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Derivation and physical interpretation of the optimum detection operators for coherent-state signals

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The optimum detection operators are derived for several linearly independent signal sets. The quantum minimax strategy is applied to the M-ary coherent state signals such as binary phase shift, ternary symmetric, ternary phase shift, and quadrature phase shift. It is shown that the detection process induces a kind of quantum interference between signal quantum states and measurement states. [S1050-2947(96)05508-4]

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I. INTRODUCTION

The quantum detection theory is an optimization theory for the signal detection process based on the full quantum-mechanical treatment. It was pioneered by Helstrom [1] and established by Yuen *et al.* and Holevo independently [2,3] and predicts much superior detection performance than conventional detection, such as homodyne or photon counting detection. This is because the detection operators which are the mathematical description of a detection process are quantum mechanically generalized to be characterized by the probability operator valued measure (POM). So far, the application of quantum detection theory was restricted to evaluating the performance of detection systems, such as the average probability of error, channel capacity, cutoff rate, etc. [4–12]. The derivation of the optimum detection operators which represent an optimum detection process has not been done [13] because of its complexity. In this paper we give the analytical representations of the optimum detection operators for some signal sets. The optimum detection operator derived here minimizes the average probability of error since it is the simplest parameter to evaluate the performance of a detection system. The signal sets treated in this paper consist of coherent states. In detail, the optimum detection operators for binary, ternary, and quadrature phase shift, and ternary symmetric signals with coherent states are derived. These results are relevant to quantum eavesdropping [14].

This paper consists of five sections. In Sec. II, the quantum detection process is summarized in comparison with the standard detection process. Section III gives two criteria to minimize the average probability of error. The quantum Bayes and minimax strategies are explained and an assumption on signal sets is applied to make the strategies solvable. Section IV gives concrete optimum detection operators and the minimum average probability of error for the signal sets mentioned above. The quantum minimax strategy is basically applied to the derivations. In Sec. V, a physical interpretation of the optimum detection processes is tried and it results in a kind of quantum interference. Section VI is the conclusion of this paper where the systematic derivation process of the optimum detection operators by the quantum minimax strategy is summarized. In the appendix, derivation of the optimum detection operators by the quantum Bayes strategy is also explained.

II. QUANTUM DETECTION PROCESS

A. Standard formalism

Before the explanation of quantum detection theory, let us show the quantum-mechanical redescription of the signal detection process in comparison with the quantum mechanically generalized one. In quantum mechanics, all signals and their processes are represented by operators on the Hilbert space. Consider an M-ary signal system, where a received signal is represented by a signal quantum state with a density operator $\hat{\rho}_i$ where $i=1,2,\ldots,M$. It is a non-negative Hermitian operator with unit trace:

$$\hat{\rho}_i \geqslant 0, \quad \forall i, \qquad \mathrm{Tr}\hat{\rho}_i = 1, \quad \forall i. \tag{1}$$

The first step of a detection process is a measurement process, followed by a decision process. The measurement process gives some value of the c-number variable x with respect to the probability density represented by a trace of a signal quantum state and a measurement operator $d\hat{M}(x)$.

$$p(x|i)dx = \mathrm{Tr}\hat{\rho}_i d\hat{M}(x). \tag{2}$$

The measurement operator is a projection valued measure (PVM) of a signal observable. The signal observable is an observable with which the transmitted signal is modulated. In practice, it may be an amplitude $\hat{X}_\phi \equiv (\hat{a}\ \exp[-i\phi] + \hat{a}^\dagger \exp[i\phi])/2$, photon numbers $\hat{n} \equiv \hat{a}^\dagger \hat{a}$, or in the case of phase modulation, it can be regarded as quadrature amplitudes $\hat{X}_c \equiv (\hat{a}+\hat{a}^\dagger)/2$ and $\hat{X}_s \equiv (\hat{a}-\hat{a}^\dagger)/2i$, where $\hat{a}$ and $\hat{a}^\dagger$ are the photon annihilation and creation operators, respectively. The measurement operator and typical receiver for each observable are given in Table I.

The decision process which follows a measurement process prescribes which signal is received based on a measurement result x. This corresponds to the division of the variable space $\mathcal{R} : x \in \mathcal{R}$ into M subspaces $\{\mathcal{R}_j\} : j=1,2,\ldots,M$. The probability that signal j is chosen when signal i is true is

TABLE I. Measurement operator for each receiver.

Signal observable	Quadrature amplitude	Photon number	Amplitude
Measurement operator	$\vert x_\phi\rangle\langle x_\phi\vert dx_\phi$	$\vert n\rangle\langle n\vert$	$\vert\alpha_0\rangle\langle\alpha_0\vert d^2\alpha_0/\pi$
Typical receiver	homodyne	photon counting	heterodyne

$$P(j|i)=\int_{\mathcal{R}_j} p(x|i)dx=\int_{\mathcal{R}_j}\mathrm{Tr}\hat{\rho}_i d\hat{M}(x). \tag{3}$$

The signal detection process consisting of the measurement and decision processes can be represented by a detection operator given as follows:

$$\hat{\Pi}_j^{\mathrm{SD}}=\int_{\mathcal{R}_j} d\hat{M}(x). \tag{4}$$

This type of detection operator is called the ''standard detection operator'' [15]. The average probability of error is defined as follows:

$$P_e=1-\sum_{i=1}^{M}\xi_i P(i|i)=1-\sum_{i=1}^{M}\xi_i \mathrm{Tr}\hat{\rho}_i\hat{\Pi}_i^{\mathrm{SD}}, \tag{5}$$

where ξ_i is a prior probability for ith signal; namely,

$$\sum_{i=1}^{M}\xi_i=1. \tag{6}$$

Since the standard detection operators are only a quantum mechanical redescription of the conventional detection process, the minimization of the average probability of error in terms of them is just equal to the standard quantum limit (SQL) which is referred to as the performance limit of a conventional receiver [16]; namely, the optimization of the standard detection operators is not for the whole detection operators but only for the decision regions $\{\mathcal{R}_j\}$.

B. Generalized detection process

The essential difference between the standard and quantum detection theories exists in the measurement process. In quantum mechanics, an observable to be measured can be mathematically generalized to an arbitrary self-adjoint operator and the measurement operator is also generalized from a projection valued measure (PVM) into a probability operator valued measure (POM). Since the POM is the resolution of identity, it can include the meaning of a decision process. Consequently, we can define the ''generalized detection operator'' based on the POM as

$$\hat{\Pi}_j=\hat{\Pi}_j^{\dagger}\geqslant 0, \qquad \sum_{j=1}^{M}\hat{\Pi}_j=\hat{I}, \tag{7}$$

where $\hat{I}$ is the identity operator. The generalized detection operator means all detection operators except for the standard detection operators given by the PVM of a signal observable [15].

