# Trapped ions as vibrational beam splitters: SU(2) states in a two-dimensional ion trap

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A scheme for preparing vibrational SU(2) states of motion in a two-dimensional ion trap is described. These anticorrelated two-mode states are formally equivalent to the output photon states of a lossless SU(2) interferometer with number-state inputs. Nontrivial statistics such as the binomial distribution and the discrete ''arc-sine'' distribution can be generated in the vibrational states of trapped ions, and detected by measuring the population inversion of the ion driven by a laser field along a specific direction. [S1050-2947(96)03308-2]

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Recent developments in the cooling and trapping ions [1–4] have opened a new research realm for both atomic physics and quantum optics. Theoretically, an ion confined in an electromagnetic trap is equivalent to a particle moving in a harmonic potential in which the center-of-mass (c.m.) motion of the ion is quantized as a harmonic oscillator. When the internal atomic states of the trapped ion are excited or deexcited by the classical laser driving field, the vibrational states of the c.m. motion are changed, as the atomic stimulated absorption or emission processes are always accompanied by momentum exchange of the driving field. For the most general case, the Schrödinger equation of this model is described by a set of linear differential equations that couple the probability amplitudes for the different vibrational states [5]. However, if the vibrating amplitude of the ion is much smaller than the laser wavelength, i.e., the Lamb-Dicke limit is satisfied, and the driving field is tuned to one of the vibrational sidebands of the atomic transition, then this model can be simplified to a form similar to the Jaynes-Cummings model (JCM) [5–7] except that the quantized radiation field is replaced by the quantized c.m. motion of the ion. As the vibrational mode in the ion trap does not couple to the external optical modes, the dissipative effects inevitable from cavity damping in the optical regime can now be significantly suppressed. This prominent feature thus leads to the possibility of realizing some cavity QED experiments without using an optical cavity. There have been several schemes proposed recently following this approach to produce nonclassical vibrational states inside an ion trap. For example, Fock states can be prepared by methods involving quantum jumps [8], adiabatic passage [9], trapping states [10], or a sequence of Rabi $\pi$ pulses driving the ion [11]. Coherent states of motion can be produced from the vacuum by a spatially uniform classical driving field or by a ''moving standing wave'' [11]. Using bichromatic Raman excitation of the ion, one is able to produce squeezed states of motion inside the trap [11–13]. In particular, quantum superpositions of two microscopically distinguishable states (the Schrödinger cat states) of the trapped ions can also be prepared [14–16].

In this paper, we describe how to prepare and observe the anticorrelated two-mode SU(2) vibrational states by using trapped ions. Consider a two-level ion of mass $M$ moving in a two-dimensional (2D) isotropic harmonic potential characterized by the trap frequency $\nu$. The vibrational quanta are

described by the annihilation (creation) operators $\hat{a}$ ($\hat{a}^\dagger$) and $\hat{b}$ ($\hat{b}^\dagger$) defined in the $X$ and $Y$ directions, respectively. Accordingly, the position operators are given by

$$\hat{x}=(2\nu M)^{-1/2}[\hat{a}+\hat{a}^\dagger], \quad \hat{y}=(2\nu M)^{-1/2}[\hat{b}+\hat{b}^\dagger], \quad (1)$$

where we have assumed that $\hbar=1$.

Using the Schwinger representation [17]

$$\hat{J}_1=(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger)/2, \quad \hat{J}_2=(\hat{a}^\dagger\hat{b}-\hat{a}\hat{b}^\dagger)/2i,$$

$$\hat{J}_3=(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b})/2, \quad \text{and} \quad \hat{J}^2=\hat{J}_1^2+\hat{J}_2^2+\hat{J}_3^2, \quad (2)$$

it follows that the two-mode Fock state $|n_x,n_y\rangle$ can be described as the pseudo-angular-momentum state $|j,\mu\rangle$, which is the common eigenstate of the angular-momentum operators $\hat{J}^2$ and $\hat{J}_3$ with

$$\hat{J}^2|j,\mu\rangle=j(j+1)|j,\mu\rangle,$$

$$\hat{J}_3|j,\mu\rangle=\mu|j,\mu\rangle, \quad (3)$$

provided $j=\frac{1}{2}(n_x+n_y)$ and $\mu=\frac{1}{2}(n_x-n_y)$.

