Direct sampling of density matrices in field-strength bases

A. Zucchetti and W. Vogel
Arbeitsgruppe Quantenoptik, Fachbereich Physik, Universität Rostock, Universitätsplatz 3, D-18051 Rostock, Germany

M. Tasche
Fachbereich Mathematik, Universität Rostock, Universitätsplatz 1, D-18051 Rostock, Germany

D.-G. Welsch
Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, Max-Wien Platz 1, D-07743 Jena, Germany
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A simple and efficient method for direct sampling of the density matrix of a signal mode in optical homodyne tomography is developed. The method is based on approximate, quasianalytical techniques of Fourier transform and yields the density matrix in an arbitrary field-strength basis. It is also suitable for direct sampling of density matrices of matter systems in generalized position or momentum bases. [S1050-2947(96)03108-3]

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The determination of the density matrices of quantum systems is of fundamental interest, since they contain all knowable information on the systems. When the density matrix of a system can be obtained from some kind of measurement, the full quantum-statistical information on it is available, at least in principle. Moreover, measurement of the quantum state in different Hilbert-space bases can give specific insight into the quantum structure of a system. In this context efficient and simple methods are required in order to reconstruct the quantum state from an appropriate set of measured data.

Recently Smithey *et al.* [1] have demonstrated the feasibility of reconstruction of the quantum state of a radiation-field mode by means of optical homodyne tomography (OHT). Measuring the difference-count statistics in balanced homodyne detection for various phase differences between signal mode and local oscillator yields the probability distributions of scaled (electric or magnetic) field strengths of the signal mode for various phase parameters [2,3]. Knowledge of the field-strength distributions for a sufficiently dense set of phase parameters within an interval of size π then allows one to reconstruct the quantum state of the signal mode in terms of s-parametrized quasidistributions [4]. In particular, the measured field-strength distributions can be expressed in terms of marginal distributions of the Wigner function. The corresponding integral relations, given for a set of phase parameters, can be inverted to reconstruct the Wigner function. Practically, this can be done by the method of inverse Radon transform [1], which corresponds to a threefold integral transform [4]. Subsequently, the density matrix in a field-strength basis can be obtained by Fourier transforming the Wigner function. It is worth noting that closely related experiments have been performed to reconstruct quantum states of molecular vibrations [5], where the freely evolving position plays the role of the radiation field strengths in optics. Moreover, a tomographic method for the reconstruction of quantum states of the trapped-ion motion has been proposed [6].

In practice methods are desired that enable one to obtain the density matrix or other representatives of the quantum state as directly as possible from the measured data including error control. As was shown by Kühn and co-workers [7,8], the density-matrix elements in a field-strength basis can simply be given in terms of twofold Fourier transforms of the field-strength distributions measured in OHT:

$$\langle \mathcal{F}_1+\mathcal{F}_2,\varphi|\hat{\varrho}|\mathcal{F}_1-\mathcal{F}_2,\varphi\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dy\; e^{-iy\mathcal{F}_1}\int_{-\infty}^{+\infty} d\mathcal{F}\, e^{i\mathcal{F}y'}p(\mathcal{F},\varphi'), \qquad (1)$$

where

$$y'=y'(y,\mathcal{F}_2)=\sqrt{\frac{\mathcal{F}_2^2}{|F|^4}+y^2}, \qquad (2)$$

$$\varphi'=\varphi'(y,\mathcal{F}_2)=\frac{1}{2}\pi+\varphi-\arg\left(-\frac{\mathcal{F}_2}{|F|^2}+iy\right). \qquad (3)$$

In Eq. (1), the $|\mathcal{F},\varphi\rangle$ are the eigenvectors of the field-strength operator $\hat{F}=F\hat{a}+F^*\hat{a}^\dagger$ defining the basis, $\hat{a}^\dagger$ and $\hat{a}$ are the photon creation and destruction operators, respectively, and $p(\mathcal{F},\varphi')=\langle\mathcal{F},\varphi'|\hat{\varrho}|\mathcal{F},\varphi'\rangle$ is the probability distribution for the field strength at phase φ'. The mode function $F=|F|e^{i\varphi}$ can be chosen to specify the kind of field (e.g., electric or magnetic) under study or, for comparing with experiments, it can be related to the shot noise [8]. Alternatively, rotated quadrature operators could be introduced by identifying $\hat{F}=\hat{x}_\varphi$ and choosing $|F|=1/2$ (or $1/\sqrt{2}$).