Since the generalized detection operators represent the whole detection process, it is difficult to distinguish the measurement and decision processes from their structure. The probability given in Eq. (3) that signal j is chosen when signal i is true is represented by a trace of a signal quantum state and a generalized detection operator:

$$P(j|i)=\mathrm{Tr}\hat{\rho}_i\hat{\Pi}_j. \tag{8}$$

Then the average probability of error results as

$$P_e=1-\sum_{i=1}^{M}\xi_i\mathrm{Tr}\hat{\rho}_i\hat{\Pi}_i. \tag{9}$$

Therefore the minimization of the average probability of error based on the generalized detection operators is not only the optimization of the decision regions but also that of an observable to be measured. This is the reason why the quantum detection theory has the possibility to predict much superior performance to the SQL. The receiver whose average probability of error is quantum mechanically optimum is called the ''optimum quantum receiver.''

III. QUANTUM DETECTION CRITERIA

The optimization criteria for the generalized detection operators are formulated based on the similar concepts of the classical signal decision criteria; namely, the quantum Bayes criterion is formulated by Helstrom [1], Yuen *et al.*, [2] and Holevo [3], and the quantum minimax criterion by Holevo [17] and Hirota and Ikehara [18]. While the quantum Bayes criterion can treat an arbitrary risk function, we restrict ourselves to considering the average probability of error.

The quantum Bayes criterion minimizes the average probability of error when signal prior probabilities are known by an observer. Hence its concept is

$$\min_{\{\hat{\Pi}\}} P_e. \tag{10}$$

On the other hand, the concept of the quantum minimax criterion is represented as follows:

$$\max_{\{\xi\}}\min_{\{\hat{\Pi}\}} P_e=\min_{\{\hat{\Pi}\}}\max_{\{\xi\}} P_e. \tag{11}$$

This criterion guarantees the minimum average probability of error for unknown signal prior probabilities although it is the worst one obtained by the quantum Bayes criterion. In the following, the necessary and sufficient conditions for the quantum Bayes and minimax criteria are summarized and some assumptions are explained to make these conditions solvable.

A. Quantum Bayes strategy

Let us explain the quantum Bayes strategy first given by Helstrom. When a binary signal is represented by quantum states $\hat{\rho}_1$ and $\hat{\rho}_2$ with their prior probabilities ξ_1 and ξ_2, respectively, the optimum detection operators can be defined in terms of the eigenvectors of the following eigenvalue equation:

$$(\hat{\rho}_2 - \lambda\hat{\rho}_1)|\eta_i\rangle = \eta_i|\eta_i\rangle, \tag{12}$$

where $\lambda \equiv \xi_1/\xi_2$, and η_i and $|\eta_i\rangle$ are an eigenvalue and an eigenvector, respectively. That is, the optimum detection operators for the binary signal set are

$$\hat{\Pi}_1 = \sum_i U(-\eta_i)|\eta_i\rangle\langle\eta_i|,$$
$$\hat{\Pi}_2 = \sum_i U(\eta_i)|\eta_i\rangle\langle\eta_i|, \tag{13}$$

where $U(x)$ is the unit step function defined as follows:

$$U(x) = \begin{cases} 1, & x>0, \\ 1/2, & x=0, \\ 0, & x<0. \end{cases} \tag{14}$$

After Helstrom, the quantum Bayes strategy was generalized to M-ary signal sets by Yuen *et al.* and Holevo independently. It results in the following necessary and sufficient conditions:

$$\hat{\Pi}_j[\xi_j\hat{\rho}_j - \xi_i\hat{\rho}_i]\hat{\Pi}_i = 0, \quad \forall i,j, \tag{15}$$

$$\hat{\Gamma} - \xi_i\hat{\rho}_i \geqslant 0, \quad \forall i, \tag{16}$$

where $\hat{\Gamma}$ is called a ''Lagrange operator'' defined by

$$\hat{\Gamma} = \sum_{i=1}^{M} \xi_i\hat{\rho}_i\hat{\Pi}_i . \tag{17}$$

Since the operator $\hat{\Gamma}$ is Hermitian, Eq. (16) indicates that the operator $(\hat{\Gamma} - \xi_i\hat{\rho}_i)$ must be a non-negative Hermitian operator.

B. Quantum minimax strategy

Let us explain here the quantum minimax strategy. The necessary and sufficient conditions of the quantum minimax criterion derived by Hirota and Ikehara are as follows [18]:

$$\mathrm{Tr}\hat{\Pi}_i\hat{\rho}_i = \mathrm{Tr}\hat{\Pi}_j\hat{\rho}_j, \quad \forall i,j,$$
$$\hat{\Pi}_j[\xi_j\hat{\rho}_j - \xi_i\hat{\rho}_i]\hat{\Pi}_i = 0, \quad \forall i,j, \tag{18}$$
$$\hat{\Gamma} - \xi_i\hat{\rho}_i \geqslant 0, \quad \forall i.$$

The second and third equations are equal to the conditions of the quantum Bayes strategy, so that the first equation give the minimax point where the average probability of error is maximized with respect to the signal prior probabilities. This quantum minimax strategy can provide a simplification of the derivation processes for the optimum detection operators compared with the cases of Bayes [13].

C. Some assumptions on quantum signal sets

In the most general case, a quantum signal set will consist of mixed or linearly dependent states. However, in such a case, the analytical derivation of the optimum detection operators is too difficult [1,19]. Hence we consider, in this paper, only linearly independent signal sets so that the following Kennedy's lemma can be applied to the derivation of the optimum detection operators.

Kennedy's lemma. When the signal quantum states are linearly independent, the optimum POM for the average probability of error is projection valued [20].

Under this condition the optimum detection operators are projectors, so that they can be supposed to be orthonormal bases of a subspace of the total Hilbert space $\mathcal{H}$. In this case, the optimum detection operators are uniquely determined from Eqs. (15) and (16) or Eq. (18) [21]. Therefore this lemma is very useful in derivation of the optimum detection operators [22]. Let us call this subspace a ''signal space'' $\mathcal{H}_s$ since it is spanned by signal quantum states $\{|\psi_i\rangle : \hat{\rho}_i = |\psi_i\rangle\langle\psi_i|\}$; namely,

$$\mathcal{H}_s \subseteq \mathcal{H},$$
$$|\phi\rangle \in \mathcal{H}_s = \sum_{i=1}^{M} c_i|\psi_i\rangle \quad |c_i \in \mathcal{C}\}, \tag{19}$$

where $\mathcal{C}$ stands for the complex number field. The signal space for an M-ary signal set is an M-dimensional Hilbert space in which the optimum detection operators can be represented by its orthonormal bases $\{|\omega_i\rangle\}$:

$$\hat{\Pi}_i = |\omega_i\rangle\langle\omega_i|,$$
$$\langle\omega_i|\omega_j\rangle = \delta_{ij}, \tag{20}$$

where $\{|\omega_i\rangle\}$ is called a ''measurement state.'' Then the detection operators given by them are generalized detection operators because they are not related to the PVM of a signal observable. The observable is not mentioned and the resolution of identity is valid only for the signal space.

