Probabilities for histories in nonrelativistic quantum mechanics

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An effort is continued to define quantum probabilities for continuous histories. The sum-over-paths approach used in Prog. Theor. Phys. **87**, 77 (1992) is critically reviewed, improved, and applied again. Consistency between wave nature and particle nature is the criterion used to judge the definability of the probabilities and is formulated as the path classifiability condition (C1) and the no-interference condition (C2). A set of classes of histories satisfying these two conditions is considered as a space-time analog of observables. In particular, Feynman's paths for a nonrelativistic particle are considered as histories and the definability of probabilities for classes of paths is investigated, where classes are defined by classifying paths according to their behavior with respect to a rectangular space-time region $\Omega \equiv \Delta X \times \Delta T$. Confining ourselves to cases where C1 is satisfied, we examine C2 for some sets of classes of paths. Although C2 does not hold in general, some examples are found where C2 holds. In all the examples, the initial wave function is restricted. In some examples, the location of Ω and/or the potential in the region also affect the success of C2. Due to these restrictions, the resultant probabilities for classes of paths take reasonable values. A puzzling example in which probabilities cannot be defined for histories contrary to intuition is resolved by considering an appropriate coarse graining of classes of paths. Considering a rectangular potential barrier, we show that the reflection and the transmission probabilities are special cases of probabilities for histories, and also that probability densities of transmission and reflection times cannot be defined. This study may be taken to be a study of the consistent-histories approach *not* with discrete histories defined by products of projection operators but with continuous histories defined by Feynman's paths in configuration space.

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I. INTRODUCTION

Quantum mechanics (QM) incorporates wave-particle duality into the theory. From a particle point of view, we have the following question.

Question 1: What is the probability of finding a particle in a space-time region? This appears to be a very simple-minded question. However, in QM, the absence of the notion of *a real physical path* along which a particle actually moves makes this question difficult to answer in a unique way. Actually, Bloch and Burba [1] showed that the probability in question takes different values depending on how the particle detection in the space-time region is designed. A similar conclusion was also obtained recently by Marolf [2]. One may feel that, among formulations of QM, Feynman's sum-over-paths formulation is better suited to answering the question. Actually, Feynman [3] himself considered the possibility of defining ''the probability that the path lies in a particular region R of space-time'' when he was developing the idea of sums over paths. This motivates us to pose the following question, which sounds more general than Question 1.

Question 2: Is it possible to define probabilities for histories of a particle? If possible, what are the values of the probabilities? This is the question we deal with in this paper.

Although the above question sounds natural from a particle point of view, it is relatively recently that clear and agreeable frameworks have appeared in which Question 2 can be answered. They are the *consistent-histories approach* constructed by Griffiths [4] with extensions by Omnès [5] and the *generalized quantum mechanics* [6,7] by Gell-Mann and Hartle. Although they are not exactly the same, the basic ideas are essentially the same, and we shall often refer to these frameworks simply as the ''histories approach.'' The histories approach has been of growing interest for several years. When seeing Question 2, one may naturally imagine continuous histories. However, most of the studies of the histories approach have been on *discrete* histories, which are snapshots of continuous histories at different moments of time. The consistent-histories approach deals only with discrete histories. The generalized quantum mechanics is capable of dealing with continuous histories, but most of the studies have focused on discrete histories. It is safe to say that the definability of probabilities for continuous histories and the interpretation of the resultant probabilities have been less well understood than that for discrete histories. In view of this, we investigate Question 2 with continuous histories of a nonrelativistic particle in one-dimensional space. Even this simplest model would help us understand how the histories approach works for continuous histories and what the resultant probabilities mean. The cases of continuous histories of a relativistic particle and those in a nonabelian gauge theory have been studied recently by Whelan [8]. We confine ourselves to nonrelativistic cases to work on a firm platform where the probability interpretation of the conventional theory is well established.

The histories approach does not always define probabilities for histories. The condition to determine when probabilities can be defined is called the *consistency condition* in the consistent-histories approach and the *decoherence condition*

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in the generalized quantum mechanics. Among several versions of the decoherence condition [9], the ''weak'' decoherence condition was also arrived at independently, but later, by the present author and Takagi [10], who called it the ''no-interference condition.'' This condition was used to search for probability-definable examples in Refs. [10–12], and some of such examples were actually found in [11,12], where the success of the no-interference condition was the result of restricting the initial condition of the particle of interest. However, those examples were not fully satisfactory (too special or a failure of normalization), the general structure of the approach was not necessarily clear, and the interpretation given of the resultant probabilities lacked generality. The aim of this paper is to overcome these shortcomings and to apply the framework to an unsettled problem in quantum physics, that is, the so-called tunneling time problem [13]. This is worthwhile in today's situation where the histories approach is of growing interest, but not much has been studied about probabilities for continuous histories.

From a wider point of view, the problem we deal with is one of ''nonlocal-in-time'' problems in QM such as the tunneling time problem, continuous measurements, quantum mechanical Zeno's paradox, quantum mechanical retrodiction, and so on, which are all ultimately related to the special role played by time in QM. To relate the present study to all of these issues is beyond the scope of this paper, and we treat one of them, the tunneling time, as an interesting application of the present method. We also point out that the no-interference condition can be understood as a consistency condition between wave nature and particle nature because, when the condition is satisfied, the superposition principle (wave nature) and the probability sum rule (particle nature) do not ''conflict'' with each other. Although this way of understanding the condition does not play any practical role (at least in this paper), it extracts the essence of our approach (and hence the essence of the histories approach) in simple physical terms, and it is of interest from the wider viewpoint, since consistency between wave and particle natures is often the core of nonlocal-in-time problems.

We proceed as follows. In Sec. II we critically review to reorganize the approach used in Refs. [10–12]. Though we have not mentioned it yet, there is another condition called the ''path classifiability condition.'' The definability of probabilities for histories is judged by checking whether or not the path classifiability condition (denoted by C1) and the no-interference condition (denoted by C2) hold. We try to clarify how the conditions are arrived at. We emphasize that a set of classes of histories satisfying the two conditions is a space-time analog of observables in Hilbert space. We also point out that the combination of the two conditions can be said to be a consistency condition between wave nature and particle nature. We restrict our attention to cases where C1 holds. In Sec. III we check C2 for some sets of classes of histories defined by their behavior with respect to a rectangular space-time region. To make connections with ''space-time coarse graining'' of histories discussed in the generalized quantum mechanics, we begin with a set of classes of paths which consists of as many classes as possible but still allows a situation where C2 holds. Section III A illustrates the method of calculations. The so-called branch wave functions are introduced and they provide clearer perspectives on the problem. Some properties of branch wave functions are discussed in Sec. III B. Section III C summarizes the probability-definable examples which the present author has found so far. In all examples, the initial condition of the particle is appropriately restricted in such a way that C2 holds. The examples include a case where a coarse graining results in the fulfillment of C2 (Sec. III C 3), cases where restrictions other than those on the initial condition also play important roles in the fulfillment of C2 (Sec. III C 4), and a case of a particle tunneling through a rectangular potential barrier (Sec. III C 6), where we find that the tunneling probabilities (the reflection and the transmission probabilities) are special cases of probabilities for continuous histories. In Sec. IV, we consider the meaning of probabilities for continuous histories. The tunneling example is very important for this purpose. We conclude that, in general, probabilities for continuous histories defined by our approach cannot be interpreted as probabilities that a particle actually follows the histories. In Sec. V the approach is applied to the tunneling time problem. Calculating interferences between classes of histories spending different times in the barrier region, we conclude that the probability densities of transmission and reflection times cannot be defined within the present framework, and hence within the framework of the histories approach, which is true no matter how we restrict the initial condition of the particle. A summary and concluding remarks are given in Sec. VI.

Although we follow the procedures of Refs. [10–12], results obtained in this paper may also be seen, unless otherwise stated, as consequences of the generalized quantum mechanics with the weak (often the medium) decoherence condition.

II. FRAMEWORK

A. The path classifiability condition and the no-interference condition

In this subsection we attempt to construct a framework which can answer Question 2, without assuming any knowledge of the histories approach. If the attempt succeeds, we will be able to define probabilities for histories, and we may then call histories ''events,'' at least in a mathematical sense. But even in the construction of the framework, it is convenient to let histories correspond to ''events'' for explanatory purpose. As histories in QM, we consider Feynman's virtual paths, i.e., the space-time paths used in configuration space path integrals for a particle. A single path (with appropriate boundary conditions) then corresponds to a simple (or indecomposable) event, and all paths (satisfying the boundary conditions) corresponds to the whole sample space. We investigate the definability of probabilities for *classes* of paths, not for individual paths. (When there is no risk of confusion, we will simply say ''probabilities for paths'' instead of ''probabilities for classes of paths.'') A class of paths corresponds to a compound event or a union of events. All the classes again correspond to the whole sample space if a path belonging to a class does not belong to any other classes; we deal with such classes. Various geometrical regions in space-time may be used to define classes of paths. For example, if we classify all paths connecting two space-time points $A=(X_A,T_A)$ and $B=(X_B,T_B)$ $(T_A<T_B)$ according to

whether or not they enter $\Omega \equiv \Delta X \times \Delta T$ at least once, we may consider two classes of paths, $\mathscr{C}_Y$ and $\mathscr{C}_N$, where $\mathscr{C}_Y$ is the class of paths which enter Ω at least once and $\mathscr{C}_N$ the class of paths never entering the region. (Read Y and N as ‘‘yes’’ and ‘‘no,’’ respectively.) The set of the two classes, $\{\mathscr{C}_Y, \mathscr{C}_N\}$, corresponds to one possible representation of the whole sample space. Finer classifications of the paths define more than two classes, and the set of all classes is written as $\{\mathscr{C}_j\}$, where j runs integers (for simplicity, we assume that classes are countable). In the case of $\{\mathscr{C}_Y, \mathscr{C}_N\}$, identify, say, $\mathscr{C}_1 = \mathscr{C}_Y$ and $\mathscr{C}_2 = \mathscr{C}_N$. Question 2 asks if probabilities $\{P_j\}$ are definable for classes of paths $\{\mathscr{C}_j\}$.

Apparently, this is too vague a question, and there should be various definitions. But whatever the definition is, $\{P_j\}$ must obey the probability axioms

$$P_j \geqslant 0, \tag{2.1a}$$

$$P_{j \vee k} = P_j + P_k \quad (j \neq k), \tag{2.1b}$$

$$\sum_j P_j = 1, \tag{2.1c}$$

where $P_{j\vee k}$ is the probability for the class $\mathscr{C}_{j\vee k}$ defined by the sum of classes $\mathscr{C}_j$ and $\mathscr{C}_k$. In addition to the probability axioms, let us require

$$\lim_{\Omega \to \Delta X} P_Y = \int_{\Delta X} dX |\Psi(X,T)|^2, \tag{2.2}$$

where the limit lets ΔT (the temporal interval of Ω) vanish, and T is the time at which the resultant spatial region ΔX is referred to. We shall call these two requirements, (2.1) and (2.2), the ‘‘minimal requirements.’’ The question which we deal with is then as follows.

Question 3: Is it possible to define quantities $\{P_j\}$ for classes of paths $\{\mathscr{C}_j\}$ in such a way that the minimal requirements are satisfied?

There still may be various definitions. One natural definition is motivated by the following observation. Let ω and $\overline{\omega}$, respectively, represent a spatial region ΔX at time T and the spatial region $\overline{\Delta X}$ also at T, where $\overline{\Delta X}$ is (the entire space) $-\Delta X$. Just as we considered $\mathscr{C}_Y$ and $\mathscr{C}_N$ by classifying paths with respect to a space-time region Ω, we consider $\mathscr{C}_y$ and $\mathscr{C}_n$ by classifying paths with respect to a spatial region ω. Let $\mathscr{C}_y$ and $\mathscr{C}_n$, respectively, be the class of paths which intersect ω on the way from A to B and the class of paths which intersect $\overline{\omega}$ on the way. Let us define the ‘‘sum-over-paths amplitude for class $\mathscr{C}_y$’’ by

$$\Phi_y(B;A) \equiv \sum_{B \leftarrow \omega \leftarrow A} \exp(iS/\hbar), \tag{2.3}$$

where the sum is over the paths in class $\mathscr{C}_y$ (see Fig. 1), and S is the action; the sum Σ is to be understood as being defined by Feynman's path integral as the well-known infinite dimensional integrals [3,14]. This sum over paths can be evaluated by factoring the paths across the surface of constant time T, noting that each path intersects the surface once and only once, giving

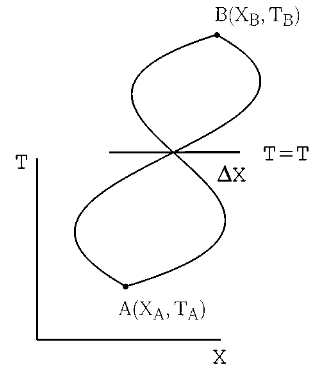

FIG. 1. Typical paths contributing to (2.3).

$$\Phi_y(B;A) = \int_{\Delta X} dX \Phi(B;X,T)\Phi(X,T;A), \tag{2.4}$$

where $\Phi(X'',T'';X',T')$ is the result of the sum over all paths from (X',T') to (X'',T'') and is therefore the Feynman kernel, or the propagator. The sum-over-paths amplitude for class $\mathscr{C}_n$, denoted by $\Phi_n(B;A)$, is defined in the same way as the sum over the paths going through $\overline{\omega}$. We define the sum-over-paths amplitude for class $\mathscr{C}_{y\vee n}$ by

$$\Phi_{y\vee n}(B;A) = \Phi_y(B;A) + \Phi_n(B;A), \tag{2.5}$$

which is finally equal to the propagator $\Phi(B;A)$, but what is important here is the property that, when classes of paths are combined into one class, the sum-over-paths amplitude for the resultant class is defined by the superposition of respective sum-over-paths amplitudes. Now, let $\mathscr{C}$ be $\mathscr{C}_y$, $\mathscr{C}_n$, or $\mathscr{C}_{y\vee n}$, and define a positive quantity for $\mathscr{C}$ by

$$P_{\mathscr{C}} \equiv \int dX_B \left| \int dX_A \Phi_{\mathscr{C}}(B;A)\Psi(A) \right|^2, \tag{2.6}$$

where $\Psi(A)$ is the wave function at time T_A, which we call the initial wave function. By using the explicit expressions for $\Phi_{\mathscr{C}}(B;A)$ [such as (2.4)] with the following property of the propagator:

$$\int dX_B \Phi^*(B;X,T)\Phi(B;X',T) = \delta(X-X'), \tag{2.7}$$

we find that (2.6) gives

$$P_y = \int_\omega dX |\Psi|^2, \quad P_n = \int_{\overline{\omega}} dX |\Psi|^2,$$

$$P_{y\vee n} = P_y + P_n, \tag{2.8}$$

where $\Psi = \Psi(X,T)$, and $P_y \equiv P_{\mathscr{C}_y}$, $P_n \equiv P_{\mathscr{C}_n}$, and $P_{y\vee n} \equiv P_{\mathscr{C}_{y\vee n}}$. At first glance, the substitution of (2.5) into (2.6) for $\mathscr{C} = \mathscr{C}_{y\vee n}$ appears to give a cross term violating the sum rule, but the cross term vanishes due to (2.7) and the sum rule is not violated. Equation (2.8) explicitly shows that P_y, P_n, and $P_{y\vee n}$ satisfy the probability axioms, provided that Ψ, and therefore $\Psi(A)$, are normalized in the usual way, i.e.,

$$\int dX_A |\Psi(A)|^2 = 1. \tag{2.9}$$

Let us call probabilities such as (2.8) *spatial* probabilities, emphasizing their relevance to spatial regions. What has been shown is that we may use (2.6), in lieu of (2.8), as the formula for spatial probabilities.[1] We call (2.6) the *sum-over-paths construction* of spatial probabilities and (2.8) the conventional construction. An essential difference between the two constructions is that probabilities are added in the conventional construction as seen in (2.8), whereas amplitudes are added in the sum-over-paths construction as seen in (2.5). This difference has appeared as the difference in how the sum rule comes out. Remember that (2.7) is essential for the recovery of the sum rule in the sum-over-paths construction. The importance of (2.7) in the sum-over-paths construction is also seen from the following point of view. The probability density that a particle is found at position X at time T, namely, $|\Psi(X,T)|^2$, is expressed as

$$\lim_{\Delta X \to 0} \frac{1}{\Delta X} \int dX_B \left| \int dX_A \left(\sum_{B \leftarrow \Delta X \leftarrow A} e^{iS/\hbar} \right) \Psi(A) \right|^2, \tag{2.10}$$

where ΔX is supposed to be taken around X at time T and the sum over paths is identical to (2.3). At first sight, $\int dX_B |\int dX_A \cdots|^2$ appears to be of the order of $(\Delta X)^2$, but it is actually of the order of ΔX due to (2.7), which guarantees that the resultant probability density is not identically zero. Also note that (2.7) guarantees that $\int dX_B |\int dX_A \cdots|^2$ gives the probability of finding a particle in ΔX at time T *regardless of* the width of ΔX.

