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ARTICLES

Entangled entanglement

Günther Krenn
Atominstitut der Österreichischen Universitäten, Schüttelstraße 115, A-1020 Wien, Austria

Anton Zeilinger
Institut für Experimentalphysik, Universität Innsbruck, Technikerstraße 25, A-6020 Innsbruck, Austria

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In entangled systems values cannot be assigned to all quantum-mechanical observables of individual members of the system independent of the measurement context of the whole system. While various cases are known where properties like spin, momentum, energy, etc. can be entangled, entanglement itself is usually considered to be an objective property of the system. We show that situations can arise where this is no longer the case, and where therefore entanglement itself becomes an entangled property. [S1050-2947(96)07008-4]

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I. INTRODUCTION

The term entanglement (in German, *Verschränkung)* was introduced by Schrödinger [1] into the discussion of the foundations of quantum physics. It describes a system composed of two or more particles, which exhibits the astonishing property that the results of measurements on one particle cannot be specified independently of the parameters of the measurements on the other particles. Although the different measurements can take place in spacelike separated regions, the results of each measurement depend on the complete experimental context [2] of the whole system. In this paper we will discuss an interesting consequence of contextuality in three-particle systems.

Usually entanglement is considered to be an objective property of the system. Although this is generally true, we will show that there can be situations in which entanglement depends on the measurement context and therefore becomes an entangled property itself.

Consider an arbitrary entangled system of three spin-$\frac{1}{2}$ particles. Spin measurements are performed on the particles in spacelike separated regions by three observers (cf. Fig. 1). The state is such that each observer obtains the result $+1$ or -1 with equal probability. In our analysis we will specifically consider data following a simple rule. Only when observer 3 obtains the result $+1$ are the corresponding results of observers 1 and 2 analyzed (cf. Fig. 2). Of course we could equally well consider the data given by the result -1 in the corresponding measurement on particle 3.

To which conclusions are we led, if we analyze the results of observers 1 and 2 which have been selected in the way specified above? Are these results correlated classically or quantum mechanically? Does the correlation depend on the measurement performed on particle 3? Is the correlation of

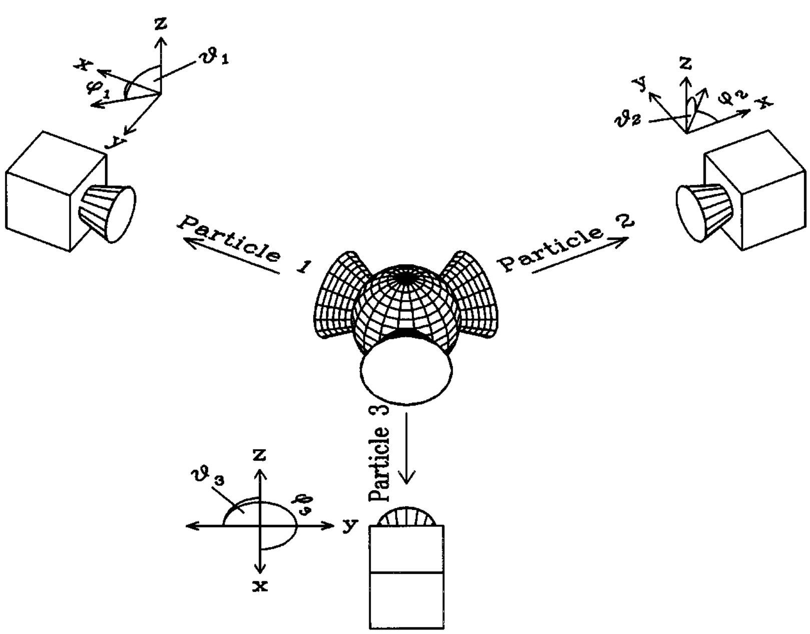

FIG. 1. Three spin-$\frac{1}{2}$ particles are emitted by a common source. Spin measurements along directions defined by the spherical coordinates $(\vartheta_i,\varphi_i)(i=1,2,$ and 3) are performed in spacelike separated regions by three observers.

the selected results the same as if we had made no selection? In answering these questions we will see that the selected results of observers 1 and 2 can be entangled under certain conditions, and not entangled under other conditions where the entanglement is entangled with the specific choice of measurement carried out on particle 3.

