BRIEF REPORTS

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Relation between quantum computing and quantum controllability

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It is demonstrated that the universality of any quantum computing element can be understood and verified via a precise mathematical criterion which tests for the controllability of an associated quantum control system. This relation between quantum computing and quantum control is deeper in that tools of coherent control of quantum dynamics may be used to arrive at specific designs for quantum computing devices. [S1050-2947(96)06607-2]

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I. QUANTUM COMPUTING AND QUANTUM CONTROL

There has recently been a spate of exciting research on quantum computing. Several authors have addressed the question of characterizing the kind of finite-dimensional quantum systems which are capable of performing the various logic operations desired of a quantum computer. In particular, in [1] it is shown that almost any quantum system is universal with respect to this property. The purpose of this Brief Report is to point out the connection between this body of work [1–3] and the notion of controllability of invariant systems on Lie groups which also plays a significant role in the control of quantum dynamics as being extensively developed for molecular and solid-state applications [4–7]. This connection goes beyond the similarities in the intervening mathematical notions in the two subjects. It is our opinion that the realization of this connection will be valuable in the actual design of logic gates for quantum computers.

The study of controllability is quite general (see [8]), but we will restrict ourselves to differential equations of the following kind, since it is these which arise both in quantum computing and molecular control and cognate fields:

$$\dot{U}=AU+\sum_{i=1}^{m} B_i U\epsilon_i(t). \tag{1.1}$$

In (1.1) U is a $N\times N$ unitary matrix, and the matrices A and B_i, $i=1,\ldots,m$ are $N\times N$ skew-Hermitian. The scalar functions $\epsilon_i(t)$, $i=1,\ldots,m$ are called the inputs (or controls), and are assumed to be amenable to laboratory adjustment (e.g., an external laser field within the semiclassical picture). Equation (1.1) thus represents the evoluation of the unitary generator of an N-dimensional quantum system. If $A=0$ in (1.1) the quantum system is said to be driftless. In the theory of controllability of general nonlinear systems there is a marked difference between driftless systems and those with drift. Fortunately, for (1.1) there is a single theorem which subsumes both situations [4,9]. We define system (1.1) to be controllable if every $N\times N$ unitary matrix U can be achieved in finite time with (1.1) initialized at any given unitary matrix U_0. Informally, an admissible control means any choice of the $\epsilon_i(t)$ as functions of time which will render (1.1) to have unique solutions for any initial condition. Note that in quantum computing one may just be interested in achieving certain unitary matrices (such as Deutsch's universal gate [3,2]), but this circumstance certainly follows if indeed all unitary matrices can be achieved. In fact, the heart of the argument in [1] is essentially a controllability argument. Furthermore, the theory of controllability also allows the identification of those unitary matrices which can be attained (see [4,9]), assuming that the necessary and sufficient condition to be stated in the theorem below does *not* hold.

A. Theorem 1.1 *A necessary and sufficient condition for (1.1) to be controllable is that the set of all matrices generated by A, the B_i, $i=1,\ldots,m$, and their commutators (i.e., the Lie algebra generated by A and the B_i, $i=1,\ldots,m$) equals all the $N\times N$ skew-Hermitian matrices. Furthermore, when this condition is satisfied any U can be attained by choosing the $\epsilon_i(t)$ to be just piecewise constant functions of time.*

The setting of Lloyd [1] corresponds to $A=0$ and $m=2$. The prescription in [1] corresponds to choosing ϵ_1 and ϵ_2 as previous constant functions of time (with values either 1 or 0). For further implications and nuances on theorem (1.1) the reader should consult Refs. [4,8,9].

It is worthwhile to address the issue of the controllability for (1.1) in full generality (i.e., also for the case $A=0$) because there seems (as yet) to be no consensus as to which quantum system will be most viable for the purposes of quantum computing. It is certainly conceivable that the molecular and solid-state quantum systems being extensively explored [10] may prove to be good candidate, in which case the study of (1.1) with $A=0$ will have added relevance.

In particular, it could transpire that choosing the ϵ_i, $i=1,\ldots,m$ as just piecewise constant (even though that choice will, in theory, be adequate for the purpose of generating a desired unitary matrix) may not be possible since the external control ϵ_i, $i=1,\ldots,m$ may not be readily amenable to being assigned constant values (as would be the case with a laser). In this regard it is worth mentioning that the apparatus of optimal control theory, which plays an important role in molecular control applications [10], is well suited for producing the desired controls. Another option is the use of the methods of tracking and its variations [11,12]. Finally, it may be worthwhile to use the structure of the Lie algebra appearing in theorem (1.1) to generate the ϵ_i *ab initio*. For driftless systems there is a substantial body of literature devoted to this question (see [13] for instance) in the controls and robotics community where it is known as the path-planning problem. For systems with $A\neq 0$ there is not much known (see, however, [9]).

II. CONCLUSION

In summary, there is considerable similarity between the issues of universality of quantum logic gates and the controllability of quantum dynamics. We submit that this connection can be further exploited to advantage by both communities.

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