Since the measurement states are orthonormal bases of a signal space, the signal quantum states $\{|\psi_i\rangle\}$ must be represented by them with variables $x_{ji} \equiv \langle\omega_j|\psi_i\rangle$ as follows:

$$\begin{bmatrix} |\psi_1\rangle \\ \vdots \\ |\psi_M\rangle \end{bmatrix} = \begin{bmatrix} x_{11} & \dots & x_{M1} \\ \vdots & \ddots & \vdots \\ x_{1M} & \dots & x_{MM} \end{bmatrix} \begin{bmatrix} |\omega_1\rangle \\ \vdots \\ |\omega_M\rangle \end{bmatrix}. \tag{21}$$

Substituting this representation into the necessary and sufficient conditions of Bayes in Eqs. (15) and (16), the operator equations for $\{\hat{\rho}_i\}$ and $\{\hat{\Pi}_j\}$ are turned into the ordinary c-number algebraic equations for $\{x_{ji}\}$. That is, the necessary and sufficient conditions for the quantum Bayes criterion can be written down as follows:

$$\xi_i x_{ii} x_{ji}^* - \xi_j x_{ij} x_{jj}^* = 0, \quad \forall i,j, \tag{22}$$

$$\mathbf{T}^{(i)} \geqslant 0, \quad \forall i, \tag{23}$$

where $\mathbf{T}^{(i)}$ is a matrix representation of the operator $(\hat{\Gamma} - \xi_i \hat{\rho}_i)$ with its elements defined as follows:

$$t_{mn}^{(i)} = \xi_m x_{mm} x_{nm} - \xi_i x_{mi} x_{ni}. \tag{24}$$

Since the operator $(\hat{\Gamma} - \xi_i \hat{\rho}_i)$ is Hermitian, $\mathbf{T}^{(i)}$ is also a Hermitian matrix. Equation (23) requires the Hermitian non-negativeness of the matrix. The necessary and sufficient conditions to be a non-negative Hermitian matrix are as follows:

$$\det \mathbf{T}^{(i)} = 0,$$

$$\det \mathbf{T}^{(i)}(k_1, k_2, \ldots, k_s) \geqslant 0$$

$$\text{for } (k_1, k_2, \ldots, k_s = 1, 2, \ldots, M; s = 1, 2, \ldots, M), \tag{25}$$

where $\det \mathbf{T}^{(i)}(k_1, k_2, \ldots, k_s)$ means a principal minor of $\mathbf{T}^{(i)}$.

In practical use of the quantum Bayes strategy, Eq. (22) gives $M(M-1)/2$ equations for M^2 variables. So we need to combine them with $M(M+1)/2$ equations for inner products of signal quantum states:

$$\kappa_{ij} = \langle \psi_i | \psi_j \rangle = \sum_{k=1}^{M} \langle \psi_i | \omega_k \rangle \langle \omega_k | \psi_j \rangle = \sum_{k=1}^{M} x_{kj} x_{ki}^*, \tag{26}$$

where we assumed that the inner products are the known observers. Then we have M^2 equations for M^2 variables, so that the solution will be obtained. Since the equations are nonlinear, the solution has some variations. Equation (23) is used to judge which of the solutions $\{x_{ji}\}$ is correct.

In the case of the quantum minimax strategy, Eq. (18) is also represented as follows:

$$|x_{ii}|^2 = |x_{jj}|^2, \quad \forall i, j,$$

$$\xi_i x_{ii} x_{ji}^* - \xi_j x_{ij} x_{jj}^* = 0, \quad \forall i, j, \tag{27}$$

$$\mathbf{T}^{(i)} \geqslant 0, \quad \forall i.$$

While the first equation in the above condition reduces $(M-1)$ unknown variables, the prior probabilities are turned into $(M-1)$ variables in the case of the minimax strategy. However, these prior probabilities are sometimes assumed to be equal to each other. This is the reason why the quantum minimax strategy can provide a simplification of the derivation process for the optimum detection operators.

Finally, the optimum measurement states can be explained by signal quantum states with the solution $\{x_{ji}\}$:

$$\begin{bmatrix} |\omega_1\rangle \\ \vdots \\ |\omega_M\rangle \end{bmatrix} = [x_{ji}]^{-1} \begin{bmatrix} |\psi_1\rangle \\ \vdots \\ |\psi_M\rangle \end{bmatrix}. \tag{28}$$

Furthermore, in practical derivation of the variables $\{x_{ji}\}$, we can assume that $\{x_{ii}\}$ is not a complex number but a real number even if some of the inner products of signal quantum states are complex numbers. This is because supposing $\{x_{ii}\}$ to be complex numbers corresponds to a phase shift of the measurement state, i.e., $|\omega_i\rangle \to |\omega_i'\rangle e^{i\varphi}$, so it makes no difference to the consequently obtained optimum detection operators.

IV. DERIVATION OF THE OPTIMUM DETECTION OPERATORS BY THE QUANTUM MINIMAX STRATEGY

Almost every application of quantum detection theory to practical signal sets has been an investigation into the average probability of error. We give the analytical solutions of the optimum detection operators for some practical quantum signal sets [13]. The quantum minimax strategy with Kennedy's lemma is used in the following derivation of the optimum detection operators for the cases of binary phase shift, ternary symmetric, ternary phase shift, and quadrature phase shift coherent state signal sets. The application of the quantum Bayes strategy with Kennedy's lemma to some of these signal sets is given in the Appendix.

A. Binary phase-shift coherent-state signal

In the case of a binary phase-shift coherent-state signal set, signal quantum states are represented as $[|\psi_1\rangle, |\psi_2\rangle] = [|\alpha\rangle, |-\alpha\rangle]$. The inner product matrix of these signal quantum states becomes

$$[\kappa_{ij}] = \begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}, \tag{29}$$

where $\kappa = \langle \alpha | -\alpha \rangle = \exp[-2|\alpha|^2]$. The equations to be solved are as follows:

$$|x_{11}|^2 = |x_{22}|^2,$$

$$\xi_1 x_{11} x_{21}^* - \xi_2 x_{12} x_{22}^* = 0,$$

$$|x_{11}|^2 + |x_{21}|^2 = 1, \tag{30}$$

$$|x_{12}|^2 + |x_{22}|^2 = 1,$$

$$x_{11} x_{12}^* + x_{21} x_{22}^* = \kappa.$$

With the assumption that the variables $\{x_{ji}\}$ are real numbers, we can obtain the solutions by elementary algebra:

$$x_{11} = x_{22} = \pm \sqrt{\frac{1 \pm \sqrt{1-\kappa^2}}{2}},$$

$$x_{12} = x_{21} = \pm \sqrt{\frac{1 \mp \sqrt{1-\kappa^2}}{2}}. \tag{31}$$

The reason why all the variables $\{x_{ji}\}$ are assumed to be real numbers is that the inner products of the signal quantum states are all real numbers. At the same time, the signal prior probabilities which give the minimax point of the average probability of error are also derived as $\xi_1 = \xi_2 = 1/2$. The solution which satisfies the condition in Eq. (23) is as follows:

$$x_{11}=x_{22}=\sqrt{\frac{1+\sqrt{1-\kappa^2}}{2}},$$

$$x_{12}=x_{21}=\sqrt{\frac{1-\sqrt{1-\kappa^2}}{2}}. \qquad (32)$$

While the negative version of the above solution, $\{-x_{ji}\}$, also satisfies the condition in Eq. (23), it corresponds to the phase shift of the measurement states given by Eq. (32).