Somewhat later, D'Ariano, Macchiavello, and Paris [9] showed that the density-matrix elements in the photon-number basis can be expressed in terms of a twofold integral as

$$\langle n|\hat{\varrho}|m\rangle=\int_{-\infty}^{+\infty} d\mathcal{F}\int_0^{\pi} d\varphi\,\mathcal{K}_{nm}(\mathcal{F},\varphi)p(\mathcal{F},\varphi). \qquad (4)$$

To avoid the involved algorithm in Ref. [9], much effort has been necessary to calculate the functions $\mathcal{K}_{nm}(\mathcal{F},\varphi)$ [9–12]. The advantage of reconstruction formulas of the form of Eq. (4) is that they allow direct sampling of the density-matrix elements by storing the measured data after multiplication with the sampling functions $\mathcal{K}_{nm}(\mathcal{F},\varphi)$.

The question therefore arises of whether or not the Fourier transform (1) can be used for direct sampling of the density-matrix elements in a field-strength basis. Clearly, changing in Eq. (1) the integration variables by using Eq. (3), so that an integration with respect to φ' is performed in place of that with respect to y, would not be very useful, since the resulting sampling functions are not bounded [10]. Moreover, the essential advantage of the formula (1), namely, the reconstruction of the density matrix by means of well-established Fourier techniques would be lost. In the following we develop a simple and efficient algorithm for direct sampling of the density-matrix elements in a field-strength basis using quasianalytical methods of Fourier transforms. The sampling functions derived in this manner are very elementary and can be computed without effort.

Let us first introduce some mathematical basic relations. It is well known that any absolutely integrable function $f(x)$ can be approximated by linear splines with compact support [13]. Choosing a finite set of nodes $\{x_n\}$, an approximating spline function $f_s(x)$ is given by

$$f_s(x)=\sum_n f(x_{n+1})B_{\Delta x_n,n}(x), \tag{5}$$

where $B_{\Delta x_n,n}(x)$ denotes the hat function [13],

$$B_{\Delta x_n,n}(x)=\begin{cases}\dfrac{x-x_n}{\Delta x_n}, & x_n\leqslant x\leqslant x_{n+1}\\ \dfrac{x_{n+2}-x}{\Delta x_{n+1}}, & x_{n+1}\leqslant x\leqslant x_{n+2}\\ 0 \quad \text{elsewhere}\end{cases} \tag{6}$$

and $\Delta x_n = x_{n+1} - x_n$. In particular, for equidistant nodes $x_n=n\Delta x$ the hat function is denoted by $N_{\Delta x,n}(x)$ and reads as

$$N_{\Delta x,n}(x)=N\left(\frac{x}{\Delta x}-n\right) \tag{7}$$

with

$$N(x)=\begin{cases}x, & 0\leqslant x\leqslant 1\\ 2-x, & 1\leqslant x\leqslant 2\\ 0 \quad \text{elsewhere,}\end{cases} \tag{8}$$

Δx being the mesh size.

The approximation $f_s(x)$ of $f(x)$ can now be used to calculate the Fourier transform

$$\tilde{f}(k)=\int_{-\infty}^{+\infty}dx e^{ikx}f(x) \tag{9}$$

explicitly. Replacing $f(x)$ in Eq. (9) by the spline function (5), we obtain

$$\tilde{f}(k)\approx\sum_n f(x_{n+1})\tilde{B}_{\Delta x_n,n}(k). \tag{10}$$

Here $\tilde{B}_{\Delta x_n,n}(k)$ is the Fourier transform of $B_{\Delta x_n,n}(x)$, which can be easily derived to be

$$\tilde{B}_{\Delta x_n,n}(k)=\begin{cases}k^{-1}[I_{\Delta x_n,n}(k)-I_{\Delta x_{n+1},n+1}(k)] & \text{for} \quad k\neq 0,\\ 2^{-1}(x_{n+2}-x_n) & \text{for} \quad k=0,\end{cases} \tag{11}$$

where

$$I_{\Delta x_n,n}(k)=\frac{e^{ikx_{n+1}}-e^{ikx_n}}{k\Delta x_n}. \tag{12}$$

In particular, for equidistant nodes Eq. (11) reduces to

$$\tilde{N}_{\Delta x,n}(k)=(\Delta x)e^{ik(n+1)\Delta x}\ \operatorname{sinc}^2\left(\frac{1}{2}k\Delta x\right) \tag{13}$$

with $\operatorname{sinc} x = \sin x/x$.

