COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on ''Why quantum mechanics cannot be formulated as a Markov process''

Piotr Garbaczewski and Robert Olkiewicz
Institute of Theoretical Physics, University of Wrocław, PL-50 204 Wrocław, Poland
(Received 12 July 1995; revised manuscript received 11 January 1996)

In the paper with the above-noted title, D. T. Gillespie [Phys. Rev. A **49**, 1607 (1994)] claims that the theory of Markov stochastic processes cannot provide an adequate mathematical framework for quantum mechanics. In conjunction with the specific quantum dynamics considered there, we give a general analysis of the associated dichotomic jump processes. If we assume that Gillespie's ''measurement probabilities'' *are* the transition probabilities of a stochastic process, then the process must have an invariant (time independent) probability measure. Alternatively, if we demand the probability measure of the process follow the quantally implemented (via the Born statistical postulate) evolution, then we arrive at the jump process which *can* be interpreted as a Markov process if restricted to a suitable duration time. However, there is no corresponding Markov process consistent with the Z_2 event space assumption, if we require its existence for all times $t \in \mathbb{R}_+$. [S1050-2947(96)01708-8]

PACS number(s): 03.65.−w

Before [1] we have contested the general statement due to Gillespie [2] about the generic contradiction between the probabilistic concepts appropriate for quantum theory and those proper to the common-sense theory of stochastic (in particular, Markov) processes. Our argument was based on invoking the standard, configuration space, Schrödinger picture quantum dynamics which, if combined with the Born statistical interpretation postulate, allows for a consistent description in terms of Markov processes of diffusion type, in conformity with the rich theory developed so far [3–9].

However, our arguments did not pertain to stochastic jump processes which were the main objective of Ref. [2]. Let us therefore consider a simple two-level quantum system undergoing the Schrödinger evolution:

$$\psi(t)=\exp(-ict)\cos\omega t|1\rangle - i\exp(-ict)\sin\omega t|2\rangle \quad (1)$$

and concentrate on its probabilistic analysis, with an additional motivation coming from the series of papers due to other authors [10–15], where the Markov property has been attributed to analogous dynamical problems; see, however, Ref. [16]. We shall slightly simplify (1) by rescaling the dimensional constants to achieve $\omega=1$.

The discussion of Ref. [2] departs from the following *epistemological* input: ''If the system is known to be in state 1 at time s, then the probabilities of finding the system at any time $t>s$ to be in states 1 and 2 are $\cos^2(t-s)$ and $\sin^2(t-s)$, respectively, and similarly that if the system is known to be in state 2 at time s then the probabilities of finding the system at any time $t>s$ to be in states 2 and 1 are $\cos^2(t-s)$ and $\sin^2(t-s)$, respectively.'' These ''measurement probabilities'' are then utilized as *transition probabilities* of a certain (presumed to be consistently defined) stochastic jump process, with the outcome that the quantum mechanical evolution is at variance with the canonical form of the master equation appropriate for the problem [Eqs. (13) in Ref. [2]].

The above transition probabilities constitute a 2×2 transition matrix $p^G(t,s)$ with elements

$$p^G_{11}(t,s)=\cos^2(t-s)=p^G_{22}(t,s), \quad (2)$$

$$p^G_{12}(t,s)=\sin^2(t-s)=p^G_{21}(t,s).$$

To avoid any possible confusion, let us recall (see, e.g., Ref. [17]) that a stochastic process on Z_2, if considered on a finite time interval, say $[0,T]$, is to be given by a hierarchy of transition probabilities (they are an easy transcription of those conventionally utilized in the framework of continuous processes):

$$p(\sigma,t), \quad 0\leq t\leq T \quad (3)$$

$$p(\sigma_1,t_1|\sigma_2,t_2), \quad 0\leq t_2<t_1\leq T$$

$$p(\sigma_1,t_1|\sigma_2,t_2,\sigma_3,t_3), \quad 0\leq t_3<t_2<t_1\leq T$$

and so on, where each index σ equals either 1 or 2.

In the above, $p(\sigma,t)=\mu\{\omega\in\Omega:X_t(\omega)=\sigma\}$, $\sigma=1,2$ defines a probability measure of the stochastic process on Z_2, i.e., probabilities with which the dichotomic random variable takes values along concrete sample (jumping) paths in the event space Ω. The probability measure of the process is then propagated (or left invariant) by the transition probability $p(\sigma_1,t_1|\sigma_2,t_2)$.

