Quantum noise reduction in stationary superradiance

Fritz Haake, Mikhail I. Kolobov, and Carsten Seeger
Fachbereich Physik, Universität-GH Essen, D-45117 Essen, Deutschland

Claude Fabre, Elisabeth Giacobino, and Serge Reynaud
Laboratoire Kastler Brossel, Université Pierre et Marie Curie, 4 place Jussieu, F-75252 Paris, France

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We discuss a laser with N three-level atoms resonantly and cooperatively interacting with two cavity modes. Its intensity is proportional to N^2, the linewidth scales as $1/N^2$, and the output intensity fluctuations display up to 100% squeezing at low frequencies. We show that quantum fluctuations of the pump do not deteriorate the squeezing significantly. We also find that the fluctuation properties of the ''passive'' cavity mode which serves to collectively relax the atoms to the ground state are similar to those of the active mode. Finally we take into account spontaneous emission. Within a certain range of frequencies good squeezing can still be found. [S1050-2947(96)08707-0]

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I. INTRODUCTION

When several identical two-level atoms are brought to their excited state with no external electromagnetic field imposed then or later, the subsequent radiative deexcitation can, certain conditions met, proceed collectively, i.e., with all N atoms acting like one rigid dipole [1,2]. The ensuing superfluorescent radiation pulse has a peak intensity $I_s \sim N^2$ while noncollective radiation would only yield $I_n \sim N$. Since the maximum energy available for radiation is N times the single-atom excitation energy $\hbar\omega$, the temporal width of a superfluorescent pulse must be inversely proportional to N ($\tau_s \sim 1/N\gamma$) while the radiative lifetime $1/\gamma$ of the excited state of a single atom would equal the duration τ_n of a pulse of normal fluorescence. For experimental realizations of superfluorescence [3] one must, roughly speaking, make sure that the characteristic times of all competing processes like inhomogeneous broadening and collisions are much longer than τ_s.

We have described in Ref. [4] a rather different type of superradiance. Like the aforementioned one, it is collectively generated by N atoms and thus has an intensity $\sim N^2$; unlike the former, it can be stationary rather than transient. An even more striking difference arises for the spectral width: while a superfluorescent pulse has a spectral width $\sim \gamma N$, the linewidth of the superradiant laser was found to be extremely small, $\Delta\nu \sim 1/N^2$. Moreover, the intensity fluctuations within an individual superfluorescent pulse are close to those of a coherent state; those of the stationary output of a superradiant laser can be much smaller and in fact can be squeezed nearly perfectly.

The model of the superradiant laser presented in Ref. [4] is the simplest one displaying cooperative behavior. However, in view of possible experiments, many complications have to be taken into account. Undertaking the first such step in the present paper, we propose to consider the noise properties of the passive mode and discuss the influence of pump fluctuations as well as of spontaneous emission on the noise spectra of the superradiant output. A forthcoming publication will be devoted to partially cooperative atomic configurations.

The paper is organized as follows. In Sec. II we present a detailed formulation of the model of the superradiant laser. In Sec. III we show that in the semiclassical limit, $N \gg 1$, the case of full cooperativity can be treated analytically for all parameter values. In Sec. IV we consider the passive mode and show that its low-frequencies noise spectra are nearly identical to those of the active mode. In Sec. V we take into account the quantum fluctuations of the pump and find that they do not deteriorate significantly the noise properties of the superradiant laser. Section VI is devoted to an investigation of the influence of spontaneous emission on the fluctuation spectra.

II. MODEL

As in Ref. [4] we consider the simplest model of a superradiant laser which accounts for N three-level atoms (see Fig. 1) placed inside a resonator. We assume a pump process $0 \rightarrow 2$ consisting of a two-photon excitation such that the resonance condition with the transititon $0 \leftrightarrow 2$ is met. A lasing process is assumed by coupling the atoms to the resonant cavity mode in tune with the transition $2 \leftrightarrow 1$. In the following we shall refer to this mode as the ''active'' mode. Finally, a certain relaxation process $1 \rightarrow 0$ has to be included to recycle the atoms back to the influence of the pump. Usually, spontaneous emission between levels 1 and 0 is such a re-

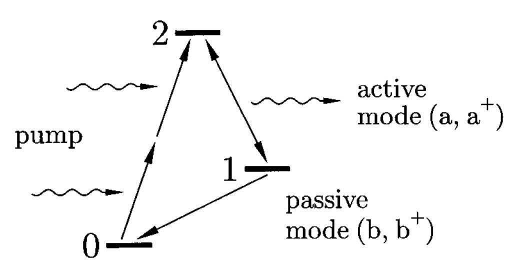

FIG. 1. Scheme of three-level superradiant laser.

laxation process. In this case one has the so-called Raman laser whose quantum noise properties were recently investigated by Ritsch, Marte, and Zoller in Ref. [5].

We here require a different type of relaxation by coupling the atoms to another cavity mode resonant with the transition $1\leftrightarrow 0$. However, with respect to the latter mode the resonator does not need to have a high finesse. In fact, in order to simplify the model we assume that this mode is damped sufficiently strongly so that it can be eliminated adiabatically. That is why we refer to this mode as to the ''passive'' mode. The situation in consideration would, in the absence of level 2 and with level 1 populated initially, entail the no-ringing limit of superfluorescence on the transition $1\rightarrow 0$, as observed in Ref. [3]. Our scheme with a third level and a stationary coherent pump can, as will be shown below, yield stationary superradiance on both transitions, $2\leftrightarrow 1$ and $1\leftrightarrow 0$.

We now turn to a more detailed specification of the outlined scheme. In order to eliminate the explicit time dependence of the pump wave we write the Hamiltonian in the interaction picture,

$$\begin{aligned} H_0 = {} & i\hbar g_{12}(aS_{21}-a^\dagger S_{12}) + i\hbar\Omega(S_{20}-S_{02}) \\ & + i\hbar g_{01}(bS_{10}-b^\dagger S_{01}). \end{aligned} \tag{2.1}$$

It displays the collectivity of the pump mechanism and of the interaction of the atoms with the active and passive modes by the appearance of the collective atomic operators $S_{ij}=\Sigma_{\mu=1}^{N}S_{ij}^{\mu}=\Sigma_\mu(|i\rangle\langle j|)^\mu$. There are nine operators S_{ij}; those with $i=j$ refer to polarizations while each ''diagonal'' one, $S_{ii}\equiv P_i$, measures the global occupation of level i; they obey $S_{ij}^\dagger=S_{ji}$ and $[S_{ij},S_{kl}]=\delta_{jk}S_{il}-\delta_{il}S_{kj}$. The operators $a,a^\dagger$, and $b,b^\dagger$ are the operators of annihilation and creation of photons in the lasing and passive cavity modes. The coupling constants g_{ij} and the pump parameter Ω are specified in the Hamiltonian (2.1) so as to have the dimension of a frequency. We assume a two-photon pump process in order to secure a ring of effectively allowed transitions $0\leftrightarrow 2\leftrightarrow 1\leftrightarrow 0$. The pump strength Ω is proportional to the product of the amplitudes of the two classical pump waves and is chosen real. Note that by taking Ω as a fixed c number we forbid pump fluctuations. This assumption will be lifted in Sec. V.