In order to apply these results to the calculation of the twofold Fourier integral in Eq. (1), we note that in OHT the field distributions $p(\mathcal{F},\varphi')$ are usually measured on an equidistant grid of points $\{\mathcal{F}_m,\varphi_n\}$. Inverting Eq. (3) yields for $\mathcal{F}_2<0$

$$y=y(\varphi',\mathcal{F}_2)=-\frac{\mathcal{F}_2}{|F|^2}\cot(\varphi'-\varphi), \tag{14}$$

from which y points that correspond to equidistant φ' points are seen to be not equidistant. Note that the density matrix is known when it is known for $\mathcal{F}_2 < 0$. Applying Eq. (5), we can express the field-strength distribution $p(\mathcal{F},\varphi'(y,\mathcal{F}_2))$ in Eq. (1) in terms of its values on a chosen grid of points $\{\mathcal{F}_m,\varphi_n\}$ as follows:

$$p(\mathcal{F},\varphi'(y,\mathcal{F}_2))\approx\sum_{m,n} p(\mathcal{F}_{m+1},\varphi_{n+1})B_{\Delta\mathcal{F}_m,m}(\mathcal{F})B_{\Delta y_n,n}(y). \tag{15}$$

In Eq. (15), the hat function $B_{\Delta y_n,n}(y)$ is given according to Eq. (6), where the nodes y_n are determined by Eq. (14),

$$y_n=y(\varphi_n,\mathcal{F}_2), \quad \varphi_{n+1}-\varphi_n=\Delta\varphi, \tag{16}$$

the φ_n being the experimentally chosen phases. In the case when $\mathcal{F}_m=m\Delta\mathcal{F}$ the hat-function $B_{\Delta\mathcal{F}_m,m}(\mathcal{F})$ reduces to $N_{\Delta\mathcal{F},m}(\mathcal{F})$ defined by Eq. (7).

Substituting in Eq. (1) for $p(\mathcal{F},\varphi')$ the approximation (15), we obtain a reconstruction formula suitable for direct sampling of the density matrix in a field-strength basis:

$$\langle\mathcal{F}_1+\mathcal{F}_2,\varphi|\hat{\varrho}|\mathcal{F}_1-\mathcal{F}_2,\varphi\rangle$$

$$\approx\sum_{m,n} K_{mn}(\mathcal{F}_1,\mathcal{F}_2,\varphi)p(\mathcal{F}_{m+1},\varphi_{n+1}), \tag{17}$$

where the sampling function, after evaluating the $\mathcal{F}$ integral, is given by