Now, what if we replace ω with Ω in the preceding paragraph to define positive quantities $P_Y \equiv P_{\mathscr{C}_Y}$, $P_N \equiv P_{\mathscr{C}_N}$, and $P_{Y\vee N} \equiv P_{\mathscr{C}_{Y\vee N}}$ as straightforward generalizations of P_y, P_n, and $P_{y\vee n}$, respectively? Since P_Y reduces to P_y as Ω approaches ω, one of the minimal requirements is manifestly satisfied by this generalization.[2] All that is left is then to check whether or not the positive quantities satisfy the probability axioms. Let us formulate this idea.

The sum-over-paths amplitude for a class of paths $\mathscr{C}$ is generally defined by

$$\Phi_{\mathscr{C}}(B;A) \equiv \sum_{\mathscr{C}} \exp(iS/\hbar), \tag{2.11}$$

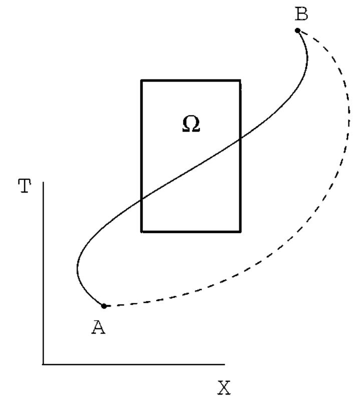

FIG. 2. Solid curve is a typical path contributing to $\Phi_{\mathscr{C}_Y}(B;A)$ and the broken one is that to $\Phi_{\mathscr{C}_N}(B;A)$.

where the sum is over the paths in class $\mathscr{C}$ (although not written out explicitly, paths are from A to B). Let now $\mathscr{C}$ be either $\mathscr{C}_1 = \mathscr{C}_Y$, $\mathscr{C}_2 = \mathscr{C}_N$, or $\mathscr{C}_{1\vee 2} (= \mathscr{C}_{Y\vee N})$. The sums for $\mathscr{C}_1$, $\mathscr{C}_2$, and for $\mathscr{C}_{1\vee 2}$ are, respectively, over those paths that enter Ω, avoid Ω, and go through $\Omega \cup \bar{\Omega}$ (see Fig. 2), where $\bar{\Omega} \equiv \overline{\Delta X} \times \Delta T$, and "to enter Ω" means to enter ΔX at least once between ΔT, and "to avoid Ω" means never to enter ΔX during ΔT. We define positive quantities $P_j \equiv P_{\mathscr{C}_j}$ and $P_{j\vee k} \equiv P_{\mathscr{C}_{j\vee k}}$ $(j \neq k)$ by substituting (2.11) into (2.6).[3] Let us check the probability axioms. Positivity is obvious. To check the sum rule, we use

$$\Phi_{j\vee k}(B;A) = \Phi_j(B;A) + \Phi_k(B;A) \quad (j \neq k), \tag{2.12}$$

where $\Phi_j \equiv \Phi_{\mathscr{C}_j}$ and $\Phi_{j\vee k} \equiv \Phi_{\mathscr{C}_{j\vee k}}$, to find that $P_{j\vee k} = P_j + P_k + 2\,\mathrm{Re}D[j;k]$, where

$$D[j;k] \equiv \int dX_B \int \int dX_A dX_{A'} \Phi_j^*(B;A) \Phi_k(B;A') \times \Psi^*(A) \Psi(A'), \tag{2.13}$$

which is nothing but the *decoherence functional* introduced by Gell-Mann and Hartle [6] in more general contexts. The sum rule is violated unless $\mathrm{Re}D = 0$. At this stage we have two choices. One is to give up further investigations, for the sum rule is generally violated. Another choice, which the author believes more interesting, is to investigate the possibility of the recovery of the sum rule in appropriately restricted situations. The condition for the recovery of the sum rule is

$$\mathrm{Re}D[j;k] = 0 \quad (j \neq k). \tag{2.14}$$

This is the weak decoherence condition in the generalized quantum mechanics, and is called the no-interference condition and denoted by C2 in Refs. [10–12]. Last, the normalization can be proved as follows:

[1] Although $\Phi_{\mathscr{C}}(B;A)$ contains information about the future of ω (i.e., information later than T), (2.6) does not depend on the future information as (2.8) shows explicitly. This property is called the "future indifference" in the generalized quantum mechanics and was also pointed out in [15].

[2] Not only do P_y, P_n, and $P_{y\vee n}$ satisfy the probability axioms, they are also physical probabilities. For example, P_y is the probability for the physical event y that the particle is found in ΔX at time T. In general, classes of paths correspond to physical events through (2.6) when classes are defined by path classifications across constant time surfaces. It is then natural to expect that classes of paths would also correspond to some physical events through (2.6) even when classes are defined by more general path classifications. What is behind the minimal requirements is this expectation.

[3] $P_{\mathscr{C}}$ thus defined also have the property of "future indifference." More generally, the decoherence functional (2.13) has this property. They do not depend on information about the future of Ω.

$$\sum_j P_j = \int dX_B \left| \int dX_A \left(\sum_j \Phi_j(B;A) \right) \Psi(A) \right|^2$$
$$= \int dX_B \left| \int dX_A \Phi(B;A)\Psi(A) \right|^2$$
$$= \int dX_B |\Psi(B)|^2 = 1, \qquad (2.15)$$

where the first equality assumes the fulfillment of C2. In passing from the first line to the second, we have used

$$\sum_j \Phi_j(B;A) = \Phi(B;A), \qquad (2.16)$$

which is because we just classified paths into classes and thus the classes reproduce all the paths when they are recombined. Thus the proof of the probability axioms formally ends. However, note that we have not shown yet that the path classification in question is actually possible, in other words, that $\Phi_j(B;A)$ can be meaningfully defined. In the case of $\{\mathscr{C}_Y, \mathscr{C}_N\}$, and also in all cases we will study in later sections, $\Phi_j(B;A)$ *are* meaningfully defined. In general, however, a bit of caution is necessary because Feynman's virtual paths, like Brownian paths, are not differentiable with respect to time (see, e.g., p. 177 of Ref. [14]); more precisely, virtual paths which are differentiable in time are of zero measure and thus do not contribute to a path integral. As a result, we cannot make a meaningful classification of paths according to, for example, how many times they cross Ω, because they may cross the ''temporal boundaries'' of Ω (by which we mean the ones of constant X) even an infinite number of times. Thus the sum-over-paths amplitude $\Phi_j(B;A)$ for crossing the space-time region for a finite number of times j vanishes, so that $\Phi_j(B;A)$ do not add up to the total propagator $\Phi(B;A)$, i.e., (2.16) fails. Equation (2.16) can thus be viewed as the condition for the path classification in question to be meaningful. Following Refs. [10–12] we call (2.16) the path classifiability condition and denote it by C1.[4] The reason C1 holds for $\{\mathscr{C}_Y, \mathscr{C}_N\}$ is because ''to enter Ω'' and ''to avoid Ω'' are ''coarser'' as information than how many times paths cross the temporal boundaries of Ω. In general, C1 fails if the path classification in question refers to the number of times paths cross the temporal boundaries, or to finer information than that. Such cases were studied in Refs. [16,10], where surfaces on which time is not constant were used to define classes of paths. In later sections, we consider such cases where C1 holds, and thus only C2 is our actual concern.

Although we have illustrated the framework with classes of paths labeled by discrete label j, the extension to cases of continuously labeled classes is straightforward, but with a bit of caution on the form of the probability density formula. Let λ be the continuous label. Dividing the λ axis into nonoverlapping regions $\Delta\lambda_j$, we have discrete classes $\{\mathscr{C}_j\}$. Writing C1 and C2 for the discrete classes and taking the limit $\Delta\lambda_j \to 0$, we find that C1 and C2 for the continuously labeled classes $\{\mathscr{C}_\lambda\}$ are, respectively,

$$\int d\lambda \Phi_\lambda(B;A) = \Phi(B;A), \qquad (2.17a)$$

$$\mathrm{Re}D[\lambda;\lambda'] = \delta(\lambda - \lambda')P_\lambda, \qquad (2.17b)$$

where P_λ, which may be called a probability density if C1 and C2 hold, is given by

$$P_\lambda = \lim_{\Delta\lambda \to 0} \frac{1}{\Delta\lambda} \int dX_B \left| \int dX_A \int_{\Delta\lambda} d\lambda \Phi_\lambda(B;A)\Psi(A) \right|^2, \qquad (2.18)$$

where $\Delta\lambda$ is taken around λ and $\Phi_\lambda(B;A)$ is the sum over the paths in class $\mathscr{C}_\lambda$. Note that P_λ is *not* $\int dX_B |\int dX_A \Phi_\lambda(B;A)\Psi(A)|^2$, which gives incorrect results when λ is the position of intersection X of a constant time surface [see (2.10)]. If C2 holds, $\int dX_B |\int dX_A \cdots|^2$ is of the order of $\Delta\lambda$ and thus the $\Delta\lambda \to 0$ limit does not vanish identically. If C2 fails, it is of the order of $(\Delta\lambda)^2$ and thus $P_\lambda \equiv 0$. Continuously labeled classes of paths are treated in Sec. V B. In cases where paths are classified into more than two classes, C2 is checked for *all* pairs of classes; it is only when ReD vanishes for all pairs that C2 holds. If ReD vanishes for some pairs but not for the others, we do not define probabilities for any classes.

To summarize, we have reached the following answer to Question 3: We can define probabilities for classes of paths according to Eq. (2.6), provided that both the path classifiability condition C1 and the no-interference condition C2 hold. We do not define probabilities if either of the two conditions fails. In cases where paths are divided into more than two classes, C2 is checked for *all* pairs of classes; it is only when ReD vanishes for all pairs that C2 holds. If ReD vanishes for some pairs but not for the others, we do not define probabilities for any classes.

B. The status of the conditions: Assuring a consistency between wave and particle natures and defining a space-time analog of observables

Before going to explicit examples, it is instructive to recall how the two conditions came to hold in the case of $\{\mathscr{C}_y, \mathscr{C}_n\}$ (more generally in cases where classes of paths are defined by path classifications across constant time surfaces). The success of C1 was the result of

$$\int dX \Phi(B;X,T)\Phi(X,T;A) = \Phi(B;A), \qquad (2.19)$$

which is the composition law of the propagator across a surface of constant time, or equivalently, the classification of the paths connecting A and B with respect to intersection X of an intermediate surface of constant time T. The nondifferentiable property of paths does not prevent us from classifying paths in this way, because they move only forward in time to cross the surface once and only once. Equation (2.19) is an alternative expression of

$$\int dX |X\rangle\langle X| = 1, \qquad (2.20)$$

[4] In the generalized quantum mechanics, Eq. (2.16) appears as the operator equation $\Sigma_\alpha C_\alpha = \exp(-iHT/\hbar)$, where H is the Hamiltonian and C_α are the class operators, which in the position representation correspond to our sum-over-paths amplitudes.

where the right-hand side represents the identity operator in Hilbert space. Condition C1, a generalization of (2.19), is therefore a generalization of the completeness (2.20) to cases of classes of continuous paths. We may alternatively call C1 the ''completeness condition.'' Next, it is Eq. (2.7) which made C2 hold for $\{\mathscr{C}_y,\mathscr{C}_n\}$, and is an alternative expression of

$$\int dX\langle X|X'\rangle=\delta(X-X'). \tag{2.21}$$

Condition C2, a generalization of (2.7), is therefore a generalization of the orthogonality to cases of classes of continuous paths. We may alternatively call C2 the ''orthogonality condition.''

In this way, C1 and C2 generalize the completeness and the orthogonality of state vectors in Hilbert space to cases of classes of continuous paths in space-time. We may say, with some abuse of language, that the two conditions define a space-time analog of observables (the word ''observables'' means observables in Hilbert space); a set of classes of paths satisfying the two conditions might be said to be a space-time analog of observables. Eigenvalue problems of observables give real eigenvalues and they directly correspond to possible outcomes of measurements. By contrast, it is not clear how classes of paths satisfying C1 and C2 can be related to measurements in a general way, although probabilities are definable for them. Nevertheless, a set of classes of paths satisfying the two conditions is expected to have *some* physical meaning, since (2.6) with C1 and C2 is a natural generalization of physical probabilities (2.8). To examine this expectation, we will study some explicit examples in Sec. III.

It should be emphasized that the decoherence functional (2.13), and therefore condition C2, have nothing to do with outer degrees of freedom, i.e., environments.[5] This is because C2 is a straightforward generalization of (2.21). When we read $\mathrm{Re}D[j;k]=0$ as ''there is no interference between two classes of paths,'' we must be aware that our usage of the word ''interference'' is different from the normal one. That is, the absence of interference between classes of paths meant by $\mathrm{Re}D=0$ is *not* the one that is achieved by coarse graining (tracing out) the variables of environments. Rather, it is a property of the system itself (a particle here), just as orthogonality (2.21) is a property of the system itself in the sense that it holds regardless of external degrees of freedom. To emphasize this, we may say that C2 is the condition for classes of paths to be *intrinsically noninterfering*. Condition C1 also does not refer to external degrees of freedom. Thus a set of classes of paths satisfying C1 and C2, a space-time analog of observables, is an intrinsic property of the system, just as ''an observable of a system'' is an intrinsic property of the system itself (although environments play important roles when it comes to how the property intrinsic to the system is probed and recorded in the process of a measurement; the intrinsic property itself is irrelevant to environments).