It may be worth pointing out that the use of coupling constants g_{12} and g_{01} identical for all atoms is justified in two limiting situations. One is typical of microwave experiments where the atomic system has a spatial extension small compared to the wavelengths involved. In the optical range of frequencies one may employ a running-wave resonator such that position dependent phase factors like $\exp[i\vec{k}_{12}\vec{x}_\mu]$ may be absorbed in the definition of the collective polarizations, $S_{12}=\Sigma_\mu\exp[i\vec{k}_{12}\vec{x}_\mu]$, etc., where $\vec{k}_{12}$ is the wave vector of the mode and $\vec{x}_\mu$ the position of the μth atom.

We finally account for two damping mechanisms due to the irreversible leakage of photons from the lasing and passive modes through nonideal mirrors. This process can be described by the following additions to the time rate of change of the Heisenberg operators $a(t)$ and $b(t)$ [6–8]:

$$(\partial a(t)/\partial t)_{\text{irr}} = -\kappa_a a(t) + \sqrt{2\kappa_a}\,\eta_a(t),$$

$$(\partial b(t)/\partial t)_{\text{irr}} = -\kappa_b b(t) + \sqrt{2\kappa_b}\,\eta_b(t). \tag{2.2}$$

Here κ_a and κ_b are the damping constants for the two modes. The quantum Langevin forces $\eta_\alpha(t)$ with $\alpha=a,b$ ensure the preservation of the Bose commutators $[a(t),a^\dagger(t)]=[b(t),b^\dagger(t)]=1$ at all times; we can take the fluctuating forces as independent with Gaussian statistics and white spectra according to

$$\begin{aligned} &[\eta_\alpha(t),\eta_\beta^\dagger(t')] = \langle \eta_\alpha(t)\eta_\beta^\dagger(t')\rangle = \delta_{\alpha\beta}\delta(t-t'), \\ &\langle \eta_\alpha(t)\rangle = \langle \eta_\alpha^\dagger(t)\eta_\beta^\dagger(t')\rangle = \langle \eta_\alpha(t)\eta_\beta(t')\rangle = 0. \end{aligned} \tag{2.3}$$

We may interpret the operators $\eta_\alpha(t)$ and $\eta_\alpha^\dagger(t)$ as representatives of the vacuum fluctuations of the electromagnetic field outside the resonators.

The Heisenberg equations of motion for the collective atomic operators $S_{ij}(t)$ are readily obtained from the Hamiltonian (2.1) as $\dot{S}_{ij}=(i/\hbar)[H_0,S_{ij}]$. The Heisenberg-Langevin equations for the photon annihilation operators of the lasing and passive modes, obtained as $\dot{a}=(i/\hbar)[H_0,a]+(\partial a/\partial t)_{\text{irr}}$ and analogously for $b(t)$, read

$$\begin{aligned} \dot{a}(t) &= -g_{12}S_{12} - \kappa_a a(t) + \sqrt{2\kappa_a}\,\eta_a(t), \\ \dot{b}(t) &= -g_{01}S_{01} - \kappa_b b(t) + \sqrt{2\kappa_b}\,\eta_b(t). \end{aligned} \tag{2.4}$$

Now, as mentioned above, we eliminate adiabatically the field variable $b(t)$ of the passive mode supposing that its damping constant κ_b is the dominant relaxation constant of the system. This gives

$$b(t) = -\frac{g_{01}}{\kappa_b}S_{01}(t) + \left(\frac{2}{\kappa_b}\right)^{1/2}\eta_b(t). \tag{2.5}$$

Substituting this expression for $b(t)$ into the atomic Heisenberg equations, we arrive at the following set of Heisenberg-Langevin equations for N three-level atoms with collective relaxation:

$$\begin{aligned} \dot{S}_{02} &= g_{12}aS_{01} - \Omega(S_{22}-S_{00}) + \gamma S_{12}S_{01} - \sqrt{2\gamma}S_{12}\eta_b, \\ \dot{S}_{12} &= -g_{12}a(S_{22}-S_{11}) + \Omega S_{10} - \gamma S_{10}S_{02} + \sqrt{2\gamma}\eta_b^\dagger S_{02}, \\ \dot{S}_{01} &= -g_{12}a^\dagger S_{02} - \Omega S_{21} + \gamma(S_{11}-S_{00})S_{01} \\ &\quad - \sqrt{2\gamma}(S_{11}-S_{00})\eta_b, \\ \dot{S}_{00} &= -\Omega(S_{02}+S_{20}) + 2\gamma S_{10}S_{01} - \sqrt{2\gamma}(S_{10}\eta_b + \eta_b^\dagger S_{01}), \\ \dot{S}_{11} &= -g_{12}(aS_{21}+a^\dagger S_{12}) - 2\gamma S_{10}S_{01} + \sqrt{2\gamma}(S_{10}\eta_b + \eta_b^\dagger S_{01}), \\ \dot{S}_{22} &= \Omega(S_{02}+S_{20}) + g_{12}(aS_{21}+a^\dagger S_{12}), \\ \dot{a}(t) &= -g_{12}S_{12} - \kappa_a a(t) + \sqrt{2\kappa_a}\,\eta_a(t). \end{aligned} \tag{2.6}$$

Here $\gamma=g_{01}^2/\kappa_b$ is a rate constant related to the collective atomic relaxation $1\rightarrow 0$. Two features of Eq. (2.6) are worth a comment, the nonlinearity of the damping and the ''multiplicative'' form of the noise. Both of these features have as their common origin the nonlinearity of the interaction of the atoms with the passive mode. It is easy to see from Eq. (2.6) that in the case of $N=1$, when the collective atomic operators become the operators of a single atom, we recover the familiar linear relaxation, through $S_{12}^{\mu}S_{01}^{\mu}=0$, $S_{10}^{\mu}S_{02}^{\mu}=S_{12}^{\mu}$, etc. [9]. We should also note that the atomic observables S_{ij} preserve their commutation relations, the presence of damping and noise notwithstanding.

