conditional α entropy has its reflection in the fact that all the inseparable T states are useful for teleportation. In addition, they have nonzero distillable entanglement, i.e., one can use them for asymptotically faithful teleportation by using the BBPSSW procedure.

Finally we believe that the results of the present paper can help in deeper understanding of the connections between the inseparability and the quantum information theory. Moreover, they may be also useful in the problem of the classification of mixed states under the nonlocality criterion [14,29,30].

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