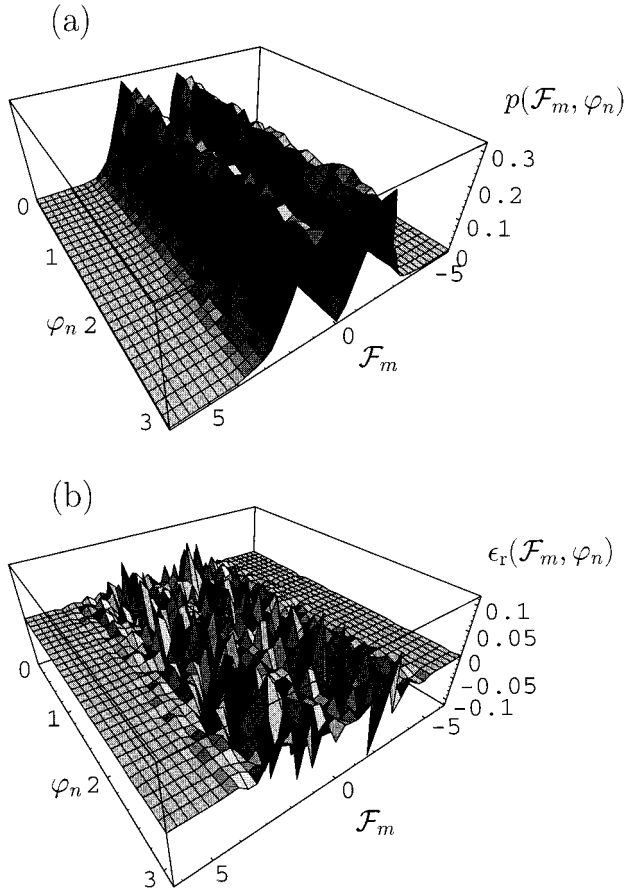

FIG. 1. Simulation of the measurement of a set of field-strength distributions $p(\mathcal{F}_m,\varphi_n)$ of a single-photon Fock state (a), where 150 000 events at 30 phase values φ_n are assumed to be recorded ($|F|=1$). The relative errors $\epsilon_r(\mathcal{F}_m,\varphi_n)$ of the simulated distributions (b) indicate the inaccuracies compared with their maximum values.

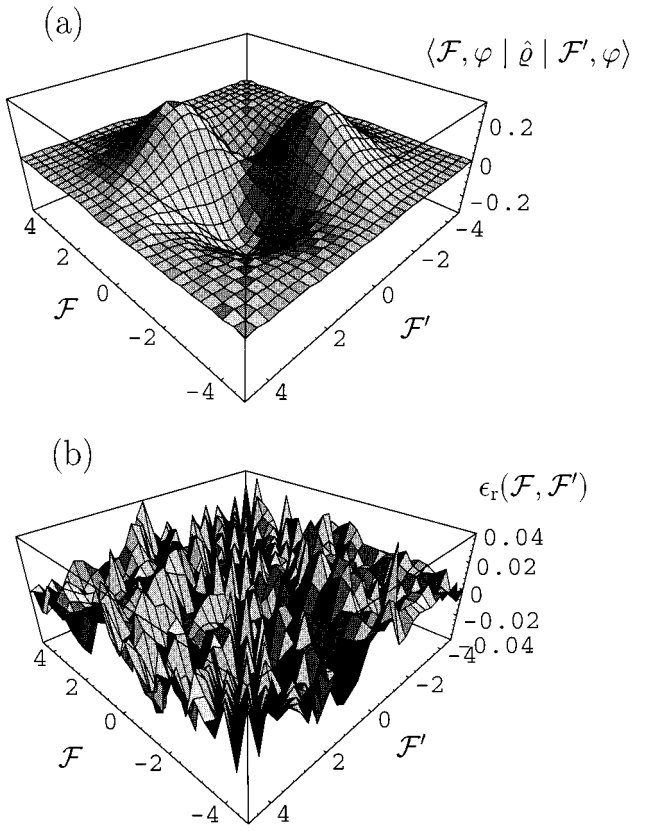

FIG. 2. Density matrix $\langle \mathcal{F},\varphi|\hat{\varrho}|\mathcal{F}',\varphi\rangle$ (a) reconstructed from the data in Fig. 1 and relative errors (b) indicating the inaccuracies compared with the absolute maximum of the density matrix ($|F|=1$). In the case of a Fock state the density matrix does not depend on the phase φ of the field strength defining the basis.

$$K_{mn}(\mathcal{F}_1,\mathcal{F}_2,\varphi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dx e^{-ix\mathcal{F}_1}\widetilde{B}_{\Delta\mathcal{F}_m,m}(y'(x,\mathcal{F}_2)) \times B_{\Delta y_n,n}(x). \tag{18}$$

Here, $\widetilde{B}_{\Delta\mathcal{F}_m,m}(y'(x,\mathcal{F}_2))$ is simply the Fourier transform of the hat function $B_{\Delta\mathcal{F}_m,m}(\mathcal{F})$ and can be obtained from Eq. (11). In particular, for $\mathcal{F}_m = m\Delta\mathcal{F}$ it reduces, according to Eq. (13), to $\widetilde{N}_{\Delta\mathcal{F},m}(y'(x,\mathcal{F}_2))$. The notation $K_{mn}(\mathcal{F}_1,\mathcal{F}_2,\varphi)$ indicates that the sampling function depends on φ, because of the dependence on φ of the y_n [see Eq. (16) together with Eq. (14)]. The Fourier integral in Eq. (18) can be calculated by applying Eq. (5) once more. In this case $\widetilde{B}_{\Delta\mathcal{F}_m,m}(y'(x,\mathcal{F}_2))B_{\Delta y_n,n}(x)$ is approximated by