Condition C2 is mathematically the condition that Eq. (2.12)—the principle of superposition in the context of sum over paths—does not result in the violation of the sum rule (2.1b). Since the principle of superposition is a reflection of wave nature of a quantum mechanical particle and the sum rule is a reflection of particle nature, C2 is physically taken to be a consistency condition between wave nature and particle nature in the space-time region used to define classes of paths. More precisely, the combination of C1 and C2 should be said to be a consistency condition between the two natures, because both of them were assumed to hold in the proof of normalization of the probabilities.

C1 may be regarded as the ''normalization'' condition on sum-over-paths amplitudes and (2.12) as representing the ''exclusiveness'' at the level of sum-over-paths amplitudes. We may therefore say that a set of classes of paths with C1 and (2.12) is an ''exhaustive set of mutually exclusive classes of paths,'' in which, however, note that the words ''exhaustive'' and ''exclusive'' are used at the level of amplitudes but not at the level of probabilities or events. This terminology is often used in the generalized quantum mechanics, but is not used in this paper. Instead, we would like to reserve the words exhaustive and exclusive for use at the level of probabilities or events, not at the level of amplitudes. If both C1 and C2 hold for a set of classes of paths, we regard the set of classes of paths as an *exhaustive set of mutually exclusive events* (ESEE), because quantities satisfying the probability axioms are considered as being defined for some ESEE and because C1 and C2 together guarantee that quantities (2.6) defined for the set of classes of paths satisfy the probability axioms. Roughly speaking, if C2 holds, an ''exhaustive set of mutually exclusive classes of paths'' turns into an ESEE in the sense that quantities (2.6) come to satisfy all the probability axioms. In general, however, we have to distinguish the two notions—an ESEE and an exhaustive set of mutually exclusive classes of paths—because C2 does not hold in general. One (maybe only one) exceptional case is when the classes of paths are defined according to where paths intersect a surface of constant time; in this case C2 holds identically as shown in Sec. II A and hence we may identify the two notions.

Looking back, we have started from virtual paths and arrived at a certain scheme to define probabilities for classes of paths. This means that we have been in effect engaged in the construction of a method which judges whether or not a set of classes of paths is an ESEE by using virtual paths instead of real physical paths.

[5]This is a bit of an exaggeration; environments can affect C2 through the preparation of the initial wave function $\Psi(A)$ and a potential is an idealization of some environments. It is more accurate to say that C2 is written fully in terms of variables of the system.

[6]This statement is intended for the present model. If we have a particle coupled to environments, we may write down the decoherence functional for the total system. Integrating out the variable of the environments, we arrive at a reduced decoherence functional for the distinguished particle. In this way, a certain type of ''environmental no-interference'' can be discussed with decoherence functionals in general; see [17].

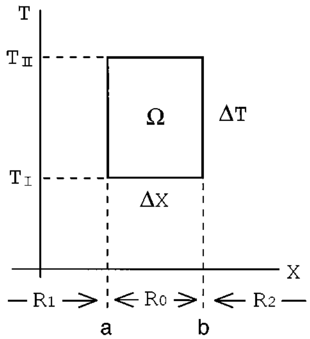

FIG. 3. Configuration of Ω in Secs. III and V.

III. EXAMPLES OF PROBABILITIES FOR HISTORIES

Here we study some explicit sets of classes of paths defined by path classifications with respect to a space-time region $\Omega=\Delta X\times\Delta T$. We take $\Delta X=[a,b](a<b)$ and $\Delta T=[T_I,T_{II}](T_I<T_{II})$, and denote the spatial regions $X<a$, $a<X<b$, and $X>b$ by R_1, R_0, and R_2, respectively (see Fig. 3). Section III A illustrates calculations, and Sec. III C summarizes probability-definable examples.

Reminder: The probability-definable situations found in this section are very simple ones, and one may be disappointed at the ''triviality'' of the results after following relatively long arguments. To prevent such improper impressions, the author would like to repeat here that probabilities for continuous histories have not been well understood in the histories approach, so that even a simple example is important. Probabilities for histories are generalizations of probabilities for observables, so that no example should be considered ''trivial.'' If we feel the results ''trivial,'' it is simply because the results are consistent with our intuition, which in turn means that our approach is a successful one.

A. Illustrating sum-over-paths calculations for particular classes

Consider all paths connecting end points A and B such that $T_A<T_B$, and classify them according to the following criterion: which spatial region R_j $(j=0,1,2)$ paths cross at times T_I and T_{II} $(T_A<T_I<T_{II}<T_B)$, and whether or not paths enter Ω. Let us denote by $\mathscr{C}_{k,\sigma,j}$ the class of paths which start from A, go through $X_I\in R_j$ at time T_I, enter Ω $(\sigma=+)$ or avoid Ω $(\sigma=-)$, go through $X_{II}\in R_k$ at T_{II}, and arrive at B.[7] Note that labels k,σ,j are time ordered with the earliest to the right. The number of times paths cross the temporal boundaries of Ω is not used as a label of classification, so that C1 holds. All paths can be classified into the following 11 nonempty classes (see Fig. 4):

$$\{\mathscr{C}_{k,\sigma,j}|(k,\sigma,j)=(0,+,0),(1,+,0),(2,+,0),(1,+,2),$$
$$(0,+,2),(2,+,1),(0,+,1),(1,+,1),$$
$$(2,+,2),(1,-,1),(2,-,2)\}. \tag{3.1}$$

[7]Symbols I and II, which, respectively, represent space-time points (X_I,T_I) and (X_{II},T_{II}), do not imply $X_I\in R_1$ nor $X_{II}\in R_2$.

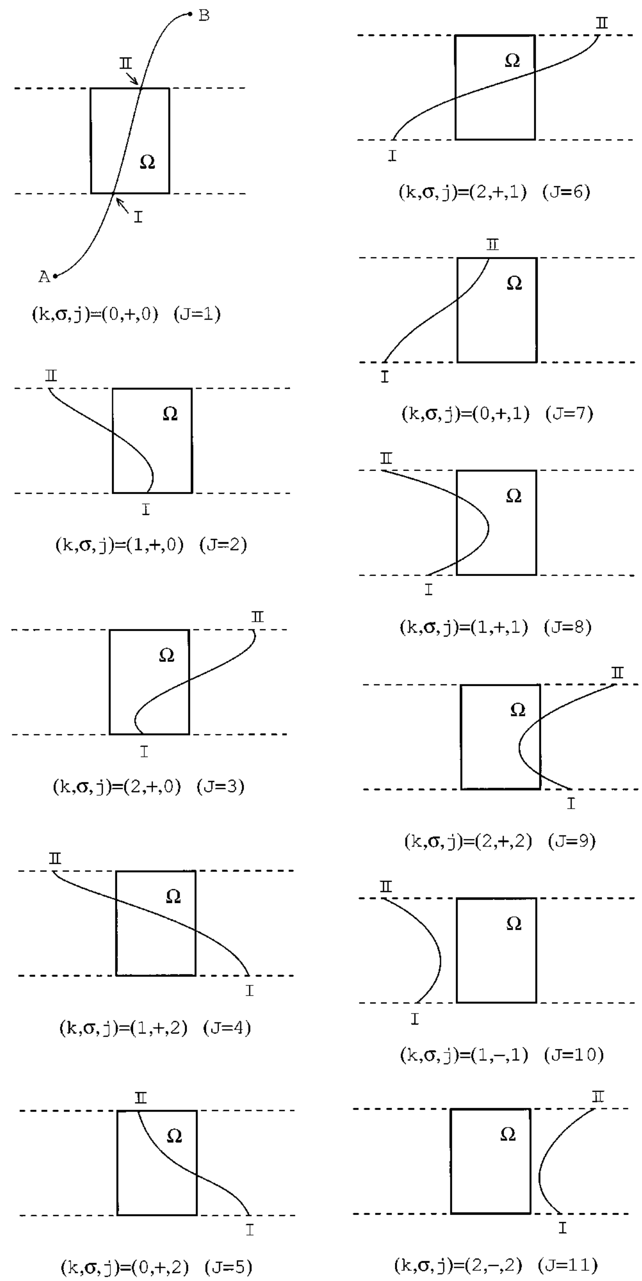

FIG. 4. Representative paths from each of the 11 classes $\{\mathscr{C}_J|J=1,2,\ldots,11\}$. Only the $[T_I,T_{II}]$ part of the path is shown for classes $\mathscr{C}_2,\ldots,\mathscr{C}_{11}$.

We express each (k,σ,j) by an integer J by letting $(0,+,0)$ correspond to $J=1$, $(1,+,0)$ to $J=2,\ldots,$ and $(2,-,2)$ to $J=11$. For example, $\mathscr{C}_3\equiv\mathscr{C}_{2,+,0}$, $\mathscr{C}_5\equiv\mathscr{C}_{0,+,2}$, $\mathscr{C}_9\equiv\mathscr{C}_{2,+,2}$, etc.

To calculate (2.11) for (3.1), decompose the sum over paths into three sums over partial paths and two integrations: a sum over partial paths from A to I, that from I to II, that from II to B, and integrations over intermediate positions X_I and X_{II}; symbolically

$$\sum_{\mathscr{C}_J}=\int_{R_k}dX_{II}\sum_{B\leftarrow II}\int_{R_j}dX_I\sum_{\mathscr{C}_J}\sum_{I\leftarrow A}, \tag{3.2}$$

where $\widetilde{\mathscr{C}}_J$ represents the $[T_I,T_{II}]$ part of $\mathscr{C}_J$, and we have used the convention that k and j are specified by J according to the correspondence between J and (k,σ,j) (for example, $J=3$ implies $k=2$ and $j=0$), the convention used in the rest of this paper. In (3.2), integration signs are used to represent ordinary integrals and summation signs to represent sums over paths. The sum $\Sigma_{I\leftarrow A}$ is over all paths from A to I and thus gives the propagator $\Phi(I;A)$. For the same reason, $\Sigma_{B\leftarrow II}$ gives $\Phi(B;II)$. As for $\Sigma_{\widetilde{\mathscr{C}}_J}$, we have three cases. (1) For $J=1,2,\ldots,7$, the sum is over all paths from I to II to give $\Phi(II;I)$. (2) For $J=10$ and 11, paths are subject to the restriction that they must not enter Ω on their way from I and II, so that the sum is over such paths that move only in R_j. Just as the sum over all paths connecting two end points on the entire space $R(\equiv R_1\cup R_0\cup R_2)$ gives the propagator Φ between the points in R, the sum over all paths connecting two end points on the restricted space R_j gives the propagator ("restricted propagator") between the points in R_j, which we shall denote by Φ^{R_j}. The sum over paths in question then gives $\Phi^{R_j}(II;I)$.[8] (For example Φ^{R_1} is the propagator in the presence of an infinite potential barrier blocking up the region $x>a$, i.e., the solution to the Schrödinger equation satisfying the δ function initial condition and the Dirichlet boundary condition at $x=a$.) (3) For $J=8$ and 9, the paths are such that they enter and then come out of Ω, which are therefore (all paths connecting I and II unrestrictedly) $-$ (all paths that connect I and II by moving only in the restricted space R_j). Thus the sum over paths in question gives

$$\varphi^{R_j}(II;I)\equiv\Phi(II;I)-\Phi^{R_j}(II;I), \tag{3.3}$$

where $X_I,X_{II}\in R_j$. Putting all these together, and using

$$\Theta_i(X)\equiv\begin{cases}1, & X\in R_i\\ 0, & \text{otherwise}\end{cases} \tag{3.4}$$

we arrive at

$$\Phi_J(B;A)=\begin{cases}\int\int dX_{II}dX_I\Theta_k(X_{II})\Theta_j(X_I)\Phi(B;II)\Phi(II;I)\Phi(I;A), & J=1,2,\ldots,7\\ \int\int dX_{II}dX_I\Theta_k(X_{II})\Theta_j(X_I)\Phi(B;II)\varphi^{R_j}(II;I)\Phi(I;A), & J=8,9\\ \int\int dX_{II}dX_I\Theta_k(X_{II})\Theta_j(X_I)\Phi(B;II)\Phi^{R_j}(II;I)\Phi(I;A), & J=10,11,\end{cases} \tag{3.5}$$

where $\Phi_J\equiv\Phi_{\mathscr{C}_J}$. From (3.5) with the help of the composition law (2.19), it is easily shown that

$$\sum_{J=1}^{11}\Phi_J(B;A)=\Phi(B;A). \tag{3.6}$$

This should be taken for granted, for we know from the earlier argument that C1 holds in the present case.

To study C2, it is very useful to introduce $\Psi_J(II)$, "branches" of $\Psi(II)$, which in the present case is given by

$$\Psi_J(II)\equiv\begin{cases}\Theta_k(X_{II})\int dX_I\Phi(II;I)\Theta_j(X_I)\Psi(I), & J=1,2,\ldots,7\\ \Theta_k(X_{II})\int dX_I\varphi^{R_j}(II;I)\Theta_j(X_I)\Psi(I), & J=8,9\\ \Theta_k(X_{II})\int dX_I\Phi^{R_j}(II;I)\Theta_j(X_I)\Psi(I), & J=10,11,\end{cases} \tag{3.7}$$

where $\Psi(I)=\int dX_A\Phi(I;A)\Psi(A)$. Branches $\Psi_J(II)$ satisfy

$$\Psi(II)=\sum_J\Psi_J(II), \tag{3.8}$$

which is a direct consequence of (3.6). The decoherence functional is now compactly expressed as

[8] One may feel that this argument is rather naive, which is basically the same argument as the method-of-image calculation of a restricted sum over paths [18]. It *is* naive but is correct; the naive handling of paths could be justified by using a sum-over-paths method on a space-time lattice [19]. Alternatively, though less rigorous than the lattice method, we can use the function F defined by integral equation (5.12) introduced later to calculate the restricted sum over paths in the manner described in Sec. 2.2 in Chap. III of [20]. All these methods give the same result.

$$D[J;J']=\int dX_{II}\int dX_{II'}\,\delta(X_{II}-X_{II'})\Psi_J^*(II)\Psi_{J'}(II')$$
$$=\int dX_{II}\Psi_J^*(II)\Psi_{J'}(II), \tag{3.9}$$

where $II'\equiv(X_{II'},T_{II'})$ such that $T_{II'}=T_{II}$, and the δ function results from the X_B integration in (2.13) carried out by use of (2.7). Since the effective integration range in the second line of (3.9) is $R_k\cap R_{k'}$,

$$D[J;J']=0 \quad \text{when} \quad R_k\cap R_{k'}=\phi, \tag{3.10}$$

which is true regardless of $\Psi(I)$ and reduces the number of D's we have to calculate to check C2. The formula for probabilities (2.6), valid when C2 holds, is simplified to be

$$P_J(\equiv P_{\mathscr{C}_J}=D[J;J])=\int dX_{II}|\Psi_J(II)|^2. \tag{3.11}$$

Equations (3.7)–(3.11) are the main results of this subsection. They, except (3.7), have generality over the present particular set of 11 classes of paths.