II. TWO-PARTICLE CORRELATIONS

Consider a three-particle system described by the *GHZ* state [3,4] of the form proposed by Mermin [5],

$$|\Psi\rangle=\frac{1}{\sqrt{2}}(|z+\rangle|z+\rangle|z+\rangle+|z-\rangle|z-\rangle|z-\rangle). \quad (1)$$

The three spin-$\frac{1}{2}$ particles are emitted by a common source into distinct directions. By adequately oriented Stern-Gerlach devices, spin measurements along arbitrary directions are performed in spacelike separated regions by three observers (cf. Fig. 1).

Observer 1 measures the spin of particle 1 along direction $\vec{e_1}$ defined by the spherical coordinates (ϑ_1,φ_1). Likewise, observers 2 and 3 perform measurements along directions $\vec{e_2}$ (ϑ_2,φ_2) and $\vec{e_3}$ (ϑ_3,φ_3), respectively. Completely independent of the measurement directions, all three observers obtain the two possible results $+1$ $(+\hbar/2)$ and -1 $(-\hbar/2)$ with equal probability.

To be able to analyze the correlations of the results obtained in such spin measurements, we have to rewrite state (1) in terms of eigenstates $|\chi_{+\vec{e_i}}\rangle$ and $|\chi_{-\vec{e_i}}\rangle$ of spin-measurements along arbitrary directions $\vec{e_i}$ (ϑ_i,φ_i). Using the identities

$$\begin{aligned}|z+\rangle&=e^{i\varphi_i/2}\left(\cos\frac{\vartheta_i}{2}|\chi_{+\vec{e_i}}\rangle-\sin\frac{\vartheta_i}{2}|\chi_{-\vec{e_i}}\rangle\right),\\|z-\rangle&=e^{-i\varphi_i/2}\left(\sin\frac{\vartheta_i}{2}|\chi_{+\vec{e_i}}\rangle+\cos\frac{\vartheta_i}{2}|\chi_{-\vec{e_i}}\rangle\right),\end{aligned} \quad (2)$$

we obtain

$$\begin{aligned}|\Psi\rangle=\frac{1}{\sqrt{2}}\Bigg\{&e^{i/2\,(\varphi_1+\varphi_2+\varphi_3)}\left[-\sin\left(\frac{\vartheta_1}{2}\right)|\chi_{-\vec{e}_1}\rangle+\cos\left(\frac{\vartheta_1}{2}\right)|\chi_{+\vec{e}_1}\rangle\right]\left[-\sin\left(\frac{\vartheta_2}{2}\right)|\chi_{-\vec{e}_2}\rangle+\cos\left(\frac{\vartheta_2}{2}\right)|\chi_{+\vec{e}_2}\rangle\right]\\&\times\left[-\sin\left(\frac{\vartheta_3}{2}\right)|\chi_{-\vec{e}_3}\rangle+\cos\left(\frac{\vartheta_3}{2}\right)|\chi_{+\vec{e}_3}\rangle\right]+e^{(-i/2)(\varphi_1+\varphi_2+\varphi_3)}\left[\cos\left(\frac{\vartheta_1}{2}|\chi_{-\vec{e}_1}\rangle+\sin\left(\frac{\vartheta_1}{2}\right)|\chi_{+\vec{e}_1}\rangle\right]\\&\times\left[\cos\left(\frac{\vartheta_2}{2}\right)|\chi_{-\vec{e}_2}\rangle+\sin\left(\frac{\vartheta_2}{2}\right)|\chi_{+\vec{e}_2}\left[\cos\left(\frac{\vartheta_3}{2}\right)|\chi_{-\vec{e}_3}\rangle+\sin\left(\frac{\vartheta_3}{2}\right)|\chi_{+\vec{e}_3}\rangle\right]\right]\Bigg\}.\end{aligned} \quad (3)$$

The *GHZ* state (1) in this general form is the basis for our further considerations. First let us investigate unconditional two-particle correlations. What is, for example, the unconditional correlation function E_{12} $(\vec{e_1},\vec{e_2})$ (the unconditional expectation value of the product of the results is independent of the measurements on particle 3) of the results of spin measurements on particles 1 and 2 along directions $\vec{e_1}$ and $\vec{e_2}$, respectively? Because the product of the results is $+1$ in case of equal results and -1 in case of different results, E_{12} $(\vec{e_1},\vec{e_2})$ can be expressed in terms of the probabilities for equal $[P_{12}(=)]$ and different results $[P_{12}(\neq)]$ in the considered measurements on particles 1 and 2:

$$E_{12}(\vec{e_1},\vec{e_2})=P_{12}(=)-P_{12}(\neq). \quad (4)$$

$P_{12}(=)$ and $P_{12}(=)$ can be easily calculated by taking into account the corresponding probability amplitudes in (3). For example, the unconditional probability for equal results $P_{12}(=)$ is calculated in the following way:

$$\begin{aligned}P_{12}(=)=\frac{1}{2}\Bigg\{&\left|e^{-i\Phi}\left(\cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2}\sin\frac{\vartheta_3}{2}\right)-e^{i\Phi}\left(\sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2}\cos\frac{\vartheta_3}{2}\right)\right|^2+\Bigg|e^{i\Phi}\left(\cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2}\sin\frac{\vartheta_3}{2}\right)\\&+e^{-i\Phi}\left(\sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2}\cos\frac{\vartheta_3}{2}\right)\Bigg|^2+\left|e^{i\Phi}\left(\cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2}\cos\frac{\vartheta_3}{2}\right)+e^{-i\Phi}\left(\sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2}\sin\frac{\vartheta_3}{2}\right)\right|^2\\&+\left|e^{-i\Phi}\left(\cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2}\cos\frac{\vartheta_3}{2}\right)-e^{i\Phi}\left(\sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2}\sin\frac{\vartheta_3}{2}\right)\right|^2\Bigg\}\\=&\frac{1}{2}+\frac{\cos(\vartheta_2-\vartheta_3)}{4}+\frac{\cos(\vartheta_2+\vartheta_3)}{4},\end{aligned}$$

where $\Phi=\varphi_1/2+\varphi_2/2+\varphi_3/2$.

With

$$P_{12}(\neq)=\frac{1}{2}-\frac{\cos(\vartheta_2-\vartheta_3)}{4}-\frac{\cos(\vartheta_2+\vartheta_3)}{4}$$

we finally obtain the result

$$E_{12}(\vartheta_1,\vartheta_2)=\cos(\vartheta_1)\cos(\vartheta_2). \tag{5}$$

This function is factored with respect to the parameters ϑ_1 and ϑ_2. Therefore the unconditional two-particle correlations do not violate Bell's inequality. This implies that the results of observers 1 and 2 are correlated classically for all directions $\vec{e_1}$ and $\vec{e_2}$, and are compatible with locality. Because of this fact one could expect that any subensemble of such ''classical'' results is also correlated classically. But this is not the case, as we will show in the following.

III. ENTANGLED SUBENSEMBLES

In Sec. I we already defined our procedure for selecting subensembles of the results of observers 1 and 2 by using the results of observer 3. Whenever observer 3 measures the result $+1$ (-1), the corresponding results of observers 1 and 2 are assigned to subensemble $+$ $(-)$ (cf. Fig. 2). As mentioned above the experiment can be arranged in such a way that all three observers obtain their results in a spacelike separated manner. Then the results of observer 3 which determine the two subensembles are totally independent of the results of observers 1 and 2.

We now consider the special case in which observers 1 and 2 perform spin measurements within the x-y plane $(\vartheta_1=\vartheta_2=\pi/2)$, whereas observer 3 measures along an arbitrary direction $\vec{e_3}$. Then, from Eq. (5), $E_{12}=0$, which means that the joint unconditional results of observers 1 and 2 are maximally random.

In order to calculate the correlation function for subensembles $+$ and $-$, respectively, we apply the projection operators $P^+=|\chi_{+\vec{e}_1}\rangle\langle\chi_{+\vec{e}_1}|$ (for subensemble $+$) and $P^-=|\chi_{-\vec{e}_1}\rangle\langle\chi_{-\vec{e}_1}|$ (for subensemble $-$) to (3) (with $\vartheta_1=\vartheta_2=\pi/2$). For subensemble $+$ we obtain