With Eqs. (28), (20), and (9), the optimum detection operators and the average probability of error derived by the quantum minimax strategy are given as follows:

$$\hat{\Pi}_1=\frac{1}{2(1-\kappa^2)}[\{1+\sqrt{1-\kappa^2}\}|\alpha\rangle\langle\alpha|+\{1-\sqrt{1-\kappa^2}\}|-\alpha\rangle \langle-\alpha|-\kappa(|\alpha\rangle\langle-\alpha|+|-\alpha\rangle\langle\alpha|)], \qquad (33)$$

$$\hat{\Pi}_2=\frac{1}{2(1-\kappa^2)}[\{1-\sqrt{1-\kappa^2}\}|\alpha\rangle\langle\alpha|+\{1+\sqrt{1-\kappa^2}\}|-\alpha\rangle \langle-\alpha|-\kappa(|\alpha\rangle\langle-\alpha|+|-\alpha\rangle\langle\alpha|)],$$

and

$$P_e^{\mathrm{OPT}}=\frac{1-\sqrt{1-\kappa^2}}{2}. \qquad (34)$$

They are just equal to those derived by the quantum Bayes strategy with $\xi_1=\xi_2=1/2$ (see the Appendix). It is easy to see that the average probability of error by the minimax criterion is the worst one by the Bayes criterion.

The average probability of error by the optimum quantum receiver in Eq. (34) is compared with that by well-known receivers, such as homodyne and Kennedy receivers [23] in Fig. 1. The homodyne receiver can be realized by current technology and its average probability of error corresponds to the SQL of this signal set. That is,

$$P_e^{\mathrm{SQL}}=\mathrm{Erfc}[\sqrt{4|\alpha|^2}], \qquad (35)$$

where Erfc$[x]$ is the complementary error function defined as follows:

$$\mathrm{Erfc}\equiv\frac{1}{\sqrt{2\pi}}\int_x^\infty \exp[-\tau^2/2]d\tau. \qquad (36)$$

Kennedy receiver is a receiver whose average probability of error is near optimum while its construction is very simple. Its average probability of error is as follows:

$$P_e^{\mathrm{Kennedy}}=\tfrac{1}{2}\exp[-4|\alpha|^2]. \qquad (37)$$

B. Ternary symmetric coherent-state signal

In the case of a ternary symmetric coherent-state signal set, signal quantum states are represented as $[|\psi_1\rangle,|\psi_2\rangle,|\psi_3\rangle]=[|0\rangle,|\alpha\rangle,|-\alpha\rangle]$. Then the inner product matrix can be given as follows:

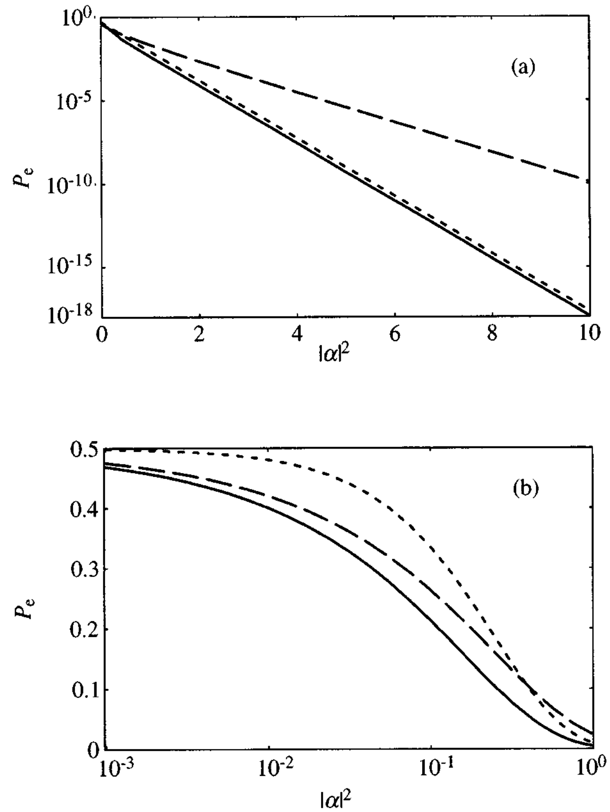

FIG. 1. Average probability of error for binary phase-shift coherent-state signal. (a) Moderate value of $|\alpha|^2$; (b) small value of $|\alpha|^2$. The solid line stands for the optimum quantum receiver, the dashed line for the homodyne receiver (SQL), and the dotted line for Kennedy receiver.

$$[\kappa_{ij}]=\begin{bmatrix} 1 & k & k \\ k & 1 & k^4 \\ k & k^4 & 1 \end{bmatrix}, \qquad (38)$$

where $k=\langle 0|\alpha\rangle=\langle 0|-\alpha\rangle=\exp[-|\alpha|^2/2]$. It is assumed that the variables $\{x_{ji}\}$ are real numbers. Furthermore, the prior probabilities for signals 2 and 3 are assumed to be equal ($\xi_2=\xi_3$) because the inner product matrix is symmetric and the minimax point for classical ternary symmetric signals has this property. Then we can reduce the number of variables as follows:

$$[x_{ji}]=\begin{bmatrix} a & d & d \\ b & c & e \\ b & e & c \end{bmatrix}, \qquad (39)$$

where the condition $a^2=c^2$ is not applied yet because its use at this stage complicates the derivation. Substituting them into Eq. (27), the equations to be solved become

$$\xi_1 ad-\frac{1-\xi_1}{2}bc=0,$$

$$a^2+2d^2=1,$$
$$b^2+c^2+e^2=1, \tag{40}$$
$$ab+cd+de=k,$$
$$b^2+2ce=k^4,$$

where ξ_1 is a prior probability for signal 1. The first equation in the above equations is used to determine the prior probabilities, and the second through the last equations can be solved in terms of b [1]. After that, the condition $a^2=c^2$ is applied to a and c to derive b as a function of k. By this procedure we have eight variations of the solution. By means of Eq. (23) the appropriate solution can be selected. It results in a change of the two solutions with respect to the square of amplitude $|\alpha|^2$.

