Reply to "Comment on 'Why quantum mechanics cannot be formulated as a Markov process'"

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In reply to the preceding Comment on an article by D. T. Gillespie [Phys. Rev. A **49**, 1607 (1994)], it is emphasized that the most plausible interpretation of the singly conditioned probability function implies that the evolution of the two-state quantum system being considered *cannot* be characterized as a Markov process on those states over *any* time interval. And it is pointed out that an alternative interpretation that is offered in the Comment appears to be in conflict with quantum mechanics. [S1050-2947(96)01608-3]

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For the two-state quantum system considered by Gillespie [1], and as dimensionally rescaled by Garbaczewski and Olkiewicz [2], quantum mechanics stipulates the following. If the system is in state 1 (2) at time s, then at any time $t\geqslant s$ the probability of finding the system in state 1 (2) is $\cos^2(t-s)$, and the probability of finding the system in state 2 (1) is $\sin^2(t-s)$. For any discrete-state stochastic process (Markovian or not), it is customary to define $p(j,t|i,s)$ to be the probability that the system will be in state j at time t given that it was in state i at time $s\leqslant t$. So, if we are to describe our two-state quantum system as some kind of stochastic process, then the preceding stipulation of quantum mechanics apparently demands that we have

$$p(j,t|i,s)=\begin{cases}\cos^2(t-s) & \text{if } j=i(=1,2)\\ \sin^2(t-s) & \text{if } j\neq i(=1,2)\end{cases}\quad (s\leqslant t). \tag{1}$$

If a stochastic process is to be *Markovian*, then its singly conditioned probability function $p(j,t|i,s)$ must satisfy the Chapman-Kolmogorov condition. For a two-state process, that condition is

$$p(j,u|i,s)=p(j,u|1,t)p(1,t|i,s)+p(j,u|2,t)p(2,t|i,s)\quad(\forall s\leqslant t\leqslant u). \tag{2}$$

Straightforward algebra will show that the p function in Eq. (1) does *not* satisfy condition (2). From this we must conclude that our quantum system *cannot* be characterized as a Markovian stochastic process over the states 1 and 2. This is precisely the conclusion reached in Ref. [1].

Garbaczewski and Olkiewicz [2] point out, in their Eqs. (17), that the function p defined by

$$p(j,t|i,s)=\begin{cases}\dfrac{\cos^2 t-\sin^2 s}{\cos 2s} & \text{if } j=i(=1,2)\\[2ex] \dfrac{\cos^2 s-\cos^2 t}{\cos 2s} & \text{if } j\neq i(=1,2)\end{cases}\quad (s\leqslant t) \tag{3}$$

satisfies the Chapman-Kolmogorov equation (2) for all $0\leqslant s\leqslant t\leqslant u\leqslant \pi/4$. They conclude from that fact that this p function describes a Markov process for times between 0 and $\pi/4$. One might cogently argue that we do not have a license to summarily "halt" the unfolding of time at $\pi/4$. But to make that argument would obscure an even more salient point: The p function in Eq. (3) does not describe our quantum system. For, if we are given that our quantum system is in state 1 at time s, then quantum mechanics stipulates that the probability of finding the system in state 2 at any time $t>s$ is $\sin^2(t-s)$, as given in Eq. (1), *not* $(\cos^2 s-\cos^2 t)/\cos 2s$ as given in Eq. (3). Since the p function in Eq. (3) does not correctly describe the temporal behavior of our quantum system, then the question of whether or not that p function describes a Markov process on *any* time interval is irrelevant for our purposes here.

I suggest that there is a flawed premise in the approach of Garbaczewski and Olkiewicz [2], namely, their premise [which is implicit in their Eq. (5) and the equation stated in text just before their Eqs. (17)] that what quantum mechanics gives us is *not* the singly conditioned probability function $p(j,t|i,s)$ but rather an *unconditioned* probability function,

$$p_j(t)=\begin{cases}\cos^2 t & \text{if } j=1\\ \sin^2 t & \text{if } j=2.\end{cases} \tag{4}$$

It should be noted that truly unconditioned probability functions play a very minor role in Markov process theory. They play no role at all in that most fundamental equation of the theory, the Chapman-Kolmogorov equation. They simply allow us to make statements such as, if at time s we are given that the probability of the system being in state i is p_i, then the probability of the system being in state j at any time $t>s$ is $\Sigma_i p(j,t|i,s)p_i$. But the salient point here is that the function $p_j(t)$ defined in Eq. (4) is *not*, in fact, an unconditioned probability. This can be seen simply by noting that $p_j(t=0)$ is 1 if $j=1$ and 0 if $j=2$; hence the probability $p_j(t)$ in Eq. (4) is actually the *conditioned* probability function $p(j,t|i,s)$ for the particular variables assignment $i=1$ and $s=0$.

In their paragraph following Eq. (4), Garbaczewski and Olkiewicz [2] criticize the approach in Ref. [1] because it characterizes the random dynamics "exclusively in terms of transition probabilities and with no reference to a probability measure of the process." But as any book on Markov process theory will attest [3], the singly conditioned probability function $p(j,t|i,s)$ embodies *all that can be known* about a Markov process, and so there is no need to supplement that function by some "probability measure." For example, the very important forward and backward master equations, which are derivable from the Chapman-Kolmogorov equation in the discrete-state context, are nothing more than time-evolution equations for $p(j,t|i,s)$. The common practice of referring to the singly conditioned probability function $p(j,t|i,s)$ as a ''transition probability'' unfortunately tends to obscure the self-sufficient and all-encompassing role of that function in Markov process theory.

In conclusion, if one accepts the eminently plausible premise that quantum mechanics prescribes for our quantum system the singly conditioned probability function (1), then our system, considered as a stochastic process over states 1 and 2, is clearly *not* Markovian. If, on the other hand, one does *not* accept that premise, then I submit that one needs first to *justify* that nonacceptance with arguments that are more penetrating and persuasive than those given in this Comment. For example, one should then explain how it can possibly make any difference to the system at time t whether the system was ''in state i'' at an earlier time s and that fact is known as a consequence of an immediately preceding measurement, or the system was ''in state i'' at time s and that fact is unknown to any observer. Such a contrived distinction applied to the elementary notion of the system being ''in state i'' certainly appears to this writer to go against the original realist motivation for trying to devise a Markovian interpretation of quantum mechanics. For, if one merely devises a Markov process interpretation of quantum dynamics that is just as weird and quirky as conventional quantum mechanics, what has been gained?

[1] D. T. Gillespie, Phys. Rev. A **49**, 1607 (1994).

[2] P. Garbaczewski and L. Olkiewicz, preceding paper, Phys. Rev. A **53**, 1733 (1996).

[3] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992); C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, Berlin, 1985); D. T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic, New York, 1991).