B. Branch wave functions

Since branches of $\Psi(II)$ play important roles in the following discussion, let us here give the general definition of branches and illustrate some of the properties. For simplicity, however, we still consider cases where classes of paths are defined by path classifications with respect to Ω. Given a set of classes of paths $\{\mathscr{C}_J\}$, branches of $\Psi(II)$ are defined by

$$\Psi_J(II)\equiv\Theta_J(X_{II})\int dX_I\left(\sum_{\tilde{\mathscr{C}}_J} e^{iS/\hbar}\right)\Theta_J(X_I)\Psi(I), \tag{3.12}$$

where $\Theta_J(X)$, a generalization of (3.4), represents class-dependent restrictions (if any) on the range of X.[9] If C1 holds, the branches satisfy (3.8). Compare (3.12) with

$$\Psi(II)=\int dX_I\left(\sum_{II\leftarrow I} e^{iS/\hbar}\right)\Psi(I), \tag{3.13}$$

where the sum in the parentheses gives the propagator $\Phi(II;I)$. If we substitute (3.12) into the right-hand side of (3.8), we get (3.13); conversely we are naturally led to (3.8) with (3.12) by decomposing all paths between the two surfaces of time T_I and time T_{II} into a sum of classes of paths $\tilde{\mathscr{C}}_J$. Just as $\Psi(II)$ is the result of the time evolution of $\Psi(I)$ by all paths from I to II (in the sense that the propagator is contributed from all paths), $\Psi_J(II)$ might be said to be the result of the "time evolution of $\Theta_J(X_I)\Psi(I)$ by the paths in $\tilde{\mathscr{C}}_J$." If we take an intermediate time T between T_I and T_{II} to consider the $[T_I,T]$ part of the paths in $\tilde{\mathscr{C}}_J$, we can define the "branch wave function" $\Psi_J(X,T)$ as the result of the time evolution of $\Theta_J(X_I)\Psi(I)$ by that part of the paths; the branch wave functions satisfy $\Sigma_J\Psi_J(X,T)=\Psi(X,T)$ at an arbitrary intermediate time T. Although we say that "$\Psi_J(X,T)$ is the result of the time evolution of $\Theta_J(X_I)\Psi(I)$," this expression should be used with caution. In general, $\Psi_J(X,T)$ does not agree with $\Theta_J(X_I)\Psi(I)$ as $T\rightarrow T_I$. [On the other hand, $\Psi_J(X,T)$ agrees with $\Psi_J(II)$ at time T_{II} for all J.] This is because the sum over the $[T_I,T]$ part of the paths does not form a propagator in general. For example, in the case of (3.7), $\Psi_8(X,T)$ is given by $\int dX_I\varphi^{R_1}(X,T;I)\Psi(I)$, where $\varphi^{R_1}(X,T;I)$ is (3.3) with II replaced by (X,T) $(X\in R_1)$; $\varphi^{R_1}(X,T;I)$ is not a propagator since it vanishes as $T\rightarrow T_I$. Therefore $\Psi_8(X,T)$ also vanishes in the limit. By contrast, $\Psi_{10}(X,T)$ tends to $\Theta_1(X_I)\Psi(I)$ in the limit because $\Phi^{R_1}(X,T;I)$ is a propagator. Equation (3.8) with (3.12) should be understood as the decomposition of $\Psi(II)\leftarrow\Psi(I)$ into the sum of branch motions. Note that it is the motion of the wave function from T_I to T_{II}, not simply the wave function at time T_{II}, that gets decomposed. This point is important.

As understood from above, there is no general law of evolution for branch wave functions. How $\Psi_J(X,T)$ evolves depends on how the relevant class $\tilde{\mathscr{C}}_J$ (and hence $\mathscr{C}_J$) is defined. If the class has some special property, $\Psi_J(X,T)$ obeys some special rule of evolution accordingly. An interesting case is, as we saw, that a class of paths forms a propagator on an appropriate space. In this case, the motion of the relevant branch wave function is norm conserving (a bit of caution is necessary at the final time T_{II} as explained below); as a result, P_J can be calculated from the information at time T_I. To illustrate this, consider again the 11 classes of paths. Then $\Sigma_{\tilde{\mathscr{C}}_J}e^{iS/\hbar}$ for $J=10$ and 11 form restricted propagators Φ^{R_1} and Φ^{R_2}, respectively, having the property

$$\int_{R_j}dX''[\Phi^{R_j}(X'',T'';X',T')]^*\Phi^{R_j}(X'',T'';X,T') = \delta(X'-X). \tag{3.14}$$

Thus (3.11) reduces to

$$P_J=\int_{R_j}dX_I|\Psi(I)|^2, \quad J=10,11. \tag{3.15}$$

The other nine branch motions are not norm conserving, so that the above expression is not valid for other J. For example, let us consider $\Psi_1(X,T)$. This branch wave function starts as $\Theta_0(X_I)\Psi(I)$ and *does* undergo unitary time evolution by the unrestricted propagator $\Phi(II;I)$ from T_I to just before T_{II}, but at time T_{II} the result of the unitary time evolution is projected onto R_0 because of the factor $\Theta_0(X_{II})$ involved in the definition of $\Psi_1(II)$. The norm conservation of $\Psi_1(X,T)$ breaks down at this moment, and thus the norms of $\Psi_1(II)$ and $\Theta_0(X_I)\Psi(I)$ are not the same. This is the reason branch motions are not norm conserving for $J=1,2,\ldots,7$. As to $J=8$ and 9, $\Sigma_{\tilde{\mathscr{C}}_J}e^{iS/\hbar}$ does not form a propagator [since (3.3) does not have the property corresponding to (2.7)], so that $\Psi_8(X,T)$ and $\Psi_9(X,T)$ do not

[9] Since $\Theta_J(X_{II})$ and $\Theta_J(X_I)$ are parts of the definition of $\tilde{\mathscr{C}}_J$, it is not necessary to write them explicitly in (3.12). But we have exhibited them to remind ourselves that, in general, the X_I integration is restricted to some ranges and also that $\Psi_J(II)$ vanishes in some ranges due to the definition of $\tilde{\mathscr{C}}_J$.

undergo unitary time evolutions even before time T_{II}, and thus the two branch motions are not norm conserving.

In terms of branch wave functions, our approach is summarized as follows: C2 is the condition that the real part of the overlap integral between branch wave functions at time T_{II} vanishes for all different pairs of branches; values of the probabilities defined when C2 holds are given by the absolute squares of the norms of branch wave functions at time T_{II}. We stressed in Sec. II B that C2 is a consistency condition between wave nature and particle nature. In terms of branch wave functions, this is understood as follows. When C2 is satisfied, we can express $P_{J\vee J'}$ in two ways: $\int dX_{II}|\Psi_J(II)+\Psi_{J'}(II)|^2$ and $\int dX_{II}|\Psi_J(II)|^2 + \int dX_{II}|\Psi_{J'}(II)|^2$. The former is in favor of wave nature and the latter is in favor of particle nature. The two expressions do not conflict when C2 holds, so that C2 can be taken to be a consistency condition between the two natures.

Equation (3.8) is the sum-over-paths counterpart of the operator identity called ''a resolution of the pure initial state into branches'' introduced by Gell-Mann and Hartle for discrete histories [see Eq. (4.10) of Ref. [17]]. (It should be understood as the resolution of the motion of a wave function from a given initial state to the final state, rather than the resolution of the initial state.) Their use of the notion of branches goes back to [6(a)]. A relativistic version of branch wave functions was used in the recent study of continuous histories by Whelan [8(a)].

C. Probability-definable examples

1. Particle initially localized in R_0

Consider (3.1). C2 requires that the real part of (3.9) be zero for 55 ($={}_{11}C_2$) combinations of J and J'. Owing to (3.10), however, we have only to check the following 15 combinations:

$$(J,J')=(1,5),(1,7),(2,4),(2,8),(2,10),(3,6),(3,9),$$
$$(3,11),(4,8),(4,10),(5,7),(6,9),(6,11),$$
$$(8,10),(9,11). \tag{3.16}$$

Although $\mathrm{Re}D\neq 0$ in general for these combinations, the dependence of $D[J;J']$ on $\Psi(I)$ through (3.7) leaves the possibility that C2 holds for restricted $\Psi(I)$, which we shall call the initial wave function from now on. One simple example showing that this occurs is when $\Psi(I)$ is localized in R_0, i.e.,

$$\int_{R_0} dX_I|\Psi(I)|^2=1, \quad \Psi(I)=0 \text{ on } R_1 \text{ and } R_2. \tag{3.17}$$

It follows that $\Psi_J(II)=0$ except $J=1,2,3$, and therefore $D[J;J']$ vanishes for all the pairs listed in (3.16). Thus we conclude that, if the initial condition of a particle satisfies (3.17), C2 holds and probabilities can be defined for the set of 11 classes of paths. Values of the probabilities are, according to (3.11),

$$P_J=\begin{cases} \int dX_{II}|\Psi_J(II)|^2=\int_{R_k} dX_{II}|\Psi(II)|^2, & J=1,2,3\\ 0, & \text{otherwise,}\end{cases} \tag{3.18}$$

where we have used $\Psi_J(II)=\Psi(II)$ for $X_{II}\in R_k$ for $J=1,2,\ldots,7$, which follows from (3.7) and (3.17). In addition to the initial localization, if $\Psi(II)$ is also assumed to be localized in, say, R_1, we have $P_2=1$. (The localization completely in R_1 at time T_{II} cannot be realized because the packet spreads as it moves, but we can think of a perfect localization in R_1 as an ideal limit.)

2. Particle moving in R_1 (''approximate probabilities'')

The above example is the only one that the present author has found so far in which C2 holds *exactly* for the set of 11 classes. If we allow an approximate satisfaction of C2, a variety of examples are available. As an example, suppose that a wave packet is localized in R_1 (far from the left boundary of Ω) at time T_I with an appropriate momentum so that it is not reflected at the left boundary of Ω (see Fig. 5). Because of the initial localization, $\Psi_J(II)$ can be nonzero only for $J=6,7,8,10$, from which together with (3.16) it follows that we have only to check $D[8;10]$. In the assumed situation, the motion of the packet is not greatly affected even if the whole region outside R_1 is covered with an infinitely high potential barrier. This means that the time evolution of the packet by the propagator $\Phi(II;I)$ on the entire space R is almost the same as that by the restricted propagator $\Phi_1(II;I)$ on the restricted space R_1, so that $\Psi_8(II)\approx 0$ and thus $D[8;10]\approx 0$. Therefore C2 holds approximately, which we shall call approximate decoherence. Consequently, we obtain $P_{10}\approx 1$, which is reasonable, and $P_{8\vee 10}\approx P_8+P_{10}$. It might be said that ''approximate probabilities'' are definable with a small violation of the probability axioms. But a caution is necessary about what this means. First, by the word approximate we never mean that probabilities (true probabilities) given by our theory can be tested only up to some precision in actual experimental situations. Rather our approximate probabilities are given by the theory itself. Second, the current situation must be distinguished from cases where probabilities can be defined for a closed system but only approximately for its subsystem due to the interaction between the subsystem and the other degrees of freedom of the closed system. Rather, our approximate probabilities are defined for a closed system (a particle) itself.

3. Particle crossing Ω (a puzzling result and its resolution)

Suppose that a wave function is localized in R_1 at time T_I, crosses Ω from left to right, and is localized in R_2 at time T_{II} (see Fig. 6). (We consider a perfect localization in R_2 at T_{II} as an ideal limit.) Since the packet crosses Ω from left to right, we expect that the probability for the class of paths crossing the region is unity, i.e., $P_6=1$. However, in the assumed situation, our approach does not define probabilities for the 11 classes of paths (3.1). To see this, note that Eq. (3.8) is now

$$\Psi(II)=\Psi_6(II)+\Psi_8(II)+\Psi_{10}(II). \tag{3.19}$$

This follows from $\Psi_J(II)=0$ for $J=1,\ldots,5,9,11$ since $\Psi(I)=0$ on R_0 and R_2 and from $\Psi_7(II)=0$ since $\Psi(II)$ is assumed to be zero on R_0. The three branches on the right-hand side are nonvanishing. By using (3.19) for $X_{II}\in R_1$, we have

$$0=\Psi_8(II)+\Psi_{10}(II), \tag{3.20}$$

which is in fact valid on the entire space due to the factor $\Theta_k(X_{II})$ involved in the definition of $\Psi_J(II)$. In spite of (3.20), we do not remove the last two terms from the right-hand side of (3.19) or, more generally, we treat the right-hand side of (3.8) in such a way that all nonvanishing $\Psi_J(II)$'s are left on it, even though some of them add up to zero as in the present case; we will then notice later an interesting correspondence between the ''naturalness'' of the decomposition into branches and the definability of probabilities for histories. It follows from (3.20) that

$$D[8;10]=-\int dX_{II}|\Psi_{10}(II)|^2=-1, \tag{3.21}$$

where the last equality follows from the property that the branch motion $\Psi_{10}(II)\leftarrow\Theta_1(X_I)\Psi(I)$ is norm conserving as argued in Sec. III B. Therefore C2 fails due to the interference between the two classes of paths, namely, the class of paths crossing Ω and the class of paths reflecting off the left boundary of Ω. Thus probabilities cannot be defined and our expectation $P_6=1$ is betrayed. In the terminology of Griffiths (extended to cases of continuous histories), the 11 classes of paths are *not consistent* in the situation where the wave packet crosses Ω from left to right. It is possible, however, to find such classes that *are* consistent in the situation and to have $P_6=1$. Such classes are constructed by coarse graining the 11 classes as follows.

Let us combine $\mathscr{C}_8$ and $\mathscr{C}_{10}$ into one class $\mathscr{C}_{8\vee 10}$, and then ask if probabilities can be defined for the resultant ten classes

$$\{\mathscr{C}_J|J=1,2,\ldots,7,8\vee 10,9,11\} \tag{3.22}$$

in the same situation as we assumed above. We already know that $D[J;J']=0$ if J and J' are chosen from $1,\ldots,7$, 9, and 11, and thus all that is left is $D[J;8\vee 10]$. To evaluate this, we go back to (2.13) and recall that the sum-over-paths amplitude for $\mathscr{C}_{J_1\vee J_2\vee\ldots}$ is the sum of the respective amplitudes [see (2.12)]. It generally follows that

$$D[J_1\vee J_2\vee\cdots;J_1'\vee J_2'\vee\cdots]=\int dX_{II}\Psi^*_{J_1\vee J_2\vee\ldots}(II)\Psi_{J_1'\vee J_2'\vee\ldots}(II), \tag{3.23}$$

where J_i and J_i' are some values of J and $\Psi_{J_1\vee J_2\vee\ldots}(II)\equiv\Sigma J_i\Psi J_i(II)$. Therefore

$$D[J;8\vee 10]=\int dX_{II}\Psi^*_J(II)[\Psi_{8\vee 10}(II)]=0 \tag{3.24}$$

due to (3.20). Thus C2 does hold and probabilities can be defined for (3.22). Values of the probabilities are found to be

$$P_6=1,\quad P_J=0 \text{ for } J\neq 6. \tag{3.25}$$

Our expectation is thus met. In this way, we cannot define probabilities for the set of 11 classes but we *can* for the set of ten classes in the situation we have assumed.

Equation (3.8) for (3.22) is

$$\Psi(II)=\Psi_6(II). \tag{3.26}$$

Compare this with (3.19). Both of them are mathematically identical, but (3.19) looks ''odd'' or ''unnatural'' in the situation where a particle crosses Ω from left to right, because two branch motions $\Psi_8(II)\leftarrow\Theta_1(X_I)\Psi(I)$ and $\Psi_{10}(II)\leftarrow\Theta_1(X_I)\Psi(I)$ do not reflect any characteristic features of the motion of the total wave function. For example, the branch motion $\Psi_{10}(II)\leftarrow\Theta_1(X_I)\Psi(I)$ is such that it bounces off the left boundary of Ω, and this reflects none of the characteristic features of the motion of the total wave function crossing Ω from left to right. By contrast, (3.26) is a very ''natural'' decomposition, since in the assumed situation the branch motion $\Psi_6(II)\leftarrow\Theta_1(X_I)\Psi(I)$ is identical to the motion of the total wave function. It can be said that (3.26) reflects characteristic features of the motion of the total wave function in the most trivial way. Although the present author does not have a definite idea of defining ''natural'' decompositions, it is suggestive that C2 fails (branch motions are inconsistent) when decomposition (3.8) takes an unnatural form but it holds (they are consistent) when the decomposition takes a natural form. This is also the case in the examples found below.