For all $|\mu|\leq j$, the states $|j,\mu\rangle$ form an orthogonal basis in a $(2j+1)$-dimensional Hilbert space $\mathcal{H}_{2j+1}$,

$$\sum_{\mu=-j}^{j}|j,\mu\rangle\langle\mu,j|=1,$$

$$\langle\lambda,j|j,\mu\rangle=\delta_{\lambda\mu}. \quad (4)$$

Thus, an arbitrary two-mode field state can be expressed as

$$|F\rangle=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F_{nm}|n,m\rangle=\sum_{2j=0}^{\infty}\sum_{\mu=-j}^{j}F_{j\mu}|j,\mu\rangle, \quad (5)$$

where the summation over $2j$ indicates that the sum includes both integer and half-integer values of $j$.

Consider the following transformation defined by the operator $\hat{U}(\theta)=\exp[\theta(\hat{a}^\dagger\hat{b}-\hat{a}\hat{b}^\dagger)]$:

$$\hat{U}^{-1}(\theta)\begin{pmatrix}\hat{a}\\\hat{b}\end{pmatrix}\hat{U}(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{a}\\\hat{b}\end{pmatrix}=\begin{pmatrix}\hat{a}_\theta\\\hat{b}_\theta\end{pmatrix},$$

$$(6)$$

which is equivalent to a rotation of the coordinate system on the $X$-$Y$ plane with angle $\theta$. According to Eq. (1), we then have

$$\begin{pmatrix} \hat{x}_\theta \\ \hat{y}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}.$$

We denote $|j,\mu\rangle_0$ as the pseudo-angular-momentum state excited along the original $X$-$Y$ axes, which can be easily prepared by exciting the ion with a sequence of Rabi $\pi$ pulses, as illustrated in Ref. [11]. According to the transformation in Eq. (6), the state $|j,\mu\rangle_\theta$ in the $X_\theta$-$Y_\theta$ direction is defined as

$$
\begin{aligned}
|j,\mu\rangle_0 &= \frac{(\hat{a}^\dagger)^{j+\mu}(\hat{b}^\dagger)^{j-\mu}}{\sqrt{(j+\mu)!(j-\mu)!}}|0\rangle, \\
&= \hat{U}(\theta)\frac{(\hat{a}_\theta^\dagger)^{j+\mu}(\hat{b}_\theta^\dagger)^{j-\mu}}{\sqrt{(j+\mu)!(j-\mu)!}}\hat{U}^{-1}(\theta)|0\rangle, \\
&= \hat{U}(\theta)|j,\mu\rangle_\theta, \quad (7)
\end{aligned}
$$

where we have used the fact that the vacuum is invariant under the rotation. Using the closure relation in Eq. (4),

$|j,\mu\rangle_0$ can be expressed as a linear combination of the pseudo-angular-momentum states observed along the directions of the $X_\theta$-$Y_\theta$ axes, i.e.,

$$
\begin{aligned}
|j,\mu\rangle_0 &= \sum_{\lambda=-j}^{j} |j,\lambda\rangle_{\theta\,\theta}\langle\lambda,j|\hat{U}(\theta)|j,\mu\rangle_\theta, \\
&= \sum_{\lambda=-j}^{j} \mathcal{D}_{\lambda\mu}^j(\cos 2\theta)|j,\lambda\rangle_\theta, \quad (8)
\end{aligned}
$$

where $\mathcal{D}_{\lambda\mu}^j(\cos 2\theta)$ is the matrix element of the rotation operator $\hat{U}(\theta)$. However, as the total quantum number is a conserved quantity for the Fock states $|n_x,n_y\rangle_0$ and $|m_x,m_y\rangle_\theta$ in different directions, we may let $N = n_x + n_y = m_x + m_y$. Therefore, in the Fock representation, Eq. (8) is expressed as

$$|n_x,N-n_x\rangle_0 = \sum_{m_x=0}^{N} \mathcal{D}_{m_x n_x}^N(\cos 2\theta)|m_x,N-m_x\rangle_\theta, \quad (9)$$

where again the matrix element is given by [17]

$$\mathcal{D}_{m_x n_x}^N(\cos 2\theta) = \sum_{k=\max(0,n_x-m_x)}^{\min(n_x,N-m_x)} \frac{(-1)^{k+m_x-n_x}\sqrt{n_x!(N-n_x)!m_x!(N-m_x)!}}{k!(n_x-k)!(N-m_x-k)!(k+m_x-n_x)!}(\cos\theta)^{N-2k+n_x-m_x}(\sin\theta)^{2k-n_x+m_x}. \quad (10)$$