The set of equations (2.6) possesses three constants of motion C_1, C_2, C_3. The first of these, the number of atoms $C_1=N$, arises since our model clearly is not wasteful of atoms. The remaining two owe their existence to the absence of any noncollective atomic process. In fact, C_2 and C_3 are the Casimir operators of the group U(3) which is a symmetry group of the cooperative dynamics of our three-level atoms [10]. Expressed in terms of the polarization and population operators S_{ij} the C_i read

$$C_1 \equiv \sum_i S_{ii}, \quad C_2 \equiv \sum_{i,j} S_{ij}S_{ji}, \quad C_3 \equiv \sum_{i,j,k} S_{ij}S_{jk}S_{ki}. \tag{2.7}$$

A third symmetry is worth mentioning. Shifting phases by the amount Δ as

$$a \to e^{i\Delta}a, \quad S_{12} \to e^{i\Delta}S_{12}, \quad S_{10} \to e^{i\Delta}S_{10}, \quad \eta_a \to e^{i\Delta}\eta_a, \quad \eta_b \to e^{-i\Delta}\eta_b, \tag{2.8}$$

leaves the system (2.6) invariant. The relative phase Δ is thus conserved as well. Therefore, the system under consideration has 11 degrees of freedom and at least four constants of motion.

Clearly, the laser model just presented is a bit of an oversimplification. We neglect for the moment such effects as pump fluctuations, spontaneous emission, inhomogenoues broadening and detunings between the fields and atomic transitions. By leaving such refinements aside we hope to exhibit most clearly but without inappropriate exaggeration the potential of cooperativity for noise reduction. We will include the quantum fluctuations of the pump into the model in Sec. V of this paper. The role of spontaneous emission which breaks the U(3) symmetry will be discussed in Sec. VI.

In the following we confine ourselves to the semiclassical limit, $N \gg 1$. Each of the eleven variables $S_{ij}, a, a^{\dagger}$ can then be represented as a sum of a dominant classical term $\overline{X} \propto N$ and a ''small'' operator valued fluctuation δX,

$$X = \overline{X} + \delta X. \tag{2.9}$$

Of course, the proportionality of the means $\overline{X}$ to N is a manifestation of the assumed collectivity. To find the $\overline{X}$ in the stationary regime we drop $\dot{X}$ and the noise forces in the Heisenberg-Langevin equations (2.6) and degrade each operator X to a c number $\overline{X}$. The dynamics of the δX's will be taken into account by the linearized Heisenberg-Langevin equations.

Since the atomic mean values are proportional to N we easily infer from (2.6) that the atoms have a relaxation rate of the order γN. We therefore may express the assumed dominance of the relaxation constant of the passive mode by

$$\kappa_b \gg \kappa_a, \quad \gamma N. \tag{2.10}$$

The solutions of the classical equations of motion depend on the expectation values of the three constants of motion C_1, C_2, C_3 which are determined by the initial atomic state. To within corrections of relative order $1/N$ we may factorize the expectation values of operator products as, e.g., $\langle S_{ij}(t)S_{kl}(t)\rangle = \overline{S_{ij}(t)}\,\overline{S_{kl}(t)}$, and obtain

$$\langle C_1\rangle = \sum_i \overline{S_{ii}} = N, \quad \langle C_2\rangle = \sum_{i,j} \overline{S_{ij}}\,\overline{S_{ji}} = c_2 N^2,$$
$$\langle C_3\rangle = \sum_{i,j,k} \overline{S_{ij}}\,\overline{S_{jk}}\,\overline{S_{ki}} = c_3 N^3. \tag{2.11}$$

We shall refer to c_2 and c_3 as cooperativity parameters.

With the help of the conservation of the number of atoms we can rewrite the cooperativity parameters as

$$c_2 = 1 - \frac{2}{N^2}[(\overline{S_{00}}\,\overline{S_{11}} - \overline{S_{01}}\,\overline{S_{10}}) + (\overline{S_{00}}\,\overline{S_{22}} - \overline{S_{02}}\,\overline{S_{20}}) + (\overline{S_{22}}\,\overline{S_{11}} - \overline{S_{21}}\,\overline{S_{12}})],$$

$$c_3 = \frac{3c_2 - 1}{2} + \frac{3}{N^3}(\overline{S_{00}}\,\overline{S_{11}}\,\overline{S_{22}} - \overline{S_{00}}\,\overline{S_{12}}\,\overline{S_{21}} - \overline{S_{11}}\,\overline{S_{20}}\,\overline{S_{02}} - \overline{S_{22}}\,\overline{S_{10}}\,\overline{S_{01}} + \overline{S_{10}}\,\overline{S_{02}}\,\overline{S_{21}} + \overline{S_{01}}\,\overline{S_{12}}\,\overline{S_{20}}). \tag{2.12}$$

To be physically acceptable, the solutions of the classical equations of motion have to obey two requirements of quantum mechanical origin. These requirements are (α) $0 \leqslant \overline{S_{ii}}/N \leqslant 1$ (since $\overline{S_{ii}}/N$ have the physical meaning of probabilities), and (β) three Schwartz inequalities, $\overline{S_{ii}}\,\overline{S_{jj}} - \overline{S_{ij}}\,\overline{S_{ji}} \geqslant 0$, which must hold for all pairs of atomic levels. Inserting the Schwartz inequalities into the expression of c_2 we now see that c_2 is restricted from above by $c_2 \leqslant 1$ with $c_2 = 1$ for $\overline{S_{ii}}\,\overline{S_{jj}} - \overline{S_{ij}}\,\overline{S_{ji}} = 0$. A simple analysis shows that the minimum value of c_2 is given by $c_2 = 1/3$, realized by $\overline{S_{ii}} = N/3$, $\overline{S_{ij}} = 0$ $(i \neq j)$, i.e., when all atomic levels are equally populated and all polarizations vanish.