The ''-1 interference'' (3.21) is essentially the same as that pointed out by Hartle in [7(a)] (the first full paragraph in the right-hand column on p. 3188). There, a coarse graining of paths was considered according to whether a particle is to

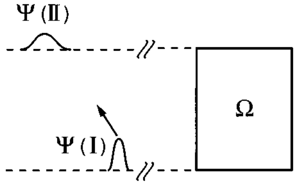

FIG. 5. A particle is localized in R_1 away from the left boundary of Ω at time T_I with a momentum as indicated by the arrow. Even if the whole region outside R_1 is covered by an infinitely high potential barrier, the motion of the packet is not greatly affected. The no-interference condition holds approximately. It might be said that approximate probabilities are definable for the set of 11 events (3.1).

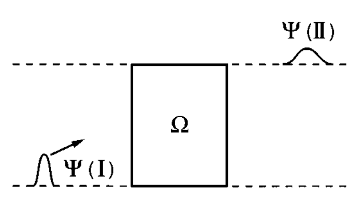

FIG. 6. Wave function initially localized in R_1 crosses Ω to arrive in R_2 at T_{II}. Naive expectation $P_6=1$ is betrayed if $\mathscr{C}_6$ is treated as a member of the set (3.1), but it is met if we coarse grain (3.1) into (3.22) to treat $\mathscr{C}_6$ as a member of the set (3.22).

the left or right of $X=0$ or both over an extended range of time T; the resulting classes of paths were denoted by $\{\mathscr{C}^H_{10},\mathscr{C}^H_{01},\mathscr{C}^H_{11}\}$, where Hartle's rule for subscript differs from ours, and the superscript H has been attached to indicate his notation. These three classes are the $b\rightarrow a$ limit of $\{\mathscr{C}_{10},\mathscr{C}_{11},\mathscr{C}_{4\vee6\vee8\vee9}\}$ with $a=0$. (Classes $\mathscr{C}_J$ for $J=1,2,3,5,7$ become empty in the limit.) The relevant nonvanishing branches are $\Psi^H_{10}(II)\equiv\Psi_{10}(II)$, $\Psi^H_{01}(II)\equiv\Psi_{11}(II)$, and $\Psi^H_{11}(II)\equiv\Psi_{4\vee6\vee8\vee9}(II)$. Consider again a wave packet initially localized in $X<0$ [the initial packet was localized in $X>0$ in [7(a)], which simply results in appropriate changes of subscript below]. In this situation, $\Psi^H_{01}(II)=0$, $\Psi^H_{11}(II)=\Psi_{6\vee8}(II)$, and we already know that the decomposition of $\Psi(II)$ into branches for (3.1) is given by (3.19); thus, for Hartle's three classes, the decomposition is $\Psi(II)=\Psi^H_{11}(II)+\Psi^H_{10}(II)$. C2 does not hold for his three classes because $\int dX_{II}[\Psi^H_{11}(II)]^*\Psi^H_{10}(II)=\int dX_{II}\Psi^*_8(II)\Psi_{10}(II)=-1$ [note that $\Psi_6(II)$ and $\Psi_{10}(II)$ do not overlap]. Thus the set of three classes considered by Hartle and the set of 11 classes considered in this paper share the same reason for the undefinability of probabilities under the assumed initial condition. The difference between the two sets lies in the following point. A set of decohering classes, (3.22), can be constructed by performing a coarse graining on the set of 11 classes that combines $\mathscr{C}_8$ and $\mathscr{C}_{10}$, whereas it is not possible to do a similar thing for his three classes because $\mathscr{C}_6$ and $\mathscr{C}_8$ are already combined into $\mathscr{C}^H_{11}$. Leaving these two classes separate but instead combining $\mathscr{C}_8$ and $\mathscr{C}_{10}$ is essential in order for us to find the decohering set of classes of paths (3.22), for which probabilities can be defined.

In passing, the right-hand side of (3.23) can be written as $\Sigma_{J_i}\Sigma_{J'_i}D[J_i;J'_i]$. An important consequence of this is that a coarse-grained set of such a set that satisfies C2 also satisfies C2. For instance, if we coarse grain (3.22) into $\{\mathscr{C}_6,\mathscr{C}_{1\vee\cdots\vee5\vee7\vee\cdots\vee11}\}$, it also satisfies C2 in the situation we assumed, and we have $P_6=1$ and $P_{1\vee\cdots\vee5\vee7\vee\cdots\vee11}=0$.

4. Antisymmetric initial conditions

Let us consider

$$\{\mathscr{C}_{1\vee2\vee\cdots\vee9},\mathscr{C}_{10},\mathscr{C}_{11}\}. \tag{3.27}$$

This is a coarse-grained set of (3.1) but not of (3.22). We already know $D[10;11]=0$. For $J=10$ and 11, we have from (3.23)

$$D[1\vee2\vee\cdots\vee9;J]=\int dX_{II}\Psi^*_{1\vee\cdots\vee9}(II). \tag{3.28}$$

If C2 holds, i.e., if the real part of (3.28) vanishes for $J=10$ and 11, probabilities can be defined for (3.27) with values given by

$$P_{1\vee2\vee\cdots\vee9}=\int dX_{II}|\Psi_{1\vee\cdots\vee9}(II)|^2=\int_{R_0}dX_I|\Psi(I)|^2, \tag{3.29a}$$

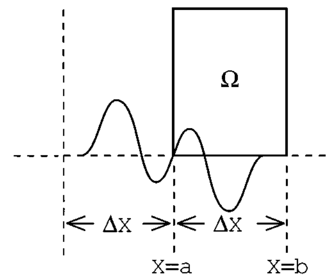

FIG. 7. A situation where probabilities can be defined for the set of three classes of paths (3.27). The wave function at time T_I is antisymmetric with respect to $X=a$ and is localized within $2\Delta X$.

$$P_J=\int_{R_j}dX_I|\Psi(I)|^2 \quad (J=10,11), \tag{3.29b}$$

where we have used (3.15) to obtain (3.29b); to obtain (3.29a), substitute $\Psi(II)=\Psi_{1\vee\cdots\vee9}(II)+\Psi_{10}(II)+\Psi_{11}(II)$ into $\int dX_{II}|\Psi(II)|^2=1$, use the assumption that C2 is satisfied, and then use (3.29b) to have $\int dX_{II}|\Psi_{1\vee\cdots\vee9}(II)|^2+\int_{R_1\cup R_2}dX_I|\Psi(I)|^2=1$, from which together with $\int dX_I|\Psi(I)|^2=1$ we obtain the final expression. Now, by insight, we can find the following two situations where C2 actually holds.

Example 1: Let R_0 be $a<X<b$. Consider a $\Psi(I)$ such that it is antisymmetric with respect to $X=a$ and is localized in $2\Delta X$ around $X=a$ (see Fig. 7). Then (3.28) vanishes for $J=11$ because $\Psi_{11}(II)=0$. We further assume no potential (or at most a potential symmetric with respect to $X=a$) throughout ΔT so that the wave function evolves keeping its initial symmetry. If we pay attention to $\Psi(II)\leftarrow\Psi(I)$ on R_1, we cannot distinguish it from $\Psi_{10}(II)\leftarrow\Theta_1(X_I)\Psi(I)$ on R_1. Thus $\Psi_{10}(II)=\Psi(II)$ on R_1, and therefore

$$\Psi_{1\vee\cdots\vee9}(II)=\Psi(II)-\sum_{J=10,11}\Psi_J(II) \tag{3.30}$$

is zero on R_1. By definition, $\Psi_{10}(II)=0$ outside R_1. Therefore $\Psi_{1\vee\cdots\vee9}(II)$ and $\Psi_{10}(II)$ are nonoverlapping and thus $D[1\vee2\vee\cdots\vee9;10]$ vanishes and C2 holds. Values of the resultant probabilities are, from (3.29),

$$P_{1\vee2\vee\cdots\vee9}=\int_a^b dX_I|\Psi(I)|^2=\frac{1}{2}, \tag{3.31a}$$

$$P_{10}=\frac{1}{2}, \quad P_{11}=0. \tag{3.31b}$$

These values look reasonable due to the conservation of the symmetry of the wave function during ΔT. Equation (3.8) is now $\Psi(II)=\Psi_{1\vee\cdots\vee9}(II)+\Psi_{10}(II)$, and this is understood as the decomposition of the motion $\Psi(II)\leftarrow\Psi(I)$ of the wave function into two branches: One is the branch motion $\Psi_{10}(II)\leftarrow\Theta_1(X_I)\Psi(I)$ and this coincides with $\Psi(II)\leftarrow\Psi(I)$ ''projected'' onto R_1, and the other branch motion is the ''sum'' of the branch motions $\Psi_J(II)\leftarrow\Theta_j(X_I)\Psi(I)$ for $J=1,2, \ldots ,9$ (the branch motions are null for $J=4$, 5, and 9) and this coincides with $\Psi(II)\leftarrow\Psi(I)$ ''projected'' onto R_0. The decomposition into these two branch motions looks ''natural'' from the viewpoint of symmetry. Again, C2 holds when decomposition (3.8) takes a ''natural'' form. By the way, it is worthwhile to note that the present example is an explicit case where the form of a potential in Ω is crucial for the success or failure of C2. Any deviations from a potential symmetric with respect to $X=a$ lead to the failure of C2 when the initial wave function is antisymmetric as assumed.

Example 2: The above example motivates us to try an initial wave function which is antisymmetric with respect to a *and* b. Such a function is periodic, so that it is not normalizable on the entire space which has been assumed to be unbounded. This is the difficulty we had in [12], which can, however, be overcome by working in a bounded space. Suppose that the entire space is now $0<X<L$. We assume that the spatial location of Ω satisfies the following commensurability condition:

$$\frac{L}{a}=p \quad \text{and} \quad \frac{L}{L-b}=q \quad \text{for positive integers } p \text{ and } q. \tag{3.32}$$

We assume no potential throughout ΔT for simplicity. We further assume that $\Psi(I)$ is antisymmetric with respect to $X=a$ and b; such a function is generally expressed as $\sqrt{2/L}\Sigma_{n=1}^{\infty}a_n \sin(npq\pi X/L)$, where $\Sigma_n|a_n|^2=1$ and we have imposed the Dirichlet boundary condition at $X=0$ and L. Because of the symmetry of the problem, $\Psi_{10}(II)$ agrees with $\Psi(II)$ on R_1 and vanishes elsewhere, $\Psi_{11}(II)$ agrees with $\Psi(II)$ on R_2 and vanishes elsewhere, and therefore $\Psi_{1\vee\cdots\vee 9}(II)$ agrees with $\Psi(II)$ on R_0 and vanishes elsewhere. Therefore $\Psi_{1\vee\cdots\vee 9}(II)$, $\Psi_{10}(II)$, and $\Psi_{11}(II)$ are nonoverlapping, and thus C2 holds. Values of the resultant probabilities are found to be

$$P_{1\vee 2\vee\cdots\vee 9}=1-\left(\frac{1}{p}+\frac{1}{q}\right), \quad P_{10}=\frac{1}{p}, \quad P_{11}=\frac{1}{q}, \tag{3.33}$$

where we have used (3.29) with symmetry considerations. These values are reasonable from the symmetry of the problem. Equation (3.8) is now $\Psi(II)=\Psi_{1\vee\cdots\vee 9}+\Psi_{10}(II)+\Psi_{11}(II)$, and this looks ''natural'' from the viewpoint of symmetry. As in Example 1, the form of a potential in Ω is crucial for the success or failure of C2. A potential symmetric with respect to $X=a,b$ and $(a+b)/2$ does not change the above results [20], but any deviations from this lead to the failure of C2 when the initial wave function has the assumed symmetry. Moreover, in this example, the spatial location of Ω is also important for C2 to hold. The two examples studied here illustrate that, along with restrictions on the initial condition, any elements involved in the decoherence functional can affect the success or failure of C2.

5. Probabilities for $\{\mathscr{C}_Y,\mathscr{C}_N\}$

$\mathscr{C}_{1\vee 2\vee\cdots\vee 9}$ is nothing but class $\mathscr{C}_Y$ considered in Sec. II, and $\mathscr{C}_{10\vee 11}$ is class $\mathscr{C}_N$. The set of two classes $\{\mathscr{C}_Y,\mathscr{C}_N\}$ is a coarse-grained set of (3.1) and also that of (3.27). From what has been studied, we can give three situations in which C2 holds for $\{\mathscr{C}_Y,\mathscr{C}_N\}$. They are (1) when $\Psi(I)$ satisfies (3.17), (2) when $\Psi(I)$ satisfies the antisymmetric initial condition considered in Example 1 in Sec. III C 4, and (3) when $\Psi(I)$ satisfies the antisymmetric initial condition and when Ω satisfies the commensurability condition considered in Example 2 in Sec. III C 4. Values of the resultant probabilities are $(P_Y,P_N)=(1,0)$, $(1/2,1/2)$, and $(1-1/p-1/q,1/p+1/q)$ in the order mentioned. Note that $\{\mathscr{C}_Y,\mathscr{C}_N\}$ is not a coarse-grained set of (3.22); in the situation considered in Sec. III C 3, C2 does not hold for $\{\mathscr{C}_Y,\mathscr{C}_N\}$ and thus P_Y and P_N cannot be defined.

6. Transmission and reflection probabilities as special cases of probabilities for histories

If the initial wave function $\Psi(I)$ is localized in R_1, $\Psi_J(II)=0$ for $J=1,2,\ldots,5,9,11$. By combining these ''null'' classes into one, and also combining classes of $J=8$ and 10 into another, we consider the following set of four classes of paths:

$$\{C_{1\vee 2\vee\cdots\vee 5\vee 9\vee 11},C_6,C_7,C_{8\vee 10}\}. \tag{3.34}$$

Equation (3.8) now becomes

$$\Psi(II)=\Psi_6(II)+\Psi_7(II)+\Psi_{8\vee 10}(II). \tag{3.35}$$

For the three terms on the right-hand side, we have from Eq. (3.7)

$$\Psi_J(II)=\Theta_k(X_{II})\int dX_I\Phi(II;I)\Psi(I) \tag{3.36a}$$

$$=\Theta_k(X_{II})\Psi(II), \quad J=6,7,8\vee 10 \tag{3.36b}$$

where the factor $\Theta_1(X_I)$ has been dropped in the first line due to the assumption of the initial localization in R_1. The three branches $\Psi_6(II)$, $\Psi_7(II)$, and $\Psi_{8\vee 10}(II)$ are thus nonoverlapping, so that C2 holds and probabilities can be defined for (3.34), whose values are found to be

$$P_J(II)=\int dX_{II}|\Psi_J(II)|^2, \quad J=6,7,8\vee 10 \tag{3.37a}$$

$$P_J(II)=0, \quad J=1\vee 2\vee\cdots\vee 5\vee 9\vee 11. \tag{3.37b}$$

Now let us further impose the following restrictions. We assume that the initial packet is moving toward Ω from the left, and also that the special region ΔX of Ω is occupied by a square potential $V(X)\equiv\Theta_0(X)V$, where V is a positive constant. This sets up a simple tunneling problem of a particle incident from the left-hand side of the potential. The standard treatment of QM tells us how the incident packet evolves in time [21]. Reaching the barrier region, the wave function begins to split into two parts, and their overlap becomes smaller as time goes on. After a sufficiently long time, the overlap becomes negligible, and we can clearly identify the reflected packet Ψ_R moving backwards in region R_1 and the transmitted packet Ψ_T moving forward in region R_2.