The probabilities (3) have to satisfy the so-called consistency conditions:

(i) $\Sigma_\sigma p(\sigma,t)=1$,

(ii) $\Sigma_{\sigma_1} p(\sigma_1,t_1|\sigma_2,t_2)=1$ and $\Sigma_{\sigma_2} p(\sigma_1,t_1|\sigma_2,t_2)\ p(\sigma_2,t_2)=p(\sigma_1,t_1)$,

(iii) $\Sigma_{\sigma_1} p(\sigma_1,t_1|\sigma_2,t_2,\sigma_3,t_3)=1$,
$\Sigma_{\sigma_2} p(\sigma_1,t_1|\sigma_2,t_2,\sigma_3,t_3)\ p(\sigma_2,t_2|\sigma_3,t_3) =p(\sigma_1,t_1|\sigma_3,t_3)$,
$\Sigma_{\sigma_3} p(\sigma_1,t_1|\sigma_2,t_2,\sigma_3,t_3)p(\sigma_2,t_2|\sigma_3,t_3)p(\sigma_3,t_3) =p(\sigma_1,t_1|\sigma_2,t_2)p(\sigma_2,t_2)$, etc.

For a Markov process we would have $p(\sigma_1,t_1|\sigma_2,t_2,\sigma_3,t_3)=p(\sigma_1,t_1|\sigma_2,t_2)$ in which case (iii) would reduce to a single identity

$$\sum_{\sigma_2} p(\sigma_1,t_1|\sigma_2,t_2)p(\sigma_2,t_2|\sigma_3,t_3)=p(\sigma_1,t_1|\sigma_3,t_3), \tag{4}$$

i.e., the Chapman-Kolmogorov equation. Then, the hierarchy is closed and the process is completely specified by giving its initial probability measure *and* its transition probabilities.

Let us point out that in Ref. [2] the random dynamics was characterized exclusively in terms of transition probabilities and with no reference to a probability measure of the process. The probabilistic description of random jumps on Z_2 patterned after [10,11] is given in terms of the ‘‘probability vector’’ (probability measure in the present case):

$$p(t)=\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}, \tag{5}$$

$$p_1(t)\geqslant 0, \quad p_2(t)\geqslant 0, \quad p_1(t)+p_2(t)=1$$

and the transition probability

$$p(t_1,t_2)=\begin{pmatrix} p_{11}(t_1,t_2) & p_{12}(t_1,t_2) \\ p_{21}(t_1,t_2) & p_{22}(t_1,t_2) \end{pmatrix}, \tag{6}$$

$$p_{ij}(t_1,t_2)=p(i,t_1|j,t_2), \quad i,j=1,2$$

$$p_{ij}(t_1,t_2)\geqslant 0, \quad \sum_i p_{ij}(t_1,t_2)=1,$$

$$\sum_j p_{ij}(t_1,t_2)p_j(t_2)=p_i(t_1).$$

The last identity is equivalent to $p(t_1,t_2)p(t_2)=p(t_1)$, understood as the matrix-vector operation in the linear space.

If Gillespie's transition matrix $p^G(t,s)$, (2), is to define a consistent stochastic process, in view of its breaking the Chapman-Kolmogorov identity, the higher rank conditional probabilities need to be introduced. Unfortunately, they are not given in Ref. [2].

Nevertheless, let us take for granted that this supplementary step can be made so that $p^G(t,s)$ *is* a transition matrix of a well defined (non-Markovian) stochastic process X_t with values in Z_2. It is natural to ask for the probability measure μ of this process, i.e., for its probability vector $p(t)$. It must satisfy the consistency (in fact, propagation) condition

$$p^G(t,s)p(s)=p(t) \tag{7}$$

for all $s<t$. Let us analyze the issue in some detail.

If $p(0)$ is an arbitrary initial density, we can always write

$$p(0)=\begin{pmatrix} a \\ 1-a \end{pmatrix}, \tag{8}$$

with $a\in[0,1]$. Then, for all $s>0$ we have

$$p(s)=p^G(s,0)p(0)=\begin{pmatrix} a\cos^2 s+(1-a)\sin^2 s \\ a\sin^2 s+(1-a)\cos^2 s \end{pmatrix} \tag{9}$$

and for every $t>s$

$$p(t)=p^G(t,0)p(0)=\begin{pmatrix} a\cos^2 t+(1-a)\sin^2 t \\ a\sin^2 t+(1-a)\cos^2 t \end{pmatrix}. \tag{10}$$