Some symmetry relations of $\mathcal{D}_{m_x n_x}^N(\cos 2\theta)$ are given below, which are useful for the further calculations,

$$
\begin{aligned}
\mathcal{D}_{m_x n_x}^N(\cos 2\theta) &= (-1)^{m_x-n_x}\mathcal{D}_{n_x m_x}^N(\cos 2\theta), \\
&= \mathcal{D}_{N-n_x N-m_x}^N(\cos 2\theta),
\end{aligned}
$$

$$\mathcal{D}_{m_x N-n_x}^N(\cos 2\theta) = (-1)^{N-n_x-m_x}\mathcal{D}_{N-m_x n_x}^N(\cos 2\theta),$$

$$\mathcal{D}_{m_x n_x}^N(-\cos 2\theta) = (-1)^{m_x}\mathcal{D}_{m_x N-n_x}^N(\cos 2\theta).$$

(11)

Essentially, Eq. (7) is just a basis transformation in the truncated Hilbert space $\mathcal{H}_{2j+1}$ in which the total quantum numbers are conserved. As a matter of fact, the square of $\mathcal{D}_{m_x n_x}^N(\cos 2\theta)$ is exactly the joint output photon-number distribution of a lossless SU(2) interferometer [18,19] when both the input ports are excited by pure number states. The photon statistics of this class of states has been investigated by several authors [19,20]. However, as the transformation of Eq. (6) coincides with a real-space rotation in the $X$-$Y$ plane, the probability distribution described by Eq. (9) can be realized and observed in the two-dimensional (2D) ion trap. Moreover, in the case of weakly excited $|n_x,n_y\rangle_0$, entangled states of importance in the quantum measurement theory can also be prepared in this manner; for example,

$$|1,0\rangle_0 = \sin\theta|0,1\rangle_\theta + \cos\theta|1,0\rangle_\theta,$$

$$|1,1\rangle_0 = \frac{1}{\sqrt{2}}[|0,2\rangle_{\pi/4} - |2,0\rangle_{\pi/4}].$$

A direct way to examine the properties of the above vibrational states is to measure the population inversion of the trapped ion. In order to distinguish the statistics determined by the angle $\theta$, one may need to construct a measuring Hamiltonian that is spatially dependent. In this case, a laser field of frequency $\omega$ is applied to the ion along the $X_\theta$ axis in the Lamb-Dicke regime, i.e., $\nu \gg \Omega$, where $\Omega$ is the Rabi frequency describing the atom-field interaction. Now, the laser frequency is turned to the first red vibrational sideband of the atomic transition so that a resonant JCM Hamiltonian can be realized along the $X_\theta$ axis [5,6]

$$\hat{H}_{\text{Jay}} = \nu\hat{a}_\theta^\dagger\hat{a}_\theta + \frac{\nu}{2}\hat{\sigma}_z + g(\hat{a}_\theta\hat{\sigma}_+ + \text{H.c.}), \quad (12)$$

where $\hat{\sigma}$'s are the Pauli matrices and $g$ is the effective coupling constant. With the help of the well-known solutions for the JCM [21], for an initially excited atom the population inversion measured in this case is given by
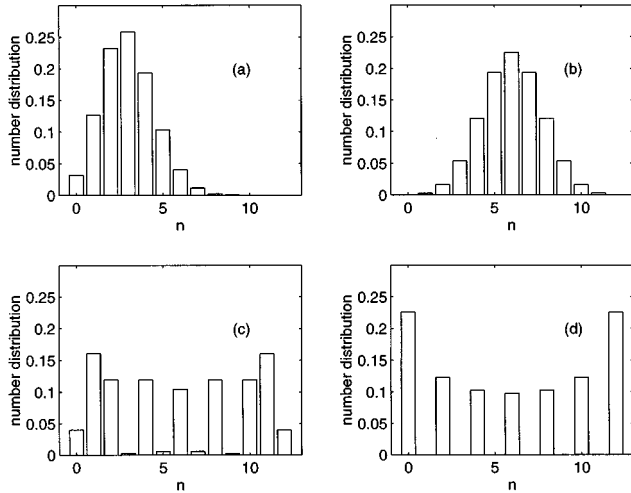
FIG. 1. Number distribution for finding $n$ vibrational quanta along the $X_\theta$ axis with (a) $n_x=12$, $n_y=0$, $\theta=\pi/3$; (b) $n_x=12$, $n_y=0$, $\theta=\pi/4$; (c) $n_a=6$, $n_y=6$, $\theta=\pi/3$; (d) $n_x=6$, $n_y=6$, $\theta=\pi/4$.