The optimum detection operators for three-dimensional signals $[|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle]$ can be represented as follows:

$$\begin{aligned}\hat{\Pi}_i=&\frac{1}{d}\{m_{i,11}|\psi_1\rangle\langle\psi_1|+m_{i,12}|\psi_1\rangle\\ &\times\langle\psi_2|+m_{i,13}|\psi_1\rangle\langle\psi_3|+m_{i,21}|\psi_2\rangle\langle\psi_1|+m_{i,22}|\psi_2\rangle\\ &\times\langle\psi_2|+m_{i,23}|\psi_2\rangle\langle\psi_3|+m_{i,31}|\psi_3\rangle\langle\psi_1|+m_{i,32}|\psi_3\rangle\\ &\times\langle\psi_2|+m_{i,33}|\psi_3\rangle\langle\psi_3|\}.\end{aligned} \tag{41}$$

In the case of the ternary symmetric signal, the parameters are

$$\begin{aligned}&d=4(1-k^2)^2(1+k^4)^2(1-k^4),\\ &m_{1,11}=4(1-k^4)(1+k^4)^2F^2,\\ &m_{1,12}=m_{1,13}=m_{1,21}=m_{1,31}\\ &\quad=4(1-k^4)(1+k^4)F\{b(1-k^2)-kF\},\\ &m_{1,22}=m_{1,23}=m_{1,32}=m_{1,33}=4(1-k^4)\{b(1-k^2)-kF\}^2,\\ &m_{2,11}=m_{3,11}=D^2,\\ &m_{2,12}=m_{2,21}=m_{3,13}=m_{3,31}=-DE_+,\\ &m_{2,13}=m_{2,31}=m_{3,12}=m_{3,21}=DE_-,\\ &m_{2,23}=m_{2,32}=m_{3,23}=m_{3,32}=-E_+E_-,\\ &m_{2,22}=m_{3,33}=E_+^2,\\ &m_{2,33}=m_{3,22}=E_-^2,\end{aligned} \tag{42}$$

where

$$\begin{aligned}b=&(-B+\sqrt{B^2-4AC})/2 \quad \text{for } |\alpha|^2\leq\ln[1+\sqrt{2}],\\ &(-B-\sqrt{B^2-4AC})/2 \quad \text{for } |\alpha|^2>\ln[1+\sqrt{2}],\end{aligned} \tag{43}$$

with

$$A=-2\{(k^4+2k^2-1)^2+8k^2\},$$

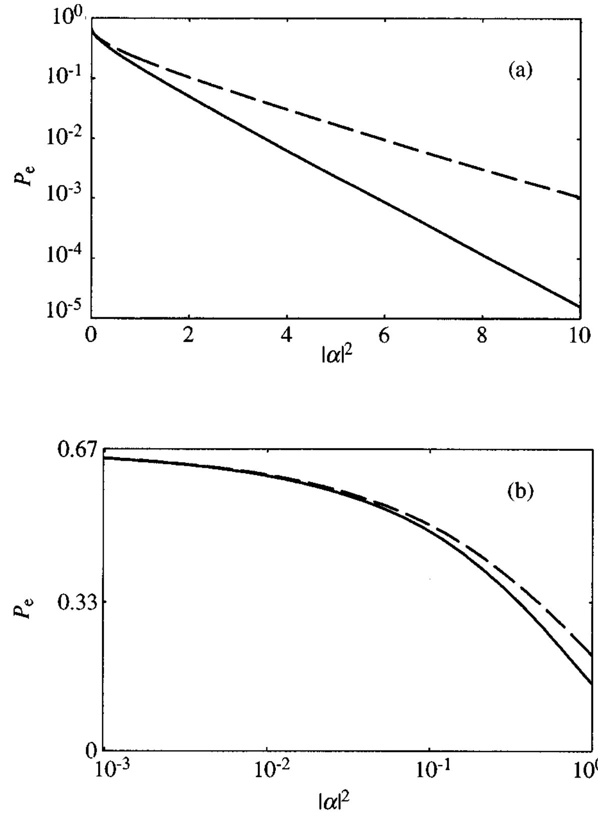

FIG. 2. Average probability of error for ternary symmetric coherent-state signal. (a) Moderate value of $|\alpha|^2$; (b) small value of $|\alpha|^2$. The solid line stands for the optimum quantum receiver and the dashed line for the homodyne receiver (SQL).

$$B=8k\sqrt{(1-k^8)(1+k^4)},$$
$$C=(1+k^4)\{(k^4+2k^2-1)^2-(1-k^8)\},$$

and

$$\begin{aligned}&D=2b(1+k^4)\sqrt{1-k^4},\\ &E_{\pm}=[(1-k^2)(1+k^4)\pm\sqrt{1-k^4}\{2bk+(1-k^2)F\}],\\ &F=\sqrt{1-2b^2+k^4}.\end{aligned} \tag{44}$$

The average probability of error is also obtained as follows:

$$\begin{aligned}P_e^{\text{OPT}}=&1-\frac{1}{2}\{1-(1/2A^2)(B^2-2AC\pm B\sqrt{B^2-4AC})\\ &+\sqrt{1-k^8-[(1-k^4)/A^2](B^2-2AC\pm B\sqrt{B^2-4AC})}\},\end{aligned} \tag{45}$$

where $+$ of $\pm$ is valid for $|\alpha|^2\leq\ln[1+\sqrt{2}]$ and $-$ is for $|\alpha|^2>\ln[1+\sqrt{2}]$.

In Fig. 2, the average probability of error by the optimum quantum receiver in Eq. (45) is compared with that by a homodyne receiver since it is the SQL of this signal set; namely,

$$P_e^{\mathrm{SQL}}=\tfrac{4}{3}\,\mathrm{Erfc}[\sqrt{|\alpha|^2}], \tag{46}$$

where the equal prior probability of signals is assumed.

C. Ternary phase-shift coherent-state signal

In the case of a ternary phase-shift coherent-state signal set $[|\psi_1\rangle,|\psi_2\rangle,|\psi_3\rangle]=[|\alpha\rangle,|\alpha e^{2i\pi/3}\rangle,|\alpha e^{-2i\pi/3}\rangle]$ the inner product matrix becomes as follows:

$$[\kappa_{ij}]=\begin{bmatrix} 1 & K_c+iK_s & K_c-iK_s \\ K_c-iK_s & 1 & K_c+iK_s \\ K_c+iK_s & K_c-iK_s & 1 \end{bmatrix}, \tag{47}$$

where $K_c=k\cos[(\sqrt{3}/2)|\alpha|^2]$, $K_s=k\ \sin[(\sqrt{3}/2)|\alpha|^2]$, and $k=\exp[-3|\alpha|^2/2]$. With the analogy of the classical minimax solution, all signal prior probabilities are assumed to be equal. Then the variable matrix can be supposed to be

$$[x_{ji}]=\begin{bmatrix} a & b+ic & b-ic \\ b-ic & a & b+ic \\ b+ic & b-ic & a \end{bmatrix}. \tag{48}$$