The condition for the maximum value of c_3, $c_3 = 1$, is also found easily: as for $c_2 = 1$ we must require $|\overline{S_{ij}}|^2 = \overline{S_{ii}}\,\overline{S_{jj}}$ and, in addition, $\arg(\overline{S_{01}}\,\overline{S_{02}}\,\overline{S_{21}}) = 0$. We refer to the case of $c_2 = c_3 = 1$ as full cooperativity. Quite interestingly, the easily realizable initial condition of all atoms in the ground state satisfies the condition of full cooperativity. The minimum conditions of c_2 are sufficient for the minimum of c_3, yielding $c_3 = 1/9$. In summary we have

$$\frac{1}{3} \leqslant c_2 \leqslant 1, \quad \frac{1}{9} \leqslant c_3 \leqslant 1. \tag{2.13}$$

The question as to whether all pairs of (c_2, c_3) values are realizable and the stationary solutions for arbitrary admissible (c_2, c_3) values will be discussed in a following publication. In this paper we concentrate on the fluctuation properties of the superradiant laser. In order to keep the calculations as simple as possible we here only consider the case of full cooperativity $c_2 = c_3 = 1$.

III. FULL COOPERATIVITY

The case of full cooperativity is of particular interest since it is, as was already mentioned, easily realizable by preparing all atoms initially in the ground state. At all subsequent times the atoms can then be described by the fully symmetric states $|n_0,n_1,n_2\rangle$ where n_i is the number of atoms in level i and $n_0+n_1+n_2=N$. In the spirit of second quantization we may introduce six atomic annihilation and creation operators $z_i, z_i^\dagger$, $(i=0,1,2)$ which lower and raise the occupation numbers n_i as $z_0|n_0,n_1,n_2\rangle=\sqrt{n_0}|n_0-1,n_1,n_2\rangle$ and $z_0^\dagger|n_0,n_1,n_2\rangle=\sqrt{n_0+1}|n_0+1,n_1,n_2\rangle$, etc. This behavior is consistent with the Bose commutation rules $[z_i,z_j^\dagger]=\delta_{ij}$. Moreover, we are led to Schwinger's representation of the Lie algebra [11] of the group U(3) by expressing our nine atomic polarization and population operators as $S_{ij}(t)=z_i^\dagger(t)z_j(t)$. Indeed, the commutation rules $[S_{ij},S_{kl}]=\delta_{jk}S_{il}-\delta_{il}S_{kj}$ are recovered and so is the action of the collective atomic operators on the symmetric basis states, $S_{01}|n_0,n_1,n_2\rangle=\sqrt{(n_0+1)n_1}|n_0+1,n_1-1,n_2\rangle$, etc. [12].

In the subspace of fully cooperative atomic states under consideration the Casimir operators C_2, C_3 take on the values $C_2=N(N+2)$, $C_3=N^2(N+3)$ with $\Sigma_{i=0}^2 z_i^\dagger z_i=N$. As a consequence, the semiclassical cooperativity parameters become maximal, $c_2=c_3=1$.

The Heisenberg equations of motion for the operators $z_i(t)$ are obtained as $\dot{z}_k=(i/\hbar)[H_0,z_k]$. From a classical point of view, after adiabatic elimination of the passive mode we have eight real equations (six for atomic variables and two for the field). However, only six of them are independent. Indeed, the number of atoms is fixed and one of the phases may be eliminated since only $\overline{z_i^* z_j}$ are of interest. This counting is in agreement with the results from the previous section. Indeed, for the case of full cooperativity we have three Schwartz equalities, $S_{ij}(t)S_{ji}(t)-S_{ii}(t)S_{jj}(t)=0$, one condition on the phases, $\arg(S_{10}S_{02}S_{21})=0$, and conservation of the number of atoms. Thus, there are a total of five constraints on 11 real equations (nine atomic and two field variables), which amounts to six independent variables.

The stationary solution of the corresponding classical equations of motion with nonvanishing mean photon numbers can be expressed in terms of an effective pump strength p and a dimensionless coupling strength c,

$$p=\frac{\Omega}{N\gamma\sqrt{c}},\qquad c=\frac{g_{12}^2\kappa_b}{g_{01}^2\kappa_a}=\frac{g_{12}^2}{\gamma\kappa_a}. \tag{3.1}$$

In the following we shall assume a pump parameter $\Omega\sim N$ so that p and c are of zero order in N. For the coupling strength c an interesting interpretation arises from the following reasoning: in a stationary regime the leakage rates of photons from the active and passive cavities must coincide, $\kappa_a\overline{n_a}=\kappa_b\overline{n_b}$. This means that

$$\sqrt{c}=\frac{g_{12}\sqrt{\overline{n_a}}}{g_{01}\sqrt{\overline{n_b}}} \tag{3.2}$$

is the ratio of the Rabi frequencies on the transitions coupled to the active and passive modes.

Due to the phase symmetry (2.8) we may choose the stationary value of the field amplitude $\overline{a}$ as real. With this choice we obtain the following stationary mean values:

$$\overline{z}_0=\left(\frac{Nc(1-p)}{1+c}\right)^{1/2},\quad \overline{z_1}=\sqrt{Np},\quad \overline{z_2}=\left(\frac{N(1-p)}{1+c}\right)^{1/2},$$

$$\overline{a}=-N\frac{g_{12}}{\kappa_a}\left(\frac{p(1-p)}{1+c}\right)^{1/2}. \tag{3.3}$$

As follows from Eqs. (3.3), the pump strength p now is allowed to range within $0\leqslant p\leqslant 1$. The absence of a threshold for the pump strength p in the laser amplitude $\overline{a}$ can be seen as due to our neglect of spontaneous emission. A less expected feature of the stationary amplitude $\overline{a}$ is the appearance of an upper limit for the pump strength, $p=1$, and of an optimal pumping, $p=1/2$, at which $\overline{a}$ is maximal. As we shall see from the stability analysis below, not all values of p between 0 and 1 are physically acceptable because the stationary solution can become unstable.

As a final remark on the mode amplitude $\overline{a}$ in Eqs. (3.3) we would like to once more underscore the proportionality $\overline{a}\propto N$ which manifests the superradiant character of the laser in discussion. However, this point deserves more explanation. If we imagine a series of realizations of our laser for various values of N, keeping all other parameters, i.e., $\gamma,\kappa_a,g_{12},\Omega$ fixed, we would go to ever weaker pumping strength p as N increases. Since $p\propto 1/N$ one would even be led to conclude $\overline{a}$ to increase only in proportion to $\sqrt{N}$. But if one insists on keeping p fixed, for instance in sticking to the optimal pumping $p=1/2$, for the whole series of imagined lasers, one gets $\overline{a}\propto N$ at the expense of increasing Ω in proportion to N. When we talk about superradiance here we think of all variables and parameters referred to their ''natural'' N dependent units that arise in a scale invariant theory.