$$W(\theta,t)=\langle \hat{\sigma}_z(t)\rangle$$

$$=\sum_{m_x=0}^{N}[\mathcal{D}_{m_x n_x}^N(\cos 2\theta)]^2\cos(2gt\sqrt{m_x+1}).$$

$$(13)$$

If, however, we wish to describe the various uncontrollable factors in real experiments, such as intensity fluctuations of the laser beams, laser phase fluctuations, or the variations in the trap drive frequency, and so forth, we can modify Eq. (13) to include dephasing [22] and obtain

$$W(\theta,t)=\sum_{m_x=0}^{N}[\mathcal{D}_{m_x n_x}^N(\cos 2\theta)]^2$$

$$\times \exp\left[\frac{-2g^2t}{\gamma}(m_x+1)\right]\cos(2gt\sqrt{m_x+1}),$$

$$(14)$$

where $\gamma$ is the decay constant that stems from the decoherence effects of the dephasing. It is easy to identify that $W(\theta,t)=W(\theta+\pi,t)$ using the symmetry relations in Eq. (11). We notice that, since the ion is confined in an isotropic 2D harmonic potential, there are lots of possibilities of choosing a coordinate system to excite the initial Fock states of motion of the ion. Nevertheless, once the relative angle $\theta$ of the probing beam is specified, the statistics of the vibrational quanta in this direction can be uniquely determined by the matrix element $\mathcal{D}_{m_x n_x}^N(\cos 2\theta)$.

Several quantum number distributions are shown in Fig. 1. Basically, the statistical properties of $\mathcal{D}_{m_x n_x}^N$ depend crucially on the initial vibrational excitation. In the following, we pay special attention to the two limiting cases: (i) $n_x>n_y=0$, $0\leq\theta\leq\pi$; and (ii) $n_x=n_y$, $\theta=\pi/4$. For the first case, the initial vibrational state is in the highest excitation angular-momentum state $|j,j\rangle_0$, which, according to Eq. (8),
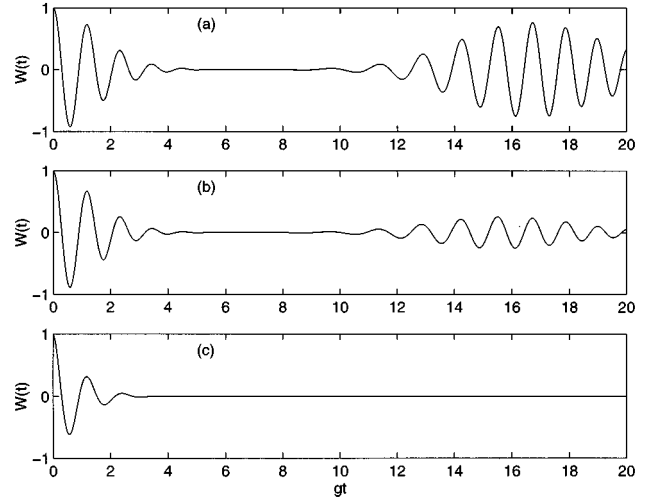


FIG. 2. Population inversion $W(\theta,t)$ as a function of $gt$, where $n_x=12$, $n_y=0$, $\theta=\pi/4$; (a) $g^2/\gamma=0$; (b) $g^2/\gamma=0.005$; (c) $g^2/\gamma=0.05$.

is transformed to the SU(2) coherent states [23]. The SU(2) coherent states are characterized by the binomial distribution

$$|\mathcal{D}_{m_x N}^N(\cos 2\theta)|^2=\frac{N!}{m_x!(N-m_x)!}(\cos^2\theta)^{m_x}(1-\cos^2\theta)^{N-m_x},$$

$$(15)$$

which results in sub-Poissonian statistics for both modes. The atomic dynamics of a JCM in the presence of SU(2) coherent states have been investigated by different authors [24,25]. In this paper, identical results are reproduced by replacing $n_x=N$ in Eq. (10). However, in order to be compatible with the real experiments, we have assumed a small initial vibrational excitation ($N=12$). The population inversions are shown in Fig. 2 with different values of the decoherence factor.