On the other hand, it holds that

$$p(t)=p^G(t,s)p(s)=\begin{pmatrix} \cos^2(t-s)[a\cos^2 s+(1-a)\sin^2 s]+\sin^2(t-s)[a\sin^2 s+(1-a)\cos^2 s] \\ \sin^2(t-s)[a\cos^2 s+(1-a)\sin^2 s]+\cos^2(t-s)[a\sin^2 s+(1-a)\cos^2 s] \end{pmatrix}. \tag{11}$$

Since (10) and (11) must be equal, we get the identity

$$(a-\tfrac{1}{2})\sin 2(t-s)\sin 2s=0 \tag{12}$$

to be valid for all $0<s<t$. It implies $a=\frac{1}{2}$ and consequently

$$p(0)=\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \tag{13}$$

is the only admissible initial choice of $p(0)$.

Moreover, $p^G(t,s)$ is a symmetric matrix, which implies the following important property: if for some t_0 we deal with $p(\sigma,t_0)=\text{const}$ which is independent of $\sigma=1,2$, then for all $t>t_0$ there is $p(\sigma,t)=p(\sigma,t_0)=\text{const}$. This follows from the observation

$$p(\sigma,t)=\sum_{\sigma_0} p(\sigma,t|\sigma_0,t_0)p(\sigma_0,t_0)=\text{const}\sum_{\sigma_0} p(\sigma,t|\sigma_0,t_0)$$
$$=\text{const}\sum_{\sigma_0} p(\sigma_0,t|\sigma,t_0)=\text{const}. \tag{14}$$

As a consequence, by utilizing $p^G(t,s)$ as a transition probability appropriate for the stochastic process, we would arrive at the process whose probability measure is conserved in time,

$$p(0)=p(t)=\begin{pmatrix}1/2\\1/2\end{pmatrix} \qquad (15)$$

for all $t>0$.

The random dynamics induced by $p^G(t,s)$ is thus appropriate exclusively for systems with an invariant probability measure. It certainly has nothing in common with the quantally implemented evolution of the probability measure associated (via the Born postulate) with the explicit solution (1), i.e., with the probability vector $p(t)$ whose components read $p(1,t)=\cos^2 t$, $p(2,t)=\sin^2(t)$.

We believe that a consistent approach towards a probabilistic reinterpretation of the quantum dynamics proper should result in the construction of a stochastic process (Markovian if possible) which is compatible with the quantum Schrödinger picture evolution (1). This issue has received attention in the literature, see, e.g., Refs. [3–9] and Refs. [10–16] in particular.

Let us follow this, alternative with respect to the reasoning of Ref. [2], idea and demonstrate that the restriction to symmetric transition matrices (Gillespie's case, see, e.g., Ref. [18] for more detailed discussion) implies that there is no consistent Markov jump process which can be associated with the Schrödinger dynamics (1) for all times.

Indeed, for the time dependent probability vector $p(t)$, at time $t_0=\pi/4$ we have

$$p(t_0)=\begin{pmatrix}\cos^2(\pi/4)\\\sin^2(\pi/4)\end{pmatrix}=\begin{pmatrix}1/2\\1/2\end{pmatrix} \qquad (16)$$

and the previous result follows for times exceeding $\pi/4$.

However, we can construct a Markov process running in the finite time interval $[0,\pi/4]$.

We have

$$p(t)=\begin{pmatrix}\cos^2 t\\\sin^2 t\end{pmatrix}.$$

Let us define $p(t,s)$:

$$p_{11}(t,s)=p_{22}(t,s)=\frac{\cos^2 t-\sin^2 s}{\cos 2s}, \qquad (17)$$

$$p_{12}(t,s)=p_{21}(t,s)=1-p_{11}(t,s)=\frac{\cos^2 s-\cos^2 t}{\cos 2s},$$

where $0\leqslant s<t\leqslant\pi/4$. All these matrix coefficients are nonnegative in the time interval $[0,\pi/4]$. Moreover, for $0\leqslant s<t<u\leqslant\pi/4$, by inspection (with some help of trigonometric identities) we find the Chapman-Kolmogorov equation to be valid,

$$p(u,t)p(t,s)=p(u,s), \qquad (18)$$

and so the Markov property is established. Obviously, the propagation formula

$$p(t,s)\begin{pmatrix}\cos^2 s\\\sin^2 s\end{pmatrix}=\begin{pmatrix}\cos^2 t\\\sin^2 t\end{pmatrix} \qquad (19)$$

holds true. Also, we have $\lim_{t\downarrow s}p(t,s)=I$ where I denotes the unit 2×2 matrix. It corresponds to the matrix element property $p(i,s|j,s)=\delta_{ij}$ for all $i,j=1,2$.