Turning now to the analysis of stability and fluctuations of the stationary solution (3.3), we linearize the equations of motion (2.6) around the classical mean values with respect to the fluctuations δz_i and δa as $z_i=\overline{z_i}+\delta z_i$, $a=\overline{a}+\delta a$. This brings about a significant simplification of the noise: through $S_{ij}(t)\eta_b(t)\to\overline{S_{ij}}\eta_b(t)$ the atomic noise forces in Eqs. (2.6) are freed of their so-called multiplicative character, i.e., become simple inhomogeneous terms. We split the operator valued fluctuations $\delta z_i,\delta a$ and the Langevin forces η_α into Hermitian ''real'' and ''imaginary'' parts as

$$\delta z_i=\delta u_i+i\,\delta v_i,\quad \delta a=-\frac{\gamma N}{g_{12}}(\delta u+i\,\delta v),\quad \eta_\alpha=\Sigma_\alpha+i\Delta_\alpha. \tag{3.4}$$

The inhomogeneous linearized equations of motion separate into two independent blocks

$$\begin{pmatrix} \delta\dot{u}_0 \\ \delta\dot{u}_1 \\ \delta\dot{u}_2 \\ \delta\dot{u} \end{pmatrix} = -\gamma N \begin{pmatrix} -p & -2\overline{z_1 z_0} & p\sqrt{c} & 0 \\ 2\overline{z_1 z_0} & |\overline{z_0}|^2 & g_{12}\overline{a}/\gamma N & -\overline{z_2} \\ -p\sqrt{c} & -g_{12}\overline{a}/\gamma N & 0 & \overline{z_1} \\ 0 & -c\overline{z_2}/\xi & -c\overline{z_1}/\xi & 1/\xi \end{pmatrix} \times \begin{pmatrix} \delta u_0 \\ \delta u_1 \\ \delta u_2 \\ \delta u \end{pmatrix} + \sqrt{2\gamma} \begin{pmatrix} -\overline{z_1}\Sigma_b \\ \overline{z_0}\Sigma_b \\ 0 \\ -\sqrt{c}\Sigma_a/\xi \end{pmatrix}, \tag{3.5}$$

$$\begin{pmatrix} \delta\dot{v}_0 \\ \delta\dot{v}_1 \\ \delta\dot{v}_2 \\ \delta\dot{v} \end{pmatrix} = -\gamma N \begin{pmatrix} -p & 0 & p\sqrt{c} & 0 \\ 0 & |\overline{z_0}|^2 & g_{12}\overline{a}/\gamma N & \overline{z_2} \\ -p\sqrt{c} & -g_{12}\overline{a}/\gamma N & 0 & \overline{z_1} \\ 0 & c\overline{z_2}/\xi & -c\overline{z_1}/\xi & 1/\xi \end{pmatrix} \times \begin{pmatrix} \delta v_0 \\ \delta v_1 \\ \delta v_2 \\ \delta v \end{pmatrix} + \sqrt{2\gamma} \begin{pmatrix} \overline{z_1}\Delta_b \\ \overline{z_0}\Delta_b \\ 0 \\ -\sqrt{c}\Delta_a/\xi \end{pmatrix}. \tag{3.6}$$

We refer to the block of the real parts $\delta u_i, \delta u$ as to the amplitude block since these fluctuations are related to the intensity fluctuations via $\delta I = 2\overline{a}\,\delta u$, etc. Conversely, the four imaginary parts $\delta v_i, \delta v$ may be interpreted as phase fluctuations through $\delta v = \overline{a}\,\delta\phi$, etc.

Beyond the coupling strength c and the pump strength p we now meet one additional dimensionless parameter ξ, the ratio between the atomic and the field decay rates

$$\xi = \frac{\gamma N}{\kappa_a}. \tag{3.7}$$

We look for dimensionless eigenvalues λ as $\delta z_i = \delta z_i(0)\exp[-\lambda\gamma N t]$, $\delta a = \delta a(0)\exp[-\lambda\gamma N t]$ and find two characteristic polynomials each of order four

$$P_u(\lambda) = \lambda\left[\lambda^3 - \lambda^2\left(\frac{p(1+2c)-c}{1+c} - \frac{1}{\xi}\right) + \lambda\left(\frac{pc(3+c-2p)}{1+c} - \frac{p}{\xi}(1-c)\right) + \frac{4pc(1-p)}{\xi}\right],$$

$$P_v(\lambda) = \lambda^2\left[\lambda^2 - \lambda\left(\frac{p(1+2c)-c}{1+c} - \frac{1}{\xi}\right) + \left(\frac{pc(c-1+2p)}{1+c} - \frac{p}{\xi}(1-c)\right)\right]. \tag{3.8}$$

The polynomial $P_u(\lambda)$ comes from the amplitude block and $P_v(\lambda)$ from the phase block. The factor λ in $P_u(\lambda)$ is due to the conservation of the norm $\Sigma_{i=0}^2 |z_i|^2 = 1$. The origin of the factor λ^2 of $P_v(\lambda)$ is the symmetry (2.8) and the invariance of the equations of motion under $z_i \to e^{i\phi} z_i$ for arbitrary real ϕ. The roots of these polynomials give the eigenvalues. Stability requires that all eigenvalues have positive real parts.

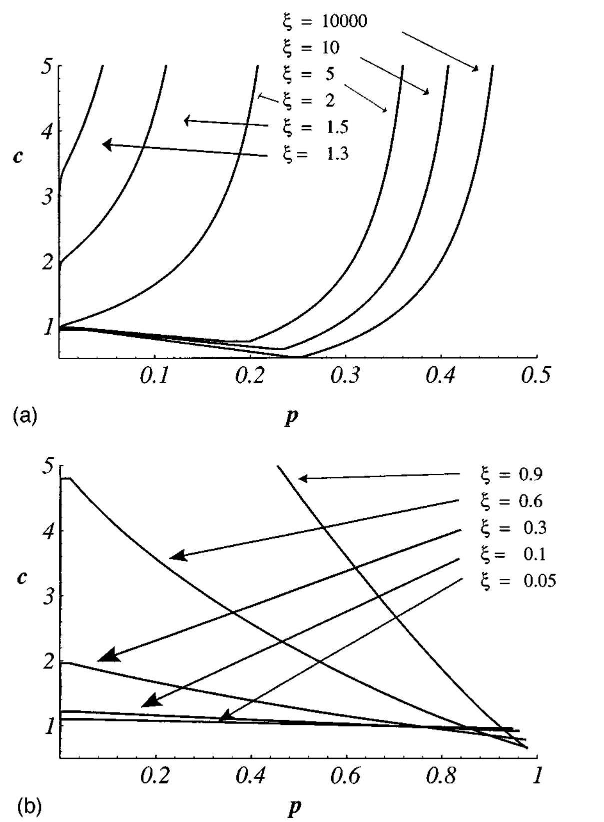

FIG. 2. The stability domain in the p,c plane for $\xi \geqslant 1$ (a) and $\xi \leqslant 1$ (b). For each ξ the solution is stable in the region above the corresponding curve.