$$\sqrt{2}\mathrm{P}^+|\Psi\rangle=e^{i/2(\varphi_1+\varphi_2+\varphi_3)}\cos\left(\frac{\vartheta_1}{2}\right)|\chi_{+\vec{e}_1}\left(-\frac{|\chi_{-\vec{e}_2}\rangle}{\sqrt{2}}+\frac{|\chi_{+\vec{e}_2}\rangle}{\sqrt{2}}\right)\left(-\frac{|\chi_{-\vec{e}_3}\rangle}{\sqrt{2}}+\frac{|\chi_{+\vec{e}_3}}{\sqrt{2}}\right)$$
$$+e^{(-i/2)(\varphi_1+\varphi_2+\varphi_3)}\sin\left(\frac{\vartheta_1}{2}\right)|\chi_{+\vec{e}_1}\rangle\left(\frac{|\chi_{-\vec{e}_2}}{\sqrt{2}}+\frac{|\chi_{+\vec{e}_2}\rangle}{\sqrt{2}}\right)\left(\frac{|\chi_{-\vec{e}_3}\rangle}{\sqrt{2}}+\frac{|\chi_{+\vec{e}_3}\rangle}{\sqrt{2}}\right). \tag{6}$$

A similar expression can be derived for subensemble $-$. By considering the corresponding probability amplitudes in (6) we may then calculate the probabilities for equal and different results in subensemble $+$ [$P_{12}^+(=)$ and $P_{12}^+(\neq)$, respectively]. According to Eq. (4) we finally obtain the correlation functions E_{12}^+ (E_{12}^-) for subensemble $+$ $(-)$,

$$\begin{aligned}E_{12}^{\pm}(\varphi_1,\varphi_2)&=P_{12}^{\pm}(=)-P_{12}^{\pm}(\neq)\\&=\pm\sin\vartheta_3\cos(\varphi_1+\varphi_2+\varphi_3).\end{aligned} \tag{7}$$

In the following we investigate if the correlation functions (7) violate Bell's inequality, and for which values of the parameters ϑ_3 and φ_3 (direction of spin-measurements on particle 3) a violation occurs. Bells inequality is given by

$$\begin{aligned}2\leqslant|&E_{12}^{\pm}(\varphi_1,\varphi_2)+E_{12}^{\pm}(\varphi_1,\varphi_2')+E_{12}^{\pm}(\varphi_1',\varphi_2)\\&-E_{12}^{\pm}(\varphi_1',\varphi_2')|,\end{aligned} \tag{8}$$

where φ_1, φ_1' and φ_2, φ_2' denote two different directions for spin measurements on particles 1 and 2, respectively. By inserting (7) into (8), we can derive conditions on which the right-hand side of (8) becomes extremal:

$$\varphi_3+\varphi_1'+\varphi_2'=\frac{k\pi}{4},\quad k=1,3,5,\ldots,$$

$$\varphi_1=\varphi_1'+\pi/2\quad\text{and}\quad\varphi_2=\varphi_2'+\pi/2.$$

According to these conditions we now define the values of the angles in Bells inequality. For $\varphi_3=0$ and $k=1$ we obtain $\varphi_1=\pi/4$, $\varphi_1{}'=3\pi/4$, $\varphi_2=0$, and $\varphi_2{}'=\pi/2$. With these values Bell's inequality reads as follows:

$$\begin{aligned}2&\leqslant\left|\pm\sin\vartheta_3\left(\cos\frac{\pi}{4}+\cos\frac{3\pi}{4}+\cos\frac{3\pi}{4}-\cos\frac{5\pi}{4}\right)\right|\\&=2\sqrt{2}\sin\vartheta_3.\end{aligned} \tag{9}$$