$$\widetilde{B}_{\Delta\mathcal{F}_m,m}(y'(x,\mathcal{F}_2))B_{\Delta y_n,n}(x) \approx\sum_k \widetilde{B}_{\Delta\mathcal{F}_m,m}(y'(x_{k+1},\mathcal{F}_2))B_{\Delta y_n,n}(x_{k+1})B_{\Delta x_k,k}(x), \tag{19}$$

where the hat function $B_{\Delta x_k,k}(x)$ can be obtained from Eq. (6) or when $x_k=k\Delta x$ from Eq. (7). Using Eq. (19) and calculating the Fourier integral in Eq. (18) yields

$$K_{mn}(\mathcal{F}_1,\mathcal{F}_2,\varphi)\approx\frac{1}{2\pi}\sum_k \widetilde{B}_{\Delta\mathcal{F}_m,m}(y'(x_{k+1},\mathcal{F}_2)) \times B_{\Delta y_n,n}(x_{k+1})\widetilde{B}_{\Delta x_k,k}(-\mathcal{F}_1), \tag{20}$$

where $\widetilde{B}_{\Delta x_k,k}(-\mathcal{F}_1)$ is obtained from Eq. (11) or when $x_k=k\Delta x$ from Eq. (13).

Equations (17) and (20) [together with Eqs. (6) and (11) or (13)] reveal that, based on Eq. (1), the reconstruction of the density matrix in a field-strength representation requires sampling functions that are very elementary. In particular, there is no need for separate and lengthy calculations of sampling functions and there are no problems of (numerical) convergence. Since the reconstruction formula (17) is valid for any phase φ, it can be used to reconstruct the density matrix in various field-strength bases simultaneously. For example, choosing two phase values $\varphi = \varphi_1$ and $\varphi = \varphi_1 + \pi/2$, the density matrix in a ''position'' and the associated ''momentum'' basis can be sampled simultaneously. Comparison of the two representations may enable one to discriminate between structures of the quantum states under study and systematic errors in the reconstruction method.

Finally, the different methods of direct sampling of the density matrix may be advantageously combined, since the usefulness of a chosen method may depend on the observables one is interested in.

To illustrate the method, we present in Figs. 1 and 2 the result of a computer simulation of direct sampling of the density matrix of a single-photon Fock state (for a realization of a single-photon Fock state in parametric decay see [14]). The reconstruction of such a state in a field-strength basis is not trivial, since the operators defining the Fock state and the basis are essentially noncommuting ones. In Fig. 1(a) the set of field-strength distributions used in Eq. (17) is shown. It has been obtained by simulating an OHT experiment in which the number of grid points $\{\mathcal{F}_m, \varphi_n\}$ and the number of sampling events are comparable with those in the experiments reported in Ref. [1]. Note that the field-strength distributions of the single-photon Fock state are phase insensitive and display the typical two-peak structure. In Fig. 1(b) the sampling noise of the field-strength distributions is plotted. The sampled density matrix is shown in Fig. 2(a), and in Fig 2(b) the relative errors are depicted, which reveal that the method yields the density matrix with suitable accuracy compared with the noise in the recorded data. Note that the relative errors in Figs. 1(b) and 2(b) are defined with respect to the maxima of the field-strength distributions and the density matrix in Figs. 1(a) and 2(a), respectively. Since field-strength distributions for different values of the phase parameter represent distributions for noncommuting observables, the information on the maximum values of the field-strength distributions for different values of the phase in Fig. 1(a) is contained in the off-diagonal elements of the density matrix, the absolute values of which are smaller than the maximum values of the diagonal elements. This explains the fact that the values of the relative errors in Fig. 2(b) are smaller than those in Fig. 1(b).

In conclusion we have derived a simple method for direct sampling of the density matrix in field-strength or generalized position (momentum) bases. It relies on approximate, quasianalytical methods of Fourier transforms, which yield very elementary expressions for the desired sampling functions. This opens the possibility of sampling the density matrix of either radiation or matter in different representations, by varying the phase of the observable defining the basis. The method developed here directly applies to the pioneering experiments [1,5].

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