The major steps of our analysis, (3)–(19), did not rely on any ''quantum measurement'' epistemology and merely invoked mathematical features of stochastic jump processes on Z_2. However, the main difference between the approach of Ref. [2] and this of Refs. [10,11] is rooted in the preferred choice of the connection between mathematics and physics (here, pertaining to the concept of measurement in quantum theory).

The theory of stochastic processes is normally regarded by physicists as a macroscopic theory in the sense that one can probe the system without significantly perturbing it. Then, it is not surprising that such a theory may be viewed as inconsistent with the ordinary quantum mechanics. This line of thought is followed in Ref. [2].

On the other hand, instead of viewing the stochastic process as a description of one system evolving from $t=-\infty$ to $t=+\infty$, one might view it as a theory for an infinite ensemble of systems, all starting either in the same initial state or with some initial probability distribution over the possible states at time $t=0$. The ensemble is allowed to evolve until we possibly decide to stop the systems. Viewed this way, the theory of stochastic processes might very well describe the evolution of a quantum system between measurements (compare, e.g., in this connection the two time localization framework of Ref. [19]). And that is the admissible interpretation of our previous discussion. In this case, on suitable time scales we have found the Markov property to persist.

Let us stress that if the zero point of the time axis is taken to be the time of the measurement on the system, where the system *was* found to be in state 1, then there is no disagreement between our example (17)–(19) and the results of Ref. [2]. The propagation from $s=0$ can be safely extended to an arbitrary time instant $T>0$ interpreted as a subsequent measurement of the system. However, to analyze the time evolution of the probability measure $p(t)$ between those fixed time interval boundaries, we must define the transition probability $p(t,s)$ for *all* intermediate time instants. Obviously, this corresponds to transitions between *unobserved* intermediate states and, as (18) shows, for not too long time intervals of interest such stochastic interpolation is Markovian until interrupted (terminated) by the measurement.

Both authors receive financial support from KBN Research Grant No. 2 P302 057 07.

[1] P. Garbaczewski and R. Olkiewicz, Phys. Rev. A **51**, 3345 (1995)
[2] D. T. Gillespie, Phys. Rev A **49**, 1607 (1994).
[3] E. Nelson, *Quantum Fluctuations* (Princeton University Press, Princeton, NJ, 1985).
[4] Ph. Blanchard, Ph. Combe, and W. Zheng, in *Mathematical and Physical Aspects of Stochastic Mechanics*, Lecture Notes in Physics Vol. 281 (Springer-Verlag, Berlin, 1987).
[5] J. C. Zambrini, J. Math. Phys. **27**, 3207 (1986).
[6] Ph. Blanchard and P. Garbaczewski, Phys. Rev. E **49**, 3815 (1994).
[7] P. Garbaczewski, J. R. Klauder, and R. Olkiewicz, Phys. Rev. E **51**, 4114 (1995).
[8] P. Garbaczewski and R. Olkiewicz, J. Math. Phys. **37**, 732 (1996).
[9] Ph. Blanchard, P. Garbaczewski, and R. Olkiewicz (unpublished).
[10] G. F. De Angelis, D. de Falco, and F. Guerra, Phys. Rev. D **23**, 1747 (1981).
[11] G. F. De Angelis and G. Jona-Lasinio, J. Phys. A **15** 2053, (1982).
[12] G. F. De Angelis, G. Jona-Lasinio, and M. Sirugue, J. Phys. A **16**, 2433 (1983).
[13] G. F. De Angelis, G. Jona-Lasinio, M. Serva, and N. Zanghi, J. Phys. A Gen. **19**, 865 (1986).
[14] B. Djehiche, J. Math. Phys. **33**, 3050 (1992).
[15] B. Djehiche, Potential Anal. **2**, 349 (1993).
[16] P. Garbaczewski, Fortschr. Phys. **38**, 447 (1990).
[17] I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes* (W. B. Saunders, Philadelphia, 1965).
[18] E. A. Carlen, S. Kusuoka, and D. W. Stroock, Ann. Inst. Henri Poincaré **2**, 245 (1987).
[19] L. S. Schulman, Ann. Phys. (N.Y.) **183**, 320 (1988).