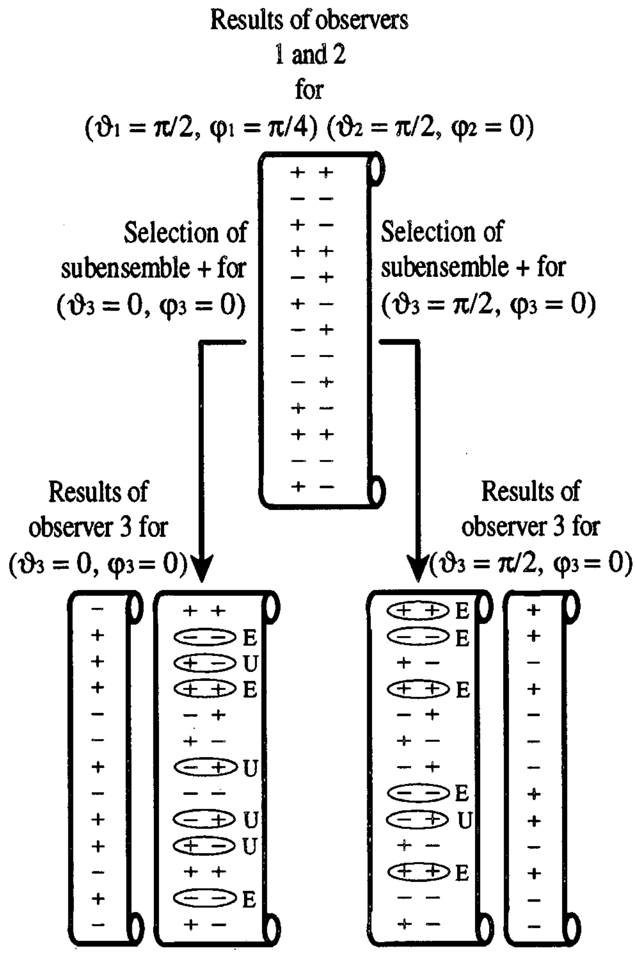

FIG. 2. The results of observers 1 and 2 shown on the list on top are separated into different subensembles depending on the measurement performed by observer 3. For two such possible measurements of observer 3 the selection of subensemble + is demonstrated. Results belonging to subensemble + are pointed out by an ellipse surrounding them. E stands for equal results, and U for unequal results. In case observer 3 chooses $\vartheta_3=0$, the numbers of equal and unequal results in subensemble + are the same. For $\vartheta_3=\pi/2$ equal results are predominate. Whereas for $\vartheta_3=0$ the results in subensemble + do not violate Bells inequality, a violation by the maximum amount permitted by quantum mechanics is possible for $\vartheta_3=\pi/2$.

First of all we notice that the value of the right-hand side of inequality (9) is the same for subensembles + and $-$. Thus the correlations in both subensembles are either classical or quantum mechanical. It can be seen easily that inequality (9) is violated for $\pi/4<\vartheta_3<3\pi/4$. For $\vartheta_3=\pi/2$, i.e., in case of spin measurements on particle 3 within the x-y plane, inequality (9) is violated by the maximum amount $(2\sqrt{2})$ permitted by quantum mechanics (cf. Fig. 2).

If we do not restrict the spin measurements on particles 1 and 2 to directions within the x-y plane, we also have to consider the parameters ϑ_1 and ϑ_2 in (6). As a consequence, (9) is modified in the following way:

$$2\leqslant 2\sqrt{2}\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3. \tag{10}$$

This means that for general directions $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ the selected results of observers 1 and 2 violate Bell's inequality only if $\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3>1/\sqrt{2}$. Therefore the condition for a maximum violation $(2\sqrt{2})$ is $\vartheta_1=\vartheta_2=\vartheta_3=\pi/2$ $(\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3=1)$, meaning that all three observers perform spin measurements within the x-y plane. It is remarkable that measurements within the x-y plane also form the basis of the arguments leading to the *GHZ* contradiction [4].

IV. DISCUSSION

By selecting the results of observers 1 and 2 with respect to the results of observer 3, one obtains two subensembles. Subensemble + $(-)$ contains the results of observers 1 and 2, which coincide with the result + $(-)$ in the corresponding measurement on particle 3. Depending on the measurement direction $\vec{e}_3$ the data within the selected subensembles can be entangled, but there are measurement directions $\vec{e}_3$ for which the results of observers 1 and 2 are selected in a way not leading to an entanglement within the subensembles (cf. Fig. 2).

Strictly speaking, the remaining state of particles 1 and 2 after a spin measurement on particle 3 will always be entangled, unless particle 3 is measured along the z direction [6]. As shown by Gisin [7], for any entangled state suitable observables can be found, which lead to a violation of a variant of Bell's inequality. But, as shown here, the violation will in general depend on the parameter of the measurement on particle 3. Therefore the entanglement of the results in subensembles + and $-$ for specific observables cannot be considered as an objective property being independent of the measurement context.

We stress that the measurements performed by observers 1 and 2 can be totally independent, for example, in a spacelike separated way, from observer 3. As a consequence the results of observers 1 and 2 are fully independent of whatever measurement observer 3 decides to perform. The interesting observation then is that, depending on the specific kind of measurement on particle 3, the same set of results of measurements on particles 1 and 2 can be divided into subensembles in very different ways, which may lead either to individual entangled subensembles or to unentangled ones. The dependence of the entanglement of the subensembles on a measurement parameter (ϑ_3) of a measurement which can take place in a spacelike separated region motivates the statement that in this situation entanglement itself is an entangled property.

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