Then we need to solve the following equations:

$$a^2+2b^2+2c^2=1,$$
$$2ab+b^2-c^2=K_c, \tag{49}$$
$$-2ac+2bc=K_s.$$

For this signal set, the solution also has some variations similarly to the case of a ternary symmetric signal set. Each variation must be checked for its Hermitian nonnegativeness. Finally, the accurate solution can be identified and the optimum detection operators and the average probability of error are obtained. The form of the optimum detection operators is as in Eq. (41). The parameters are as follows:

$$d=18S^2I^2,$$

$$m_{1,11}=m_{2,22}=m_{3,33}=G^2,$$

$$m_{1,12}=m^*_{1,13}=m^*_{1,21}=m_{1,31}=m_{2,12}=m_{2,23}=m^*_{2,21}=m^*_{2,32}$$
$$=m^*_{3,13}=m_{3,23}=m_{3,31}=m^*_{3,32}=GH^*, \tag{50}$$

$$m_{1,22}=m_{1,33}=m_{2,11}=m_{2,33}=m_{3,11}=m_{3,22}=|H|^2,$$

$$m_{1,23}=m^*_{1,32}=m^*_{2,13}=m_{2,31}=m_{3,12}=m^*_{3,21}=H^2,$$

where

$$G=\sqrt{2}J(S+\sqrt{2}I),$$

$$H=-J(I-\sqrt{2}S)+3iK_s\sqrt{1+2K_c},$$

$$I=\sqrt{(1+2K_c)}J, \tag{51}$$

$$J=\sqrt{1+S-K_c},$$

$$S=\sqrt{(1-K_c)^2-3K_s^2}.$$

The average probability of error becomes

$$P_e^{\mathrm{OPT}}=\frac{2(3-S-\sqrt{2}I)}{9}. \tag{52}$$

On the other hand, the SQL of this signal set is given by the heterodyne receiver. It results in

$$P_e^{\mathrm{SQL}}=\frac{pe_1+2(pe_{23}+pe_2)-2pe_{23}pe_2}{3}, \tag{53}$$

where

$$pe_1=\mathrm{Erfc}\left[\frac{\sqrt{2}(9|\alpha|^2-4\ln 2)}{12|\alpha|}\right],$$

$$pe_{23}=\mathrm{Erfc}\left[\frac{\sqrt{2}(9|\alpha|^2+4\ln 2)}{12|\alpha|}\right], \tag{54}$$

$$pe_2=\mathrm{Erfc}\left[\frac{\sqrt{6}|\alpha|}{2}\right].$$

The average probability of error by the optimum quantum receiver in Eq. (52) and that by the heterodyne receiver in Eq. (53) are compared in Fig. 3.

D. Quadrature phase-shift coherent-state signal

In the case of a quadrature phase-shift coherent-state signal set $[|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle]=[|\alpha\rangle,|i\alpha\rangle,|-\alpha\rangle,|-i\alpha\rangle]$, the inner product matrix becomes

$$[\kappa_{ij}]=\begin{bmatrix} 1 & K_c+iK_s & k^2 & K_c-iK_s \\ K_c-iK_s & 1 & K_c+iK_s & k^2 \\ k^2 & K_c-iK_s & 1 & K_c+iK_s \\ K_c+iK_s & k^2 & K_c-iK_s & 1 \end{bmatrix}, \tag{55}$$

where

$$k=\exp[-|\alpha|^2],$$

$$K_c=k\ \cos[|\alpha|^2], \tag{56}$$

$$K_s=k\ \sin[|\alpha|^2].$$

With the symmetry of the inner product matrix and the assumption that all prior probabilities are equal, the variables to be determined can be supposed to be

$$[x_{ji}]=\begin{bmatrix} a & b+ic & d & b-ic \\ b-ic & a & b+ic & d \\ d & b-ic & a & b+ic \\ b+ic & d & b-ic & a \end{bmatrix}. \tag{57}$$

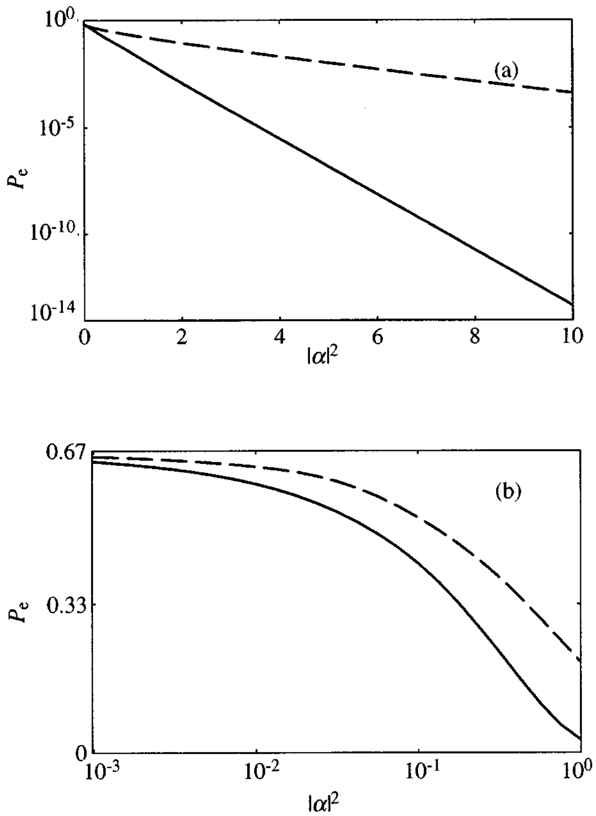

FIG. 3. Average probability of error for ternary phase-shift coherent-state signal. (a) Moderate value of $|\alpha|^2$; (b) small value of $|\alpha|^2$. The solid line stands for the optimum quantum receiver and the dashed line for the heterodyne receiver (SQL).

Then we have the equations to be solved as follows:

$$
\begin{aligned}
&a^2+d^2+2(b^2+c^2)=1,\\
&2b(a+d)=K_c,\\
&-2c(a-d)=K_s,\\
&2(ad+b^2-c^2)=k^2.
\end{aligned}
\tag{58}
$$

There are 16 variations of the solution and some of them satisfy the Hermitian non-negative condition in Eq. (23) partially. In detail, similarly to the case of the ternary symmetric signal, the optimum detection operators have two variations; one is valid for $n\pi \leqslant |\alpha|^2 < \pi/2+n\pi$ and the other is for $\pi/2+n\pi \leqslant |\alpha|^2 < (n+1)\pi$.