As can be seen in Fig. 1, the basic difference between cases (i) and (ii) is that the quantum number distribution of
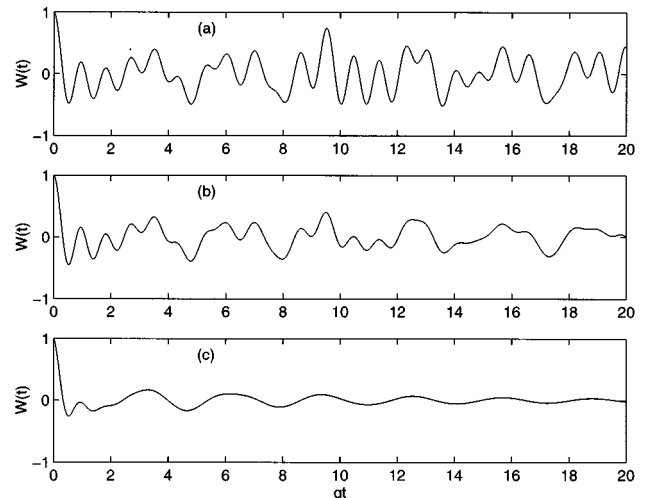


FIG. 3. Population inversion $W(\theta,t)$ as a function of $gt$, where $n_x=6$, $n_y=6$, $\theta=\pi/4$; (a) $g^2/\gamma=0$; (b) $g^2/\gamma=0.005$; (c) $g^2/\gamma=0.05$.

the latter becomes oscillatory between odd and even counts rather than remain given by the binomial nature of the former. This is because, when $n_x = n_y$, the amplitude $\mathcal{D}^N_{m_x N/2}(\cos 2\theta)$ is proportional to the associated Legendre function $P^{m_x-N/2}_{N/2}$,

$$\mathcal{D}^N_{m_x N/2}(\cos 2\theta) = \sqrt{\frac{(N-m_x)!}{m_x!}} P^{m_x-N/2}_{N/2}(\cos 2\theta). \quad (16)$$

The oscillatory character of the quantum number distribution implies some nonclassical properties engendered by the strong quantum interference. Particularly, if $\theta = \pi/4$, it follows immediately that the number distribution vanishes when $m_x$ is an odd integer, indicating that odd numbers of vibrational quanta do not exist in both modes, as shown in Fig. 1(d). When $m_x$ is an even integer, the number distribution is given by

$$|\mathcal{D}^N_{m_x N/2}(0)|^2 = 2^{-N} \frac{m_x!(N-m_x)!}{\left[\left(\frac{m_x}{2}\right)!\left(\frac{N-m_x}{2}\right)!\right]^2}, \quad (17)$$

which is known as the discrete arcsine distribution of order $N/2$ in probability theory [26]. In the presence of the number distribution, Eq. (17), the population inversions of the ion are plotted assuming various values of the decoherence factor in Fig. 3. We see that the Rabi oscillations shown in Fig. 3 exhibit chaotic behavior rather than the distinct quantum revivals and collapses of the Rabi oscillations in Fig. 2. This is because the spectral components of the Rabi oscillations that originate from the odd-count Fock states are absent, due to the vanishing of the odd number of the initial quantum number distribution. In this way the time evolution of the inversion is similar to that seen for a squeezed vacuum, where again odd photon numbers are absent. The clear revivals of the coherent state JCM depend on the precise interrelationship of the adjacent spectral Rabi components; this is lacking for the cases investigated here.

In summary, we have described a simple scheme to prepare vibrational SU(2) states within a 2D ion trap. The SU(2) states are important in quantum optical theory not only because they are employed in interferometers, but because they can also be visualized as the bosonic realization of the collective atomic states [27]. For example, the pseudo-angular-momentum state $|j,\mu\rangle$ and the SU(2) coherent states discussed in this paper correspond to the Dicke state and Bloch states in the context of the collective atomic interactions, respectively [27]. The vibrational SU(2) states discussed in this paper are formally identical to the photon states on the output ports of a lossless SU(2) interferometer with number-state inputs [19]. The main differences between these two systems are (1) the photons are replaced by the localized vibrational quanta, and (2) the mode-mixing processes on the output ports of the interferometer are replaced by a real-space rotation. The statistical properties of the vibrational SU(2) states are spatially dependent, which can be determined by measuring the atomic population inversion with the atom being driven in a specific direction.

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