Hence, if we choose time T_{II} to be such a time at which the overlap is negligible, we may write

$$\Psi(II)=\Psi_R(II)+\Psi_T(II). \tag{3.38}$$

Comparing this with (3.36b), we find that

$$\Psi_6(II)=\Psi_T(II), \quad \Psi_7(II)=0, \quad \Psi_{8\vee 10}(II)=\Psi_R(II). \tag{3.39}$$

Therefore

$$P_6=\int dX_{II}|\Psi_T(II)|^2=\text{transmission probability}, \tag{3.40a}$$

$$P_{8\vee 10}=\int dX_{II}|\Psi_R(II)|^2=\text{reflection probability}, \tag{3.40b}$$

$$P_J=0, \quad J=7,1\vee 2\vee\cdots\vee 5\vee 9\vee 11. \tag{3.40c}$$

It has now turned out that the tunneling probabilities (the reflection and the transmission probabilities) are special cases of probabilities for classes of continuous histories. This is the main result in this subsection. It should be noted that the reflection probability is contributed not only from $\mathcal{C}_8$ but also from $\mathcal{C}_{10}$ entering Ω.

By the way, Eq. (3.40c) shows that there is no reason to keep $\mathscr{C}_{1\vee 2\vee\cdots\vee 5\vee 9\vee 11}$ and $\mathscr{C}_7$ uncombined so long as we deal with asymptotic situations where T_{II} is sufficiently later than T_I. But in general, $\mathscr{C}_7$ is of the same importance as $\mathscr{C}_6$ and $\mathscr{C}_{8\vee 10}$. If T_{II} is not sufficiently later than T_I, Eq. (3.37), which is valid for an arbitrary T_{II} ($>T_I$), gives a nonzero value of P_7. This probability, which decreases as T_{II} becomes large, deserves to be called the ''trapped probability.'' The normalization of probabilities is generally such that the sum of three time-dependent probabilities, the reflection, the transmission, and the trapped probabilities, is equal to one. (From the standard terminology, however, it is an abuse of language to call P_6 and $P_{8\vee 10}$ the transmission and the reflection probabilities, respectively, when the wave function is not yet split into two nonoverlapping parts.)

IV. INTERPRETATION

From the result of Sec. III C 6, we can draw an important conclusion as to the interpretation of probabilities for continuous histories. That is, *probabilities for continuous histories defined by our approach (and hence by the generalized quantum mechanics) are, in general, not probabilities for a particle to actually follow the histories*. More precisely, it is not generally the case that the fulfillment of C2 means the existence of a real physical path which falls within class $\mathscr{C}_J$ with probability P_J. This is because we know that the reflection and the transmission probabilities are *not* the probabilities that a particle actually follows such paths that reflect off or penetrate the potential barrier.

Nevertheless, the probability-definable examples we have found in Sec. III C give us an impression that probabilities for continuous histories defined by the present approach are more or less informative about the particle's behavior before time T_{II}, or in other words, an impression that the probabilities can be used for retrodiction. The present author has no idea to what extent such an impression has generality. It would be that the no-interference condition is too broad a condition and other requirements such as restrictions on the type of classes of paths are needed in addition to C2 to say something definite as to what the probabilities mean. For example, if a class $\mathscr{C}$ is such that $\Sigma_{\mathscr{C}} e^{iS/\hbar}$ forms a propagator on an appropriate region of space, the time evolution of the relevant branch wave function is norm conserving. This special property enables us to retrodict that the particle had been in that spatial region with such and such probability. This is indeed the case in Sec. III C 4. Without such special properties, it would be that only a formal (but general) statement is available. A formal statement is as follows. Let us note that $\int dX_{II}|\Psi(II)|^2=1$ expresses the existence of a particle at time T_{II}. If C2 holds, the left-hand side becomes the sum of square norms of $\Psi_J(II)$ over all classes, and thus

$$\sum_J P_J=\int dX_{II}|\Psi(II)|^2=1. \tag{4.1}$$

C2 also guarantees

$$P_{J\vee J'\vee\cdots}=P_J+P_{J'}+\cdots. \tag{4.2}$$

These equations, just Eq. (2.15) rewritten in terms of branches, read as follows: *Each class $\mathscr{C}_J$ contributes to the existence of a particle at time T_{II} by the amount P_J, and the contributions from respective classes are additive and sum up to unity*.[10] One might be tempted to propose more detailed and physical statements, for instance that ''the motion of the total wave function $\Psi(X,T)$ agrees with that of a branch wave function $\Psi_J(X,T)$ with probability P_J.'' However, Eqs. (4.1) and (4.2) do not warrant such statements beyond the formal one given above.

V. APPLICATION TO TUNNELING TIME PROBABILITY DENSITIES

A. Introduction

How long a quantum particle spends in a potential region, the tunneling time problem, has been controversial for more than 60 years [13]. Although several characteristic time scales have been proposed based upon different ideas of measuring (or defining) the time of tunneling, no tunneling time as a universally acceptable concept has been established.[11] In view of this, some authors argue that there is no such thing as ''the tunneling time'' and one can only associate some characteristic time scales with tunneling processes. Undoubtedly, it is of practical importance to study such characteristic time scales, but, at the same time, the search for ''the tunneling time'' should be continued as well because of its fundamental importance in QM.

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[10]Although we have been using a rectangular space-time region, this statement would remain unchanged even if more than one rectangular space-time region or regions which are not necessarily rectangular are used to define classes of paths, provided that time T_{II} is identified to be the latest time of the regions.

[11]This is very similar to the situation surrounding Question 1. It does not have a unique answer in QM, as mentioned in Sec. I; different answers (different values of the probability) are obtained depending on how the particle detection in the space-time region is designed. Looking for an answer to Question 1 and looking for an answer to the tunneling time are considered different aspects of the same desire, that is, the desire to see things from the particle point of view in a space-time region.

''The tunneling time,'' if definable, cannot be completely unique. It should depend on the shape of the potential barrier and on the initial condition of the incident wave packet. Just as the position where a particle is found is described by a probability density and its mean position under a given initial condition is calculated by using the density function, it is reasonable to expect, given an initial condition, that the tunneling time is calculated as a quantum mechanical average with respect to a probability density of tunneling time which depends on the shape of the potential barrier. In an attempt to define such a probability density, the following two points should be taken into account. The first point is that the density should be independent of the method of measuring the time of tunneling, or more generally external degrees of freedom, just as the probability density of a particle's position is independent of how we measure the position; this means that we expect the tunneling time to have an observablelike property. The second point is that the method of processing amplitudes (tunneling time amplitudes) into probabilities (tunneling time probabilities) should have generality in QM. That is, when the method is applied to cases where correct answers are known, it must reproduce the correct answers. As such cases, we can think of the probability density of a particle's position and, especially in connection with tunneling, the reflection and the transmission probabilities.

Thinking in this way, and looking back at the arguments of Secs. II B and III C 6, we find that our approach is suited to investigating the definability of tunneling time probability densities.

B. Calculation

We again consider the simple tunneling situation considered in Sec. III C 6. Starting from (3.34), we classify paths connecting two surfaces of constant time T_I and T_{II} according to the amount of time paths spent in the barrier region. Before doing this, let us note the following. If probability densities are definable for tunneling times (the reflection time and the transmission time), these densities integrated over possible times should reproduce the tunneling probabilities (the reflection and the transmission probabilities). Since the tunneling probabilities can be defined only asymptotically (i.e., when T_{II} is sufficiently later than T_I),[12] it is a contradiction if tunneling time probabilities should be definable for an arbitrary T_{II} later than T_I. Taking T_{II} sufficiently later than T_I, and combining $\mathscr{C}_7$ and $\mathscr{C}_{1\vee2\vee\cdots\vee5\vee9\vee11}$ into $\mathscr{C}_0\equiv\mathscr{C}_{1\vee2\vee\cdots\vee5\vee7\vee9\vee11}$, we consider the following three classes of paths:

$$\{\mathscr{C}_0,\mathscr{C}_c|c=T,R\}, \tag{5.1}$$

where $\mathscr{C}_T\equiv\mathscr{C}_6$ and $\mathscr{C}_R\equiv\mathscr{C}_{8\vee10}$; read c as ''channel.'' Let us now ''fine grain'' $\mathscr{C}_c$ with respect to the amount of time τ paths spend in the barrier region. The resultant classes are labeled by the discrete index c and the continuous index τ. The set (5.1) is fine grained into

$$\{\mathscr{C}_0,\mathscr{C}_{c,\tau}|c=T,R,0\leqslant\tau\leqslant T_{II}\}, \tag{5.2}$$

where we have set $T_I=0$; $\mathscr{C}_{c,\tau}$ is the class of paths which belong to class $\mathscr{C}_c$ *and* spend time τ in the barrier region. It follows that[13]

$$\sum_{\mathscr{C}_c} e^{iS/\hbar}=\int_0^{T_{II}}d\tau\sum_{\mathscr{C}_{c,\tau}} e^{iS/\hbar}, \tag{5.3}$$

which is also true if we replace $\mathscr{C}_c$ and $\mathscr{C}_{c,\tau}$ by $\tilde{\mathscr{C}}_c$ and $\tilde{\mathscr{C}}_{c,\tau}$ ($\tilde{\mathscr{C}}$ is the $[0,T_{II}]$ part of $\mathscr{C}$), respectively, and thus

$$\Psi_c(II)=\int_0^{T_{II}}d\tau\Psi_{c,\tau}(II), \tag{5.4}$$

where

$$\Psi_{c,\tau}(II)\equiv\int dX_I\left(\sum_{\tilde{\mathscr{C}}_{c,\tau}} e^{iS/\hbar}\right)\Psi(I). \tag{5.5}$$

This gives nonzero branches of $\Psi(II)$ for (5.2). (The other branch $\Psi_0(II)$ is zero because we know from Sec. III C 6 that $\Psi_{1\vee2\vee\cdots\vee5\vee9\vee11}(II)=0$ [since $\Psi(I)$ is localized in R_1] and $\Psi_7=0$ (since T_{II} is sufficiently later than T_I).) Noting (3.39) and substituting (5.4) into (3.35), we have

$$\Psi(II)=\sum_{c=T,R}\int_0^{T_{II}}d\tau\Psi_{c,\tau}(II), \tag{5.6}$$

which is Eq. (3.8) written for (5.2). Now the no-interference condition for (5.2) is given by, according to (2.17b),

$$\mathrm{Re}D[(c,\tau);(c',\tau')]=\delta_{c,c'}\delta(\tau-\tau')P_{c,\tau}, \tag{5.7}$$

where

$$D[(c,\tau);(c',\tau')]=\int dX_{II}\Psi^*_{c,\tau}(II)\Psi_{c',\tau'}(II), \tag{5.8}$$

and $P_{c,\tau}$ is given by, according to (2.18),

$$P_{c,\tau}=\lim_{\Delta\tau\to0}\frac{1}{\Delta\tau}\int dX_B\left|\int dX_A\int_{\Delta\tau}d\tau\Phi_{c,\tau}(B;A)\Psi(A)\right|^2 \tag{5.9a}$$

$$=\lim_{\Delta\tau\to0}\frac{1}{\Delta\tau}\int dX_{II}\left|\int_{\Delta\tau}d\tau\Psi_{c,\tau}(II)\right|^2, \tag{5.9b}$$

where $\Phi_{c,\tau}(B;A)\equiv\Sigma_{\mathscr{C}_{c,\tau}}e^{iS/\hbar}$ and we have used

$$\sum_{\mathscr{C}_{c,\tau}} e^{iS/\hbar}=\int dX_{II}\int dX_I\Phi(B;II)\left(\sum_{\tilde{\mathscr{C}}_{c,\tau}} e^{iS/\hbar}\right)\Phi(I;A) \tag{5.10}$$

[12]This is not true in the case of tunneling by a particle having a fixed energy. But such a case is an ideal limit of tunneling by a particle described by a wave packet, which is the case we deal with.

[13]As will be explained later, a careful treatment is necessary for $\tau=0$ for $c=R$. The following equations in this paragraph need to be modified accordingly. We will do this later. For now, with the following somewhat inaccurate equations, the author would like to sketch the essence of how the history approach is going to be applied to the tunneling time problem.

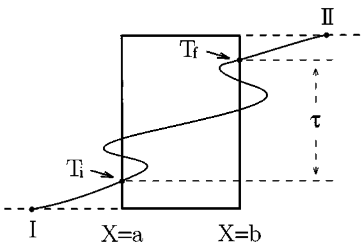

FIG. 8. The time τ a Feynman's path spends in the barrier region is defined as the difference between the time T_f at which it leaves the region for the last time and the time T_i at which it enters the region for the first time. The case $c=T$ is illustrated. Though a smooth path is drawn here for legibility, Feynman's paths are nondifferentiable in time.

to obtain (5.9b). If C2 holds, $P_{c,\tau}$ deserves to be called the probability density of transmission time for $c=T$ and the probability density of reflection time for $c=R$; C2 guarantees that the densities integrated from $\tau=0$ to T_{II} reproduce the transmission and the reflection probabilities. Unfortunately, however, C2 does not hold and the probability densities cannot be defined. We prove this below. Now we must implement (5.3) explicitly. [Although the conclusion is negative, how the decomposition (5.6) can actually be accomplished with a slight modification is an interesting problem in its own right. Our secondary aim in this subsection is to show the process of decomposition in detail.]