To obtain the stability conditions we apply the Hurwitz criterion to both polynomials. The resulting stability conditions read

$$p < \frac{c+1/\xi}{1+2c},$$

$$\frac{1-c+(1-c^2)/(\xi c)}{2} < p < \frac{3+c-(1-c^2)/(\xi c)}{2}, \tag{3.9}$$

$$0 < -\frac{1-c}{\xi^2} + \frac{p(1+2c)-2c}{\xi} - c\,\frac{[p(1+2c)-c](3-2p+c)}{(1+c)^2}.$$

In the good-cavity limit, $\xi \gg 1$, these are the conditions already given in Ref. [4]. In Fig. 2 we show the stability regions in the p,c plane for various ξ values. Starting with $\xi \gg 1$ the domain of stability in the p,c plane shrinks for decreasing ξ and vanishes for $\xi = 1$. Another domain of stability appears for $\xi < 1$. In the bad-cavity case, $\xi \ll 1$, the domain of stability is determined by $c > 1$. In this case, two of the eigenvalues of the amplitude block turn out to be $\lambda = p(1-c)/2 \pm \sqrt{p^2(1-c)^2/4 - 4pc(1-p)}$. They become complex for $p \ll 1$. The one with the positive imaginary part is responsible for the appearance of a peak at the frequency $\mathrm{Im}\lambda$ with the width $\mathrm{Re}\lambda$ in the amplitude fluctuation spectrum in the bad-cavity case. For $p \ll 1$ the frequency is approximately given by $\mathrm{Im}\lambda \approx 2\xi\sqrt{pc}\kappa_a \ll \kappa_a$ and the width $\mathrm{Re}\lambda = \xi p(c-1)\kappa_a \ll \kappa_a$. The physical origin of this peak is in the phenomenon of relaxation oscillations well known for usual (nonsuperradiant) bad-cavity lasers [13].

To solve the linearized Heisenberg-Langevin equations we perform a Fourier transform of the fluctuations as

$$\delta X(\omega) = \int_{-\infty}^{+\infty} dt \quad e^{i\omega t}\delta X(t). \tag{3.10}$$

The linearized differential equations of motion now become linear algebraic equations which we solve with the help of Cramer's rule.

Inasmuch as we are interested in the noise properties of the field transmitted outside the cavity through the outcoupling mirror, we should consider the annihilation operator $a_{\mathrm{out}}(t)$ of photons outside the cavity. That operator is related to its intracavity partner $a(t)$ by the input-output transformation [7]

$$a_{\mathrm{out}}(t) = \sqrt{2\kappa_a}a(t) - \eta_a(t). \tag{3.11}$$

It is worth pointing out that the operators $a_{\mathrm{out}}(t)$ and $a(t)$ have different normalizations and dimensions. The mean value $\langle a^\dagger(t)a(t)\rangle$ gives a mean photon number inside the cavity while $\langle a^\dagger_{\mathrm{out}}(t)a_{\mathrm{out}}(t)\rangle$ represents a photon flux (in photons/sec) outside the cavity. We use such a normalization of the extracavity field $a_{\mathrm{out}}(t)$ since it entails a more natural expression for the standard quantum-noise limit which does not include the round-trip time of the cavity (see below). The explicit form of the output fluctuations of the active mode is written out in the Appendix.

Particularly simple results arise in the good-cavity limit, $\xi \gg 1$. The amplitude fluctuation spectrum then reads

$$\langle \delta u_{\mathrm{out}}(\omega)\delta u_{\mathrm{out}}(\omega')\rangle = (1/4)\delta(\omega+\omega')\left\{1 - \frac{S_0(p,c)}{1+\omega^2\tau_a^2}\right\}, \tag{3.12}$$

with the squeezing strength $S_0(p,c)$

$$S_0(p,c) = \frac{1}{2} + \frac{2c}{(1+c)^2} - \frac{p^2}{2(1-p)^2}, \tag{3.13}$$

and the width

$$\frac{1}{\tau_a} = 4\kappa_a \frac{(1-p)(1+c)}{3+c-2p}. \tag{3.14}$$

Positive values of the squeezing strength indicate noise reduction below the vacuum level. Ideal squeezing is incurred at $p \to 0$, $c=1$ (see Fig. 3), i.e., when the pump is weak and when the Rabi frequencies associated with the transitions $2 \leftrightarrow 1$ and $1 \leftrightarrow 0$ coincide. Since zero pumping is admittedly not a particularly interesting working point of a laser it is important to infer from Fig. 3 that the squeezing strength S_0 has a rather flat maximum at that point so that good squeezing prevails for weak pumping.

Another quantity of interest is the low-frequency asymptotic version of the fluctuation spectrum of the phase quadrature since it gives the linewidth $\Delta\nu_a$ of the laser output through $\langle \delta v_{\mathrm{out}}(\omega)\delta v_{\mathrm{out}}(\omega')\rangle \to \delta(\omega+\omega')(\overline{a})^2\Delta\nu_a/\omega^2$ for $\omega \to 0$. The low-frequency divergence of $\langle \delta v_{\mathrm{out}}(\omega)\delta v_{\mathrm{out}}(\omega')\rangle$ as $\sim 1/\omega^2$ is characteristic of phase diffusion (see also [14]). In the limit $\xi \gg 1$ we obtain the linewidth

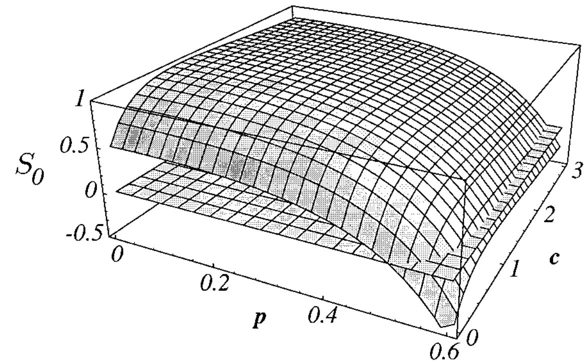

FIG. 3. The squeezing function $S_0(p,c)$ of the superradiant laser without pump fluctuations; $S_0=1$ corresponds to optimal squeezing, $S_0=0$ to a coherent state.