The full description of the optimum detection operators for a quaterary signal set becomes too long, so we simplify it as follows:

$$\hat{\Pi}_i=\sum_{j=1}^{4}\sum_{k=1}^{4} t_{ij}t_{ik}^*|\psi_j\rangle\langle\psi_k|. \tag{59}$$

The optimum parameters are

$$
\begin{aligned}
&t_{11}=t_{22}=t_{33}=t_{44}=\frac{A}{4(A^2-B^2)}-\frac{C}{4(C^2-D^2)},\\
&t_{12}=t_{23}=t_{34}=t_{41}=t_{14}^*=t_{21}^*=t_{32}^*=t_{43}^*\\
&\quad=-\frac{B}{4(A^2-B^2)}-i\frac{D}{4(C^2-D^2)},\\
&t_{13}=t_{24}=t_{31}=t_{42}=\frac{A}{4(A^2-B^2)}+\frac{C}{4(C^2-D^2)},
\end{aligned}
\tag{60}
$$

and

$$
\begin{aligned}
&A=\frac{B(1+k^2-4B^2)}{K_c},\\
&B=\frac{\sqrt{1+k^2-\sqrt{(1+k^2)^2-4K_c^2}}}{2\sqrt{2}},\\
&C=\frac{D(1-k^2-4D^2)}{K_s},\\
&D_{\pm}=\pm\frac{\sqrt{1-k^2-\sqrt{(1-k^2)^2-4K_s^2}}}{2\sqrt{2}},
\end{aligned}
\tag{61}
$$

where the $-$ of $\pm$ is for $n\pi \leqslant |\alpha|^2 < \pi/2+n\pi$ and $+$ is for $\pi/2+n\pi \leqslant |\alpha|^2 < (n+1)\pi$. The average probability of error by these optimum detection operators is

$$P_e^{\mathrm{OPT}}=1-(A-C)^2. \tag{62}$$

The average probability of error in Eq. (62) is compared with the SQL, i.e., that of the heterodyne receiver, in Fig. 4. The average probability of error by the heterodyne receiver is as follows:

$$P_{\mathrm{e}}^{\mathrm{SQL}}=2\,\mathrm{Erfc}[\sqrt{|\alpha|^2}]-(\mathrm{Erfc}[\sqrt{|\alpha|^2}])^2. \tag{63}$$

V. PHYSICAL INTERPRETATION OF THE OPTIMUM DETECTION PROCESS

The average probability of error derived in the previous section is much superior to the standard quantum limit which is given by the optimum standard detection operators derived from the conventional receivers in Table I. It is important to analyze the reason why the standard quantum limit is overcome. In this section, we would like to interpret the physical meaning of such improvement as a quantum interference because overcoming the standard quantum limit is a purely quantum mechanical phenomenon [24,25]. That is, the detection process based on standard detection operators can be completely described by a projection valued measure, so that the measurement operators never introduce quantum interference. On the other hand, in the case of the detection process by the generalized detection operators introduced in Sec. III, the quantum interference is obvious since the detection operators can be decomposed to the measurement states which are linear combinations of the signal quantum states; namely, a measurement state is represented from Eq. (28) as follows:

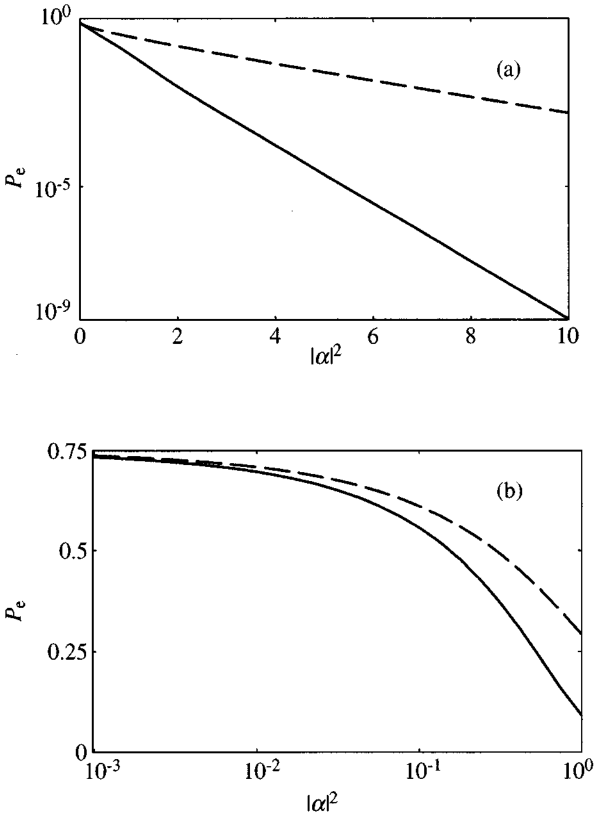

FIG. 4. Average probability of error for quaternary phase-shift coherent-state signal. (a) Moderate value of $|\alpha|^2$; (b) small value of $|\alpha|^2$. The solid line stands for the optimum quantum receiver and the dashed line for the heterodyne receiver (SQL).

$$|\omega_j\rangle = \sum_{k=1}^{M} c_{kj}|\psi_k\rangle, \tag{64}$$

where $\{c_{kj}\}$ is an element of the matrix $[x_{ji}]^{-1}$ in Eq. (28). Therefore the quantum interference in a detection process by some generalized detection operators can be visualized as follows:

$$P(j|i) = \mathrm{Tr}\hat{\rho}_i\hat{\Pi}_j = |\langle\psi_i|\omega_j\rangle|^2 = \left|\left\langle \psi_i \left| \left(\sum_{k=1}^{M} c_{kj} \left|\psi_k\right\rangle\right)\right.\right\rangle\right|^2. \tag{65}$$

For the binary phase shift coherent state signal, one of the error probabilities is

$$\begin{aligned} P(2|1) &= \mathrm{Tr}\hat{\rho}_1\hat{\Pi}_2 = |\langle\alpha|\omega_2\rangle|^2 = |\langle\alpha|(c_{12}|\alpha\rangle + c_{22}|-\alpha\rangle)|^2 \\ &= c_{12}^2 + c_{22}^2\langle\alpha|-\alpha\rangle^2 + 2c_{12}c_{22}\langle\alpha|-\alpha\rangle, \end{aligned} \tag{66}$$

where

$$c_{12} = -\sqrt{\frac{1-\sqrt{1-\kappa^2}}{2(1-\kappa^2)}},$$

$$c_{22} = \sqrt{\frac{1+\sqrt{1-\kappa^2}}{2(1-\kappa^2)}}, \tag{67}$$

$$\kappa = \langle\alpha|-\alpha\rangle = \exp[-2|\alpha|^2].$$

The third term of Eq. (66) explicitly represents the quantum interference. This term is essential to explain the difference between the optimum detection and a near optimum one. In detail, the near optimum detection for the binary signal is well known as the Kennedy receiver [23]. The detection operators of the Kennedy receiver become as follows [26]:

$$\hat{\Pi}_1 = |\alpha\rangle\langle\alpha|,$$
$$\hat{\Pi}_2 = \hat{I} - |\alpha\rangle\langle\alpha|. \tag{68}$$

In this representation, however, it may be difficult to see the quantum interference explicitly while the average probability of error is also much superior to the standard quantum limit.