Equation (5.3) assumes that the amount of time τ a given path spends in the barrier region can be meaningfully defined. This is not obvious at all, for Feynman's paths are nondifferentiable in time. In this paragraph, we argue that a meaningful way of defining τ is the following (see Fig. 8):

$$\tau \equiv T_f - T_i, \tag{5.11}$$

where T_i is the first time the path hits $X=a$ (the left boundary of Ω) for both $c=T$ and R, while T_f is the last time the path hits $X=b$ (the right boundary of Ω) for $c=T$ and $X=a$ for $c=R$.[14] Of course, T_i and T_f must be proved to be meaningful when paths are nondifferentiable in time. As evidence of the meaningfulness of T_i (it is enough to consider either T_i or T_f), we show that paths can be classified according to the first time they hit a fixed position.[15] From a space-time point of view, it is convenient to call a fixed position X a ''surface'' of constant X, which we shall denote by S_X. Consider if the paths from (X',T') to (X'',T'') can be classified according to the time T_i of their first hit of S_X. We consider the case $X'<X<X''$. If this classification is possible, the following composition law should hold:[16]

$$\Phi(X'',T'';X',T') = \int_{T'}^{T''} dT_i \Phi(X'',T'';X,T_i) F(X,T_i;X',T'), \tag{5.12}$$

where function $F(X,T_i;X',T')$ is defined to be the result of the sum over paths from (X',T') to (X,T_i) never hitting S_X before the arrival at (X,T_i). This composition law can be regarded as the Volterra integral equation of the first kind for the unknown function F. Does (5.12) have a unique solution? If it does, we may take it as a proof of the meaningfulness of T_i.[17] Because of the time translation invariance of the problem, we may write

$$\Phi(X'',T'';X',T') = \Phi(T''-T'|X'',X'),$$

$$F(X,T_i;X',T') = F(T_i-T'|X,X'). \tag{5.13}$$

Equation (5.12) is now of convolution type and we can solve it for F by performing Laplace transformation on both sides. The formal solution is found to be[18]

$$F(T|X,X') = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{sT} \frac{\tilde{\Phi}(s|X'',X')}{\tilde{\Phi}(s|X'',X)}, \tag{5.14}$$

where γ is an appropriate real number, and $\tilde{\Phi}$ is the Laplace transform of Φ, i.e.,

$$\tilde{\Phi}(s|X'',X') \equiv \int_0^\infty dT e^{-sT} \Phi(T|X'',X'). \tag{5.15}$$

Unless the potential is ill behaved, the complex integral (5.14) is expected to exist to give a unique F. As an illustration of this, and also for our later purpose, we calculate (5.14) for a free particle. In this case we may further write

[14]As pointed out by Schulman and Ziolkowski [22], this definition of τ overestimates the duration, because the path can exit and reenter the region between T_i and T_f. To do the estimation ''correctly,'' we must know when and how many times the path crosses the temporal boundaries of Ω. This is, however, impossible because Feynman's paths are nondifferentiable in time; they cross a ''surface'' of constant X even an infinite number of times. T_i and T_f are the most detailed information we can meaningfully talk about.

[15]This is a quantum mechanical version of the so-called ''first passage (hitting) time,'' which is one of the standard subjects in the field of stochastic processes.

[16]Studies of the integral equation (5.12) in the context of path classifications go back to [23]. Recently it was rediscovered in the study of multidimensional tunneling and called the path decomposition expansion (PDX) [24].

[17]Strictly speaking, the existence of a unique solution is only a necessary condition for T_i to be meaningful. More rigorously, we could discretize both space and time (real time) to define T_i on the space-time lattice in the manner described in Sec. 2.2 in Chap. III of [20]. We can then complete the path classification in question on the lattice, which is an easy task because we have only to deal with a countable number of paths on the lattice, to obtain an equation expressing the lattice-path classification. Equation (5.12) is then derived by taking an appropriate continuum limit. We can then follow the present procedure to arrive at (5.14). Hence the meaningfulness of T_i and the existence of a unique solution of (5.12) can be regarded as the same thing. Working on the lattice, we could also derive (5.19) directly [i.e., without using (5.12)] as a sum over lattice paths.

[18]As usual, the integration contour for (5.14) is an infinite vertical line in the complex s plane, and the constant γ must be chosen in such a way that all the singularities of the integrand are on the left-hand side of the contour.

$$\Phi(T''-T'|X'',X')=\Phi(T''-T'|X''-X'),$$

$$F(T_i-T'|X,X')=F(T_i-T'|X-X'). \qquad (5.16)$$

The Laplace transform of the free propagator $\Phi(T|X)$ is [we use the units $m/\hbar=1$ (m is the particle's mass)]

$$\tilde{\Phi}(s|X)=\sqrt{\frac{1}{2is}}e^{i\sqrt{2isX^2}}, \qquad (5.17)$$

and thus

$$\frac{\tilde{\Phi}(s|X''-X')}{\tilde{\Phi}(s|X''-X)}=e^{i\sqrt{2is(X-X')^2}}=-i\frac{\partial}{\partial X}\tilde{\Phi}(s|X-X'). \qquad (5.18)$$

Substituting this result into (5.14) and changing the order of $\int ds$ and $\partial/\partial X$, which can be justified, we finally obtain

$$F(T|X)=-i\frac{\partial}{\partial X}\Phi(T|X) \qquad (5.19a)$$

$$=\left(\frac{X^2}{2\pi iT^3}\right)^{1/2}\exp\left(i\frac{X^2}{2T}\right), \qquad (5.19b)$$

where Φ is the free propagator. This is the explicit expression for the ''first hitting amplitude'' $F(T|X)$ for a free particle; it is the sum of $e^{iS_{free}/\hbar}$ (S_{free} is the free action) over the paths starting from (0,0) and ending at (X,T) without hitting the surface S_X before time T. Although the variable X in $F(T|X)$ is supposed to be positive in the context of our derivation, the final result (5.19b) is valid regardless of the sign of X. The explicit expression for F for a free particle with arbitrary end points follows immediately from (5.19b) with (5.13) and (5.16). In passing, (5.19b) satisfies the following equation:

$$\int_0^{T'}dTF(T'-T|X')F(T|X)=F(T'||X|+|X'|). \qquad (5.20)$$

Hence the first hitting amplitude F evolves, as it were, in space, just as the propagator Φ evolves in time according to Eq. (2.19). What is behind (5.20) is the following path classification. Let us assume $X,X'>0$ for simplicity. The right-hand side corresponds to the paths which start from (0,0) and end at $(X+X',T')$ without hitting $S_{X+X'}$ before time T'. Such paths intersect the intermediate surface S_X, and the number of times of intersection is, in general, more than one. By classifying the paths according to the first time T they hit S_X, we have (5.20).[19]

Definition (5.11) is incomplete for $c=R$, for which T_i and T_f cannot be defined for the paths in $\mathscr{C}_{10}$. Since they spend no time in the barrier region, we define that $\tau=0$ for them. This, together with (5.11), completes our definition of τ. Accordingly, Eq. (5.3) for $c=R$ has to be replaced by

$$\sum_{\mathscr{C}_R}e^{iS/\hbar}=\sum_{\mathscr{C}_{10}}e^{iS/\hbar}+\int_0^{T_{II}}d\tau\sum_{\mathscr{C}_{8,\tau}}e^{iS/\hbar}, \qquad (5.21)$$

where $\mathscr{C}_{8,\tau}$ is the subclass of $\mathscr{C}_8$ consisting of the paths spending time τ in the barrier region. For $\tau\neq 0$, we may write $\mathscr{C}_{R,\tau}$ for $\mathscr{C}_{8,\tau}$, because a path in $\mathscr{C}_R$ spending nonzero time in the barrier region inevitably belongs to $\mathscr{C}_8$. By contrast, we may not write $\mathscr{C}_{R,0}$ for $\mathscr{C}_{10}$, because those paths in $\mathscr{C}_8$ for which $T_i=T_f$ also belong to $\mathscr{C}_{R,0}$; as will be clarified later, their contribution needs a careful treatment, and we will be led to consider the contribution from those paths in $\mathscr{C}_R$ that spend less than $\epsilon>0$ in the barrier region, which is formally expressed as

$$\sum_{\mathscr{C}_{R,0\le\tau\le\epsilon}}e^{iS/\hbar}\equiv\sum_{\mathscr{C}_{10}}e^{iS/\hbar}+\int_0^{\epsilon}d\tau\sum_{\mathscr{C}_{8,\tau}}e^{iS/\hbar}. \qquad (5.22)$$

Taking this point into account in advance and noting $\mathscr{C}_{8,\tau}=\mathscr{C}_{R,\tau}$ for $\tau\neq 0$ we rewrite Eq. (5.21) as

$$\sum_{\mathscr{C}_R}e^{iS/\hbar}=\sum_{\mathscr{C}_{R,0\le\tau\le\epsilon}}e^{iS/\hbar}+\int_{\epsilon}^{T_{II}}d\tau\sum_{\mathscr{C}_{R,\tau}}e^{iS/\hbar}. \qquad (5.23)$$

Accordingly, the reflected packet can be decomposed as

$$\Psi_R(II)=\Psi_{R,0\le\tau\le\epsilon}(II)+\int_{\epsilon}^{T_{II}}d\tau\Psi_{R,\tau}(II), \qquad (5.24)$$

where $\Psi_{R,\tau}(II)$ is defined by (5.5) and

$$\Psi_{R,0\le\tau\le\epsilon}(II)\equiv\int dX_I\left(\sum_{\tilde{\mathscr{C}}_{R,0\le\tau\le\epsilon}}e^{iS/\hbar}\right)\Psi(I). \qquad (5.25)$$

This modification means that we consider the set of classes of paths

$$\{\mathscr{C}_0,\mathscr{C}_{R,0\le\tau\le\epsilon},\mathscr{C}_{c,\tau}|c=T,R\ \ \epsilon<\tau\le T_{II}\quad\text{for } c=R,$$
$$0\le\tau\le T_{II}\quad\text{for } c=T\ \} \qquad (5.26)$$

rather than (5.2); class $\mathscr{C}_0$ can be dropped from elements because it is null in the sense of $\Psi_0(II)=0$ and thus irrelevant for C2. Condition C2 for (5.26) is (5.7), in which $\tau,\tau'>\epsilon$ for reflection, plus

$$\text{Re}\int dX_{II}\Psi^*_{R,0\le\tau\le\epsilon}(II)\Psi_{R,\tau}(II)=0, \qquad (5.27)$$

where $\tau>\epsilon$ for $\Psi_{R,\tau}(II)$. In passing, if C2 holds for (5.26), the probability that the reflection time is less than ϵ is given by $P_{R,0\le\tau\le\epsilon}=\int dX_{II}|\Psi_{R,0\le\tau\le\epsilon}(II)|^2$.

Now let us implement (5.3) for $c=T$ and (5.23) explicitly to obtain branches $\Psi_{R,0\le\tau\le\epsilon}(II)$, $\Psi_{R,\tau}(II)$, and $\Psi_{T,\tau}(II)$. Any path in $\tilde{\mathscr{C}}_T$ can be split into three parts: the part from I to the first hit of S_a at time T_i, the part from (a,T_i) to the last hit of S_b at time T_f, and the part from (b,T_f) to II. The last part can be reinterpreted as the first hit of S_b at time $T_{II}-T_f$ after starting from $X=X_{II}$ at $T=0$. For reflection, a similar splitting can be applied only to those paths in class $\mathscr{C}_8$. Thus we have

[19]In the present context, (5.20) has appeared in connection with a free particle in one-dimensional space. Interestingly, the same identity appears in connection with a particle in a weak potential in three-dimensional space (see, e.g., p. 131 of Ref. [14]).

$$\sum_{\tilde{\mathscr{C}}_T} e^{iS/\hbar}=\Theta_2(X_{II})\Theta_1(X_I)\int_0^{T_{II}}dT_f\int_0^{T_f}dT_iF(T_{II}-T_f|X_{II}-b)\Phi(T_f-T_i|b,a)F(T_i|a-X_I). \tag{5.28a}$$

$$\sum_{\tilde{\mathscr{C}}_R} e^{iS/\hbar}=\Theta_1(X_{II})\Theta_1(X_I)\int_0^{T_{II}}dT_f\int_0^{T_f}dT_iF(T_{II}-T_f|a-X_{II})\Phi(T_f-T_i|a,a)F(T_i|a-X_I). \tag{5.28b}$$

Note that F's are those defined in the absence of a potential [i.e., given by (5.19)] because the relevant paths stay out of the barrier region before T_i and after T_f, whereas Φ's are the propagators in the presence of the rectangular potential [25]. By changing the integration variables T_f and T_i to τ and $u\equiv T_i$, we can rewrite (5.28) into the form of τ integrations, whose integrands are found to be

$$\sum_{\tilde{\mathscr{C}}_{T,\tau}} e^{iS/\hbar}=\Theta_2(X_{II})\Theta_1(X_I)\Phi(\tau|b,a)\int_0^{T_{II}-\tau}duF(T_{II}-\tau-u|X_{II}-b)F(u|a-X_I)$$

$$=-i\Theta_2(X_{II})\Theta_1(X_I)\Phi(\tau|b,a)\frac{\partial}{\partial X_{II}}\Phi_{free}(T_{II}-\tau|X_{II}-X_I-b+a), \tag{5.29a}$$

$$\sum_{\tilde{\mathscr{C}}_{8,\tau}} e^{iS/\hbar}=\Theta_1(X_{II})\Theta_1(X_I)\Phi(\tau|a,a)\int_0^{T_{II}-\tau}duF(T_{II}-\tau-u|a-X_{II})F(u|a-X_I)$$

$$=i\Theta_1(X_{II})\Theta_1(X_I)\Phi(\tau|a,a)\frac{\partial}{\partial X_{II}}\Phi_{free}(T_{II}-\tau|2a-X_I-X_{II}), \tag{5.29b}$$

where we have used Eq. (5.20) to carry out the u integrations, and then used Eq. (5.19a); Φ_{free} is the free propagator. Now it is clear why we have to make a special treatment for $\tau=0$ for reflection; due to $\Phi(\tau|a,a)$, (5.29b) behaves as $\tau^{-1/2}$ for small τ and thus diverges at $\tau=0$, but converges when integrated from $\tau=0$. This means that it does not make sense to consider the contribution from those paths in $\mathscr{C}_8$ whose T_i and T_f are exactly the same, which in turn means that $\Psi_{8,0}(II)$ and thus $\Psi_{R,0}(II)$ do not make sense. [We need well-defined branches of $\Psi(II)$ to check C2.] As mentioned earlier, a proper treatment is to consider the contribution from those paths in $\mathscr{C}_8$ whose T_i and T_f differ up to ϵ; their contribution converges and thus it makes sense to consider $\Psi_{R,0\leqslant\tau\leqslant\epsilon}(II)$ defined by (5.25) and (5.22) with $\mathscr{C}$ replaced by $\tilde{\mathscr{C}}$. Now the nonvanishing branches of $\Psi(II)$ for (5.26) are obtained as follows. $\Psi_{T,\tau}(II)$ and $\Psi_{R,\tau}(II)$ are obtained by substituting (5.29) into (5.5), noting that $\tilde{\mathscr{C}}_{R,\tau}=\tilde{\mathscr{C}}_{8,\tau}$ for $\tau\neq0$; $\psi_{R,0\leqslant\tau\leqslant\epsilon}(II)$ is obtained by substituting (5.29b) and $\Sigma_{\tilde{\mathscr{C}}_{10}}e^{iS/\hbar}=\Phi^{R_1}(II;I)$, which is the restricted propagator on R_1, into (5.22) with $\mathscr{C}$ replaced by $\tilde{\mathscr{C}}$ and then using (5.25). Finally, going back to the standard notation [i.e., writing $\Phi_{free}(T_{II}-\tau|X_{II}-X_I-b+a)=\Phi_{free}(X_{II}-(b-a),T_{II}-\tau;X_I,0)$, etc.] and recovering m and $\hbar$, we arrive at

$$\Psi_{T,\tau}(II)=\Theta_2(X_{II})\Phi(b,\tau;a,0)\ \frac{\hbar}{im}\frac{\partial}{\partial X_{II}}\Psi_{free}(X_{II}-(b-a),T_{II}-\tau), \tag{5.30a}$$

$$\Psi_{R,\tau}(II)=-\Theta_1(X_{II})\Phi(a,\tau;a,0)\ \frac{\hbar}{im}\frac{\partial}{\partial X_{II}}\Psi_{free}(2a-X_{II},T_{II}-\tau), \tag{5.30b}$$

$$\Psi_{R,0\leqslant\tau\leqslant\epsilon}(II)=\Theta_1(X_{II})\int dX_I\Phi^{R_1}(II;I)\Psi(I)+\int_0^{\epsilon}d\tau\,\Psi_{R,\tau}(II), \tag{5.30c}$$