$$\Delta\nu_a = \frac{\kappa_a}{(\overline{a})^2}\frac{p^2(1+c)^2+(1-p)^2(1-c)^2}{c-1+2p}. \tag{3.15}$$

It is interesting to compare the linewidth (3.15) of the superradiant laser with the Schawlow-Townes linewidth $\Delta\nu_{ST}$ for an ordinary incoherently pumped laser. The Schawlow-Townes linewidth in our notations would yield $\Delta\nu_{ST} = \kappa_a/(\overline{a})^2$. The linewidth of a superradiant laser thus differs from the Schawlow-Townes result only by a dimensionless factor. For some special cases, $c=1$, or $p=0$ and $p=1$, we have $\Delta\nu_a = \Delta\nu_{ST}$. For $p=1/2$ which provides the maximum value of $\overline{a}$, $\Delta\nu_a = \Delta\nu_{ST}(1+c^2)/(2c^2)$; for large coupling strength $c \gg 1$ this gives the linewidth $\Delta\nu_a$ one half the Schawlow-Townes result.

It is worth underscoring once more that due to the superradiant character of the laser field, $\overline{a} \sim N$, the linewidth $\Delta\nu$ scales as $1/N^2$. Thus not only the amplitude fluctuations exhibit an interesting potential of noise reduction; the linewidth does as well.

Using the general expressions for the output fluctuations given in the Appendix, we have numerically evaluated the fluctuation spectrum for an arbitrary quadrature component $x_{\mathrm{out}}(\theta) = a_{\mathrm{out}}e^{-i\theta} + a^\dagger_{\mathrm{out}}e^{i\theta}$ of the field outside the cavity:

$$\langle \delta x_{\mathrm{out}}(\omega)\delta x_{\mathrm{out}}(\omega')\rangle = (1/4)\delta(\omega+\omega')I(\theta,\omega). \tag{3.16}$$

Minimizing the spectrum $I(\theta,\omega)$ with respect to θ for each ω we have obtained the optimum-squeezing spectrum $I_-(\omega)$. The fluctuations in the corresponding conjugate quadrature are described by the maximum-stretching spectrum $I_+(\omega)$. These spectra are displayed in Fig. 4 together with their product $I_-(\omega)I_+(\omega)$ which approaches the quantum limit unity for large frequencies as well as for certain finite values of ω. Figure 4(a) refers to the good-cavity limit $\xi \gg 1$ for which simple analytic results arise, while Fig. 4(b) was obtained for the bad-cavity limit $\xi \ll 1$.

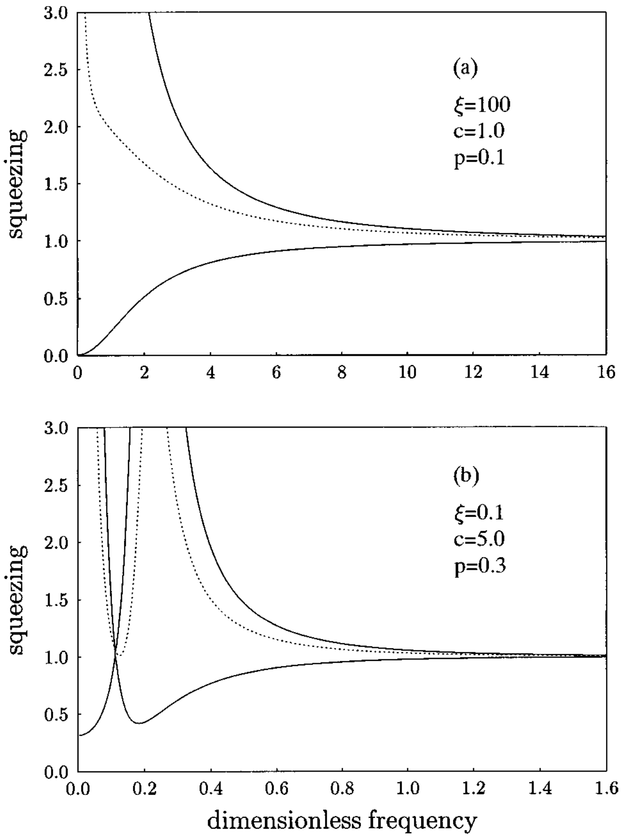

FIG. 4. Optimum squeezing (lower solid curve) and stretching (upper solid) spectra $I_-(\omega)$ and $I_+(\omega)$ and their product (dots) for the superradiant laser: (a) good-cavity case, (b) bad-cavity case; the dimensionless frequency is defined as ω/κ_a.

As it turns out, the optimum-squeezing spectrum at zero frequency does not depend on ξ and is determined solely by p and c as long as the stationary solution is stable. It follows from Fig. 2, that for parameters close to the point of optimal squeezing, $p\ll 1$ and $c\approx 1$, the stationary solution is stable for both $\xi\gg 1$ and $\xi\ll 1$. As mentioned above, in the bad-cavity case there appears a peak in the squeezing spectrum at the frequency $2\xi\sqrt{pc}$. With ξ decreasing this peak shifts towards smaller frequencies that will make the low-frequency squeezing in this regime hardly observable.

We would like to conclude with a simple assessment of the limits of applicability of our semiclassical approximation. Clearly, our separation of observables X into a classical term $\overline{X}$ and a small quantum fluctuation δX [see Eq. (2.9)] is valid only when $\langle(\delta X)^2\rangle\ll\overline{X}^2$. We will check on this condition for the example of the amplitude quadrature component u of the field inside the resonator.