According to the detailed analysis of the effect of the type of quantum interference in Eq. (66), rather the reduction of the average probability of error is caused by the second term with $\langle\alpha|-\alpha\rangle^2$ as well. Here this term should be regarded as a kind of quantum interference because of the following. *The detection operators by which the standard quantum limit is overcome do not commute with the signal observable, and it has been shown that the above statement corresponds to quantum interference* [15]. Thus our optimum quantum detection process is absolutely interpreted as the quantum interference.

VI. CONCLUSION

The optimum detection operators and the average probability of error for several linearly independent signal sets were derived by the quantum minimax strategy with Kennedy's lemma. The effectiveness of Kennedy's lemma and the minimax strategy lies in the following.

Kennedy's lemma can convert the operator equations with necessary and sufficient conditions into ordinary c-number algebraic equations because the optimum detection operators are regarded as projection valued measures (PVM) on the signal space. The signal space spanned by signal quantum states is a closed subspace of the total Hilbert space. Therefore the derived optimum detection operators are orthonormal resolutions of the identity on the signal space. The uniqueness of the solution is also guaranteed. In practice, some of the variables can be assumed to be real numbers even if the inner products are complex numbers. Furthermore, when the inner products are all real numbers, all variables can be assumed to be real. This assumption has no effect on the optimum detection operators and the minimum average probability of error.

The quantum minimax formula can reduce the number of variables with some analogy to the classical theory. That is, the signal prior probabilities are governed by the signal arrangement so as to satisfy the minimax strategy. Then some of the signal prior probabilities can be supposed to be equal when the signal arrangement is symmetric. In practice, for ternary and quadrature phase-shift coherent-state signal sets the prior probabilities for all signals, and for a ternary symmetric coherent-state signal set those for signals 2 and 3 are assumed to be equal before the derivations while the equal signal prior probability is derived for a binary phase-shift coherent-state signal set.

The equations to be solved are nonlinear, so that the solution has some variations. Although Eq. (23) can identify the accurate solutions, some of them must be switched between two solutions with respect to the square of the signal amplitude.

Finally, the optimum detection operators and the minimum average probability of error are obtained with Eqs. (28), (20), and (9). They are related to quantum eavesdropping. The derived optimum detection operators guarantee the minimum average probability of error for unknown signal prior probabilities, although it is the worst one attained by the quantum Bayes strategy. It is true that the quantum Bayes strategy can give the solutions for some of the above signal sets by eigenvalue equations. However, the quantum minimax strategy can derive more solutions than the quantum Bayes strategy and its derivation process is much systematic.

A physical interpretation of the optimum detection operator is tried and it can be regarded as a kind of quantum interference. That is, the optimum detection process induces the quantum interference in a measurement process with optimum control so as to reduce the average probability of error.

These derivations are based on Kennedy's lemma where the signal quantum states is assumed to be linearly independent. Therefore the derivation of the optimum detection operators for signals with linearly dependent or mixed states is impossible with the above algorithm. We will find another algorithm to derived the optimum detection operators with linearly dependent signals.

ACKNOWLEDGMENT

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APPENDIX: DERIVATION OF THE OPTIMUM DETECTION OPERATORS BY THE QUANTUM BAYES STRATEGY

Even in the case of a binary phase-shift coherent-state signal set $[|\psi_1\rangle,|\psi_2\rangle]=[|\alpha\rangle,|-\alpha\rangle]$ the c-number equations in Eqs. (22) and (23) based on the quantum Bayes strategy are too difficult to be solved. However, the eigenvalue equation in Eq. (12), which is the alternative equation by the quantum Bayes strategy, can give the solution for arbitrary prior probabilities. The eigenvalue equation becomes

$$(|-\alpha\rangle\langle-\alpha|-\lambda|\alpha\rangle\langle\alpha|)|\eta_i\rangle=\eta_i|\eta_i\rangle \tag{A1}$$

where $\lambda\equiv\xi_1/\xi_2$. Two eigenvalues are obtained from the above equation:

$$\eta_1=\frac{1-\lambda}{2}-R,$$
$$\eta_2=\frac{1-\lambda}{2}+R, \tag{A2}$$

where

$$R=\sqrt{\left(\frac{1+\lambda}{2}\right)^2-\lambda\kappa^2},\quad \kappa=\langle\alpha|-\alpha\rangle=\exp[-2|\alpha|^2]. \tag{A3}$$

Since $\eta_1<0$ and $\eta_2>0$, the optimum detection operator should be represented by each eigenstate:

$$\hat{\Pi}_1=|\eta_1\rangle\langle\eta_1|,$$
$$\hat{\Pi}_2=|\eta_2\rangle\langle\eta_2|. \tag{A4}$$

In accordance with Kennedy's lemma these eigenstates correspond to the measurement states

$$|\eta_1\rangle\Rightarrow|\omega_1\rangle,$$
$$|\eta_2\rangle\Rightarrow|\omega_2\rangle. \tag{A5}$$

Substituting the relation of Eq. (21) into Eq. (A1) with the assumption that all the variables $\{x_{ji}\}$ are real numbers, we can obtain the solutions although the calculation process is very tedious and requires a skillful algorithm. The accurate solution is selected by the condition in Eq. (23).

Finally, with Eq. (20) and Eq. (28), the optimum detection operators for arbitrary signal prior probabilities can be derived as follows:

$$\hat{\Pi}_1=\frac{1}{4R(1-\kappa^2)}[\{2R-2\lambda\kappa^2+(1+\lambda)\}|\alpha\rangle\langle\alpha|+\{2R+2\kappa^2-(1+\lambda)\}|-\alpha\rangle\langle-\alpha|-\kappa(2R+1-\lambda)(|\alpha\rangle\langle-\alpha|+|-\alpha\rangle\langle\alpha|)], \tag{A6}$$

$$\hat{\Pi}_2=\frac{1}{4R(1-\kappa^2)}[\{2R+2\lambda\kappa^2-(1+\lambda)\}|\alpha\rangle\langle\alpha|+\{2R-2\kappa^2+(1+\lambda)\}|-\alpha\rangle\langle-\alpha|-\kappa(2R-1+\lambda)(|\alpha\rangle\langle-\alpha|+|-\alpha\rangle\langle\alpha|)].$$

The average probability of error by these optimum detection operators becomes

$$P_e^{\mathrm{OPT}}=\frac{1-\sqrt{1-4\xi_1\xi_2\kappa^2}}{2}. \tag{A7}$$

It is also possible to derive the optimum detection operators by the quantum Bayes strategy for ternary phase shift and quaternary phase-shift coherent-state signal sets, where all the signal prior probabilities are assumed to be equal [1]. However, this assumption is just equal to the results obtained by the quantum minimax strategy. Furthermore, in the case of the ternary symmetric signal set, the derivation of the optimum detection operators is too difficult to obtain the analytical solution. Hence we can say that the quantum minimax strategy is more effective in the derivation of the optimum detection operators because it can provide more solutions than the quantum Bayes strategy.

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