where

$$\Psi_{free}(X,T)\equiv\int dX'\Phi_{free}(X,T;X',0)\Psi(X',0), \tag{5.31}$$

where the factor $\Theta_1(X')$ has been dropped from the integrand because the initial packet has been assumed to be localized in R_1. $\Psi_{T,\tau}(II)$ is a part of the transmitted packet that ‘‘spends the amount of time τ in the barrier region’’ in the sense that the paths contributing to it spend that amount of time in the region, and analogously for $\Psi_{R,\tau}(II)$ and $\Psi_{R,0\leqslant\tau\leqslant\epsilon}(II)$. In (5.30b), $2a-X_{II}$ is the mirror reflection of X_{II} with respect to $X=a$. Note that the contributions from the motions inside and outside the barrier region are factorized in (5.30a) and (5.30b); in the former, for example, the factor $\Phi(b,\tau;a,0)$ corresponds to the motion of the particle spending τ in the potential region, and $\Psi_{free}(X_{II}-(b-a),T_{II}-\tau)$ corresponds to the free motion before and after the tunneling; the appearance of $(im)^{-1}\hbar\,\partial/\partial X_{II}$, the ‘‘velocity operator,’’ is the direct consequence of (5.19a) and is natural, since the first time a particle hits S_X should be related to its velocity, and it implies that higher momentum components of the initial packet make greater contributions to $\Psi_{T,\tau}(II)$. By the very construction of the branches of $\Psi(II)$, they satisfy

$$\Psi_{R,0\leqslant\tau\leqslant\epsilon}(II)+\int_{\epsilon}^{T_{II}}d\tau\ \Psi_{R,\tau}(II)+\int_0^{T_{II}}d\tau\ \Psi_{T,\tau}(II)$$

$$=\Psi(II), \tag{5.32}$$

which is taken for granted since C1 holds. As to C2, there is a possibility that (5.27) holds (at least approximately) for a properly chosen ϵ. However, even if it should hold, (5.7) must also hold in order for C2 to hold. Condition (5.7) has to be checked only for $c=c'$, since $\Psi_{T,\tau}(II)$ and $\Psi_{R,\tau}(II)$ are nonoverlapping. Substituting (5.30a) and (5.30b) into (5.8) and changing the variable of integration from X_{II} to $X_{II}-b+a$ for transmission and to $-X_{II}+2a$ for reflection, we have

$$D[(T,\tau);(T,\tau')]=\Phi^*(b,\tau;a,0)\Phi(b,\tau';a,0)I(\tau,\tau';a), \tag{5.33a}$$

$$D[(R,\tau);(R,\tau')]=\Phi^*(a,\tau;a,0)\Phi(a,\tau';a,0)I(\tau,\tau';a), \tag{5.33b}$$

where $\tau,\tau'>\epsilon$ for the latter and

$$I(\tau,\tau';a)\equiv\left(\frac{\hbar}{m}\right)^2\int_a^\infty dX\left[\frac{\partial}{\partial X}\Psi^*_{free}(X,T_{II}-\tau)\right]\times\frac{\partial}{\partial X}\Psi_{free}(X,T_{II}-\tau'). \tag{5.34}$$

$D[(c,\tau);(c,\tau')]$ depends on the initial condition through $I(\tau,\tau';a)$ and on the potential through $\Phi(b,\tau;a,0)$ or $\phi(a,\tau;a,0)$.

Apparently the real parts of (5.33) do not vanish in general. All that is left for the success of C2 is the possibility that the real parts vanish in some restricted situations. What can be restricted are the initial condition $\Psi(I)$ and the parameters of the potential, a, b, and V. However, (5.33) explicitly shows that, regardless of such restrictions, $D[(c,\tau);(c,\tau')]$ is a continuous function of τ and τ', and so its real part. Thus $\mathrm{Re}D[(c,\tau);(c,\tau')]$ can never be proportional to $\delta(\tau-\tau')$. Therefore, C2 does not hold irrespective of $\Psi(I)$, a, b, and V. We conclude that tunneling time probability densities cannot be defined within the present framework. [Incidentally, $P_{c,\tau}$ vanishes identically since C2 fails. See the comment below (2.18).] The decoherence functionals would also be continuous in τ and τ' for more general potential barriers, although their explicit expressions depend on the shape of potential. Thus the negative conclusion reached here for a rectangular barrier would also be true for more general potential barriers.

Our conclusion is in sharp contrast with that of Sokolovski, Connor, and Brouard [26]. They have proposed several probability distributions (probability densities in our terminology) of traversal time. A detailed review of their work is out of the scope of this paper, and the present author would like to point out only the essential differences between their approach and ours. They also work within the framework of path integral, but start from a different definition of tunneling (traversal) time which is not (5.11) and use a different prescription to process an amplitude into a probability density [our prescription is (2.18)]. The biggest difference is, however, the absence of a criterion like our C2 which is used to determine probability-definable situations. Probability densities of tunneling times have also been discussed by Dumont and Marchioro [27]. Their discussion implicitly relies on the assumption that a positive quantity $\int dX\int dT|\Psi(X,T)|^2$ with appropriate integration ranges is meaningful as a ‘‘dwell time’’ that a particle is in the barrier region irrespective of whether it is eventually reflected or transmitted. Although this assumption has been accepted by many authors, it has been questioned by Olkhovsky and Recami [28]. Hauge and Støvneng [13(c)] also warn that the assumption is not universally accepted for wave packets, and the present author shares the same opinion. Leavens, Aers, and McKinnon [29] use the Bohm trajectory approach [30] to define tunneling time probability densities. Unlike Feynman's paths, Bohm trajectories are associated with positive weights, and thus the definition of tunneling time probability densities is straightforward. However, the results heavily rely on the properties of Bohm trajectories, whose physical meaning is not necessarily clear, so that the reproduction of the same results by some independent methods is hoped. The Bohm trajectory approach by Leavens and co-workers and the Feynman path approach by Sokolovski and co-workers have been compared in [29(d)]. The present author has recently learned that Leavens [31] had come to the conclusion that mean transmission and reflection times are meaningless concepts within the consistent-histories approach.

VI. SUMMARY AND CONCLUDING REMARKS

The main points are as follows.

(1) Most of the studies of the histories approach have been on discrete histories defined by products of projection operators in Hilbert space. By contrast, we have studied continuous histories defined by Feynman's paths in configuration space.

(2) Probabilities can be defined for a given set of classes of paths, provided that both C1 (the path classifiability condition) and C2 (the no-interference condition) hold. The combination of C1 and C2 is considered as a consistency condition between wave and particle natures. A set of classes of paths satisfying the two conditions is considered as a space-time analog of observables. We have focused on cases where C1 holds.

(3) In general, the probability for a class of paths cannot be regarded as the probability that the particle actually follows the paths in the class.

(4) We have found examples of sets of classes of paths for which C2 holds and probabilities can be defined.

(5) In all the examples, the initial condition of the particle of interest is appropriately restricted. The examples show, however, that not only the initial condition but also other elements entering into the definition of the decoherence functional can be crucial to the success or failure of C2.

(6) In the examples, restrictions on the initial condition and on the other elements are essential for the resultant probabilities to take reasonable values consistent with intuition.

(7) The examples imply that C2 holds when decomposition (3.8) takes a ‘‘natural’’ form reflecting characteristic features of the motion of Schrödinger's wave function with respect to Ω.

(8) The transmission and the reflection probabilities (for a rectangular potential barrier) are special cases of probabilities for classes of continuous histories.

(9) Probability densities for the transmission and the reflection times cannot be defined for a rectangular potential barrier no matter how the initial condition is restricted.

The present approach does not always define probabilities for histories. One may feel that this is uncomfortable; especially, one may insist that probabilities should be defined, if they are definable at all, for an arbitrary initial condition. This insistence arises perhaps from an implicit assumption that in the conventional QM probabilities are defined for an arbitrary initial condition. This is true for probabilities for observables, but is not necessarily the case in general; such probabilities that are not always definable also play important roles in the conventional QM. For example, tunneling probabilities, which are very important in understanding tunneling phenomena, are not always definable (consider, for instance, the case when the initial wave packet is localized in the barrier region). Tunneling probabilities are special cases of probabilities for histories, and hence it is not surprising that probabilities for histories in general are not always definable. However, it should be pointed out that, although C1 and C2 were explained earlier as conditions which define a space-time analog of observables, the resultant probabilities are not always definable. The property that probabilities are always definable for observables is not owned by our analog of observables. This is what is lost by our generalization of observables. What is gained is an understanding that the combination of C1 and C2 makes it possible to introduce the concept of probabilities for histories into the sum-over-paths formulation of QM as a natural generalization of probabilities for observables.

All the examples found in Secs. III C are such that not only C2 but also the medium decoherence condition [Eq. (2.14) without ''Re''] is satisfied. It remains a problem to find cases, if any, in which the medium decoherence condition fails but C2 holds. In all the examples found in Sec. III C, restrictions on the initial condition of a particle (and also on other elements entering into the definition of the decoherence functional) are essential for the success of C2, while C2 always holds when classes of paths are defined by path classifications across constant time surfaces. This implies that it is only when constant time surfaces are used to classify paths into classes that C2 holds without restrictions. Investigation of this implication is of great interest. As to tunneling time probability densities, we plan to investigate if appropriate coarse graining of transmission and reflection times leads to the success of C2. Analysis of other approaches to tunneling time probability densities from the viewpoint of the histories approach is also interesting. More broadly, investigations of the usefulness of the histories approach in other nonlocal-in-time problems are interesting.

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APPENDIX: ''BY-HAND'' NORMALIZATION

One might ask if there are other ways of defining probabilities for histories in such a way that the minimal requirements are satisfied. There *are* other ways, and one of them is the following: Abandon (2.6) for $\mathscr{C}_{j\vee k}$ $(j\neq k)$, while maintaining it for $\mathscr{C}_j$ and $\mathscr{C}_k$, and drop condition C2 (we assume that C1 holds). Equation (2.15) is then invalid, since the first equality fails. Let us now define

$$\overline{P_j}\equiv\frac{P_j}{\sum_k P_k}, \tag{A1a}$$

$$\overline{P_{j\vee k}}\equiv\overline{P_j}+\overline{P_k} \quad (j\neq k). \tag{A1b}$$

The probability axioms are satisfied by $\overline{P}$. Considering $\{\mathscr{C}_Y,\mathscr{C}_N\}$, we can easily prove that $\overline{P_Y}$ tends to $\int dX|\Psi|^2$ as $\Omega\rightarrow\Delta X$. Thus the above construction of $\overline{P}$ meets the minimal requirements. When C2 is satisfied, (2.6) and (A1) are the same thing. But otherwise there is a big difference; (A1) defines probabilities but our approach does not. Why have we not chosen (A1), which looks much simpler than our approach? One reason is that the values which (A1) gives are not acceptable, as illustrated later. Besides, the present author is not satisfied with (A1) for the following two reasons. (1) The ''amplitude sum rule'' (i.e., the superposition principle) has not been taken into account, whereas the probability sum rule has been forced to hold by adopting it as a definition. By contrast, neither of the two sum rules is given priority over the other in our approach; recall that C2 is the condition for the two sum rules not to conflict. (2) The way of obtaining normalized probabilities is artificial. Compare the ''by-hand'' normalization (A1a) with our ''automatic'' normalization (2.15), in which the normalization is a consequence of the normalization at an initial time. By-hand normalizations may be used if it is *a priori* clear that we are dealing with an ESEE.[20] But it is this very point that needs careful treatment when we deal with histories, and thus by-hand normalizations should not be used. It should also be pointed out that the by-hand normalization is not possible for

[20] This is because an ESEE is defined in this paper as a set of classes of Feynman's paths satisfying both C1 and C2 and because the automatic normalization and the by-hand normalization give the same result if C1 and C2 are satisfied. Note that the present author is not opposed to the use of by-hand normalizations in general but to the by-hand normalization of the positive quantities defined by (2.6). In fact, by-hand normalizations can be applied to classical unnormalized probabilities and also to quantum unnormalized probabilities for instantaneous events to get normalized probabilities. In such cases, the notion of exhaustiveness and exclusiveness is *a priori* clear and this, in the present author's opinion, justifies the use of by-hand normalizations.

probability densities P_λ, because $P_\lambda \equiv 0$ if C2 fails, as shown in Sec. II A. Thus, the by-hand normalization cannot be a universal procedure of normalization. It seems to the present author that the ways of defining probabilities for histories are fairly restricted if we require the following. (1) The approach must be simple and meet the minimal requirements. (2) In constructing probabilities from amplitudes, neither the probability axioms nor the superposition principle must be discarded. (3) The normalization of probabilities should be a direct consequence of the existence of a particle at an initial time. All these are satisfied by our approach.

Now let us see by an example that (A1) is not a good choice. We use results obtained in Sec. III A. The sum-over-paths amplitude for $\mathscr{C}_Y$ is calculated as

$$\Phi_Y(B;A)=\sum_{J=1}^{9} \Phi_J(B;A)=\Phi(B;A)-\sum_{j=1,2} \int_{R_j} dX_{II} \int_{R_j} dX_I \Phi(B;II)\Phi^{R_j}(II;I)\Phi(I;A), \tag{A2}$$

where we have used (3.5) and (3.6). The second term (after the minus sign) on the right-hand side of (A2) is equal to $\Phi_N(B;A)$. Assume that $\Psi(I)$ is localized in R_1 and also assume ideally that $\Psi(II)$ is completely localized in R_2 (the same situation as shown in Fig. 6). We put (A2) into (2.6) for $\mathscr{C}=\mathscr{C}_Y$. Noting $\int dX_A \Phi(I;A)\Psi(A)=\Psi(I)$ and defining

$$\chi_j(II) \equiv \int_{R_j} dX_I \Phi^{R_j}(II;I)\Psi(I), \tag{A3}$$

which vanishes for $j=2$ because of the localization of $\Psi(I)$, we obtain

$$P_Y=\int dX_B|\Psi(B)|^2-2\ \mathrm{Re}\int dX_B\Psi^*(B)\int_{R_1} dX_{II}\Phi(B;II)\chi_1(II)+\int dX_B\left|\int_{R_1} dX_{II}\Phi(B;II)\chi_1(II)\right|^2. \tag{A4}$$

The right-hand side is evaluated as follows. The first term is unity. The second term is zero because $\int dX_B \Psi^*(B)\Phi(B;II)=\Psi^*(II)$ and this is zero on R_1. The third term is unity; to see this, use (2.7) to carry out the X_B integration to get $\int_{R_1} dX_{II}|\chi_1(II)|^2$, and then use (3.14) to carry out the X_{II} integration to have $\int_{R_1} dX_I|\Psi(I)|^2$, which is unity because of the assumption of the initial localization. Consequently we have $P_Y=2$, and in the same way we get $P_N=1$. Normalizing these results according to (A1a), we obtain

$$\overline{P_Y}=\frac{2}{3}, \quad \overline{P_N}=\frac{1}{3}. \tag{A5}$$

It thus follows that the probabilities take these fixed values when the initial and the final packets are localized in R_1 and R_2, respectively. It is not reasonable that only these constraints completely determine the values of the probabilities. In conclusion, not only is the by-hand normalization unsatisfactory in spirit, it also produces unacceptable results in practice.

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