Since we have availed ourselves of the output fluctuations $\langle\delta u_{\rm out}(\omega)\delta u_{\rm out}(\omega')\rangle=(1/4)\delta(\omega+\omega')\langle(\delta u_{\rm out})^2\rangle_\omega$ it is convenient to obtain the variance $\langle(\delta u)^2\rangle$ by reading the input-output relation (3.11) backwards and integrating over ω,

$$\langle(\delta u)^2\rangle=1+\frac{1}{2\kappa_a}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}[\langle(\delta u_{\rm out})^2\rangle_\omega-1]. \quad (3.17)$$

This integration is easily performed in the good-cavity limit, $\xi\gg 1$, when the spectrum $\langle(\delta u_{\rm out})^2\rangle_\omega$ is a simple Lorentzian [see Eq. (3.12)]. This gives for $\langle(\delta u)^2\rangle$

$$\langle(\delta u)^2\rangle=1-S_0(p,c)\frac{(1-p)(1+c)}{3+c-2p}=1+Q, \quad (3.18)$$

where Q is Mandel's parameter calculated in Ref. [4]. In the interesting region of p and c where we have squeezing, $S_0\geq 0$, the variance is thus seen to be of order unity. The condition $\langle(\delta u)^2\rangle\ll\overline{u^2}$ is then equivalent to $\overline{u}\gg 1$. For the coupling strength c near its optimal value $c=1$ we obtain the following condition of validity of the semiclassical approximation:

$$p\gg\frac{1}{N\xi}, \quad (3.19)$$

which is equivalent to $\Omega\gg\kappa_a/N$. Thus, for the good-cavity case the semiclassical approximation holds true up to Rabi frequencies of the order of $1/N$ times the rate of the cavity decay.

This assessment changes in the bad-cavity limit, $\xi\ll 1$, due to the presence of a high peak of relaxation oscillations in the spectrum $\langle(\delta u_{\rm out})^2\rangle_\omega$. The height of this peak scales as $1/p^2$ and its width as ξp so that the area underneath is proportional to ξ/p. Thus, for small values of the pump strength p, when $\xi/p\gg 1$, the peak in question will bring a dominant contribution to the integral in Eq. (3.17). The condition $\langle(\delta u)^2\rangle\ll\overline{u^2}$ is then equivalent to

$$p\gg\frac{1}{\sqrt{N}}, \quad (3.20)$$

which is stronger than the one given by Eq. (3.19).

Apart from the "non-trivial" stationary solution found above with nonvanishing mean photon number, given by Eq. (3.3), there are "trivial" solutions of the classical equations of motion with vanishing stationary photon number. One is given by

$$\overline{z}_1=\sqrt{N}, \quad \overline{z_2}=\overline{z_0}=\overline{a}=0, \quad (3.21)$$

and obviously has all atoms in the intermediate level 1. A second such solution has the intermediate level empty and displays Rabi oscillations between the upper and lower levels driven by the pump,

$$\overline{z}_1=\overline{a}=0,$$

$$\overline{z}_0(t)=\overline{z_0}(0)\cos(p\sqrt{c}N\gamma t)-\overline{z_2}(0)\sin(p\sqrt{c}N\gamma t), \quad (3.22)$$

$$\overline{z}_2(t)=\overline{z_0}(0)\sin(p\sqrt{c}N\gamma t)+\overline{z_2}(0)\cos(p\sqrt{c}N\gamma t),$$

with two complex parameters $\overline{z_0}(0),\overline{z_2}(0)$ obeying $|\overline{z_0}(0)|^2+|\overline{z_2}(0)|^2=N$. It follows from the linearized stability analysis that for each of these solutions at least one degree of freedom is not damped. Hence, their stability must be checked by a nonlinear treatment. Our numerical investigations of the classical equations of motion suggest that whenever the system is incapable of sustaining finite photon number for large times it ends up in one of these trivial solutions.

IV. SQUEEZING IN THE PASSIVE MODE

We proceed to calculating the fluctuation spectra of the passive mode. When we adiabatically eliminated that mode at the beginning we expressed the annihilation operator $b(t)$ through the collective atomic polarization $S_{01}(t)$ and the Langevin force $\eta_b(t)$ [see Eq. (2.5)]. For the mean value of the stationary field amplitude $\overline{b}$ we obtain

$$\overline{b} = \frac{N\gamma}{g_{01}}\left(\frac{cp(1-p)}{1+c}\right)^{1/2} = -\left(\frac{\kappa_a}{\kappa_b}\right)^{1/2}\overline{a}, \tag{4.1}$$

and infer that the radiation field of the passive mode is also collective in character since $\overline{b} \sim N$.

From Eq. (2.5) the fluctuations of the field mode b are given by

$$\delta b(t) = -\frac{g_{01}}{\kappa_b}\,\delta S_{01}(t) + \left(\frac{2}{\kappa_b}\right)^{1/2}\eta_b(t). \tag{4.2}$$

Again, we split the field fluctuations δb into real and imaginary parts as $\delta b = \delta\widetilde{u} + i\,\delta\widetilde{v}$. By invoking the previous results for $\delta u_i(\omega)$ and $\delta v_i(\omega)$ we obtain the amplitude fluctuations of the passive mode.

The annihilation operator $b_{\rm out}(t)$ of the passive mode outside the cavity is related to the intracavity operator $b(t)$ in the same fashion as is the case for the active mode in Eq. (3.11),

$$b_{\rm out}(t) = \sqrt{2\kappa_b}\,b(t) - \eta_b(t). \tag{4.3}$$

The explicit form of the output fluctuations of the passive mode is presented in the Appendix. In the good-cavity limit $\xi \gg 1$ and for the frequencies $\omega \ll \gamma N$ the fluctuation spectrum of the amplitude quadrature component of the passive mode outside the cavity simplifies to

$$\langle \delta\widetilde{u}_{\rm out}(\omega)\,\delta\widetilde{u}_{\rm out}(\omega')\rangle = (1/4)\,\delta(\omega+\omega')[1 - S(p,c,\omega)], \tag{4.4}$$

with

$$S(p,c,\omega) = \frac{S_0(p,c) + \omega^2/[2\kappa_a^2(1-p)(1+c)]}{1+\tau_a^2\omega^2}, \tag{4.5}$$

with the squeezing function $S_0(p,c)$ and the width τ_a of the lasing mode given by Eqs. (3.14) and (3.15). For frequencies $\omega \ll \kappa_a$ this result may be approximated by a Lorentzian with a width

$$\frac{1}{\tau_b} = \frac{1}{\tau_a}\left[1 - \frac{8(1-p)(1+c)}{[p(1+c)+(3+c)(1-p)]^2 S_0(p,c)}\right]^{-1/2}. \tag{4.6}$$

We see that for frequencies $\omega \ll \kappa_a$ the amplitude fluctuation spectrum of the passive mode is flatter than that of the lasing mode. Not surprisingly, the width of the passive mode is, like the width of the active mode, proportional to κ_a. This is simply due to the fact that on the time scale $1/\kappa_a$ the ''fast'' passive mode adiabatically follows the slow relaxation of the active mode.

We have calculated the linewidth $\Delta\nu_b$ of the laser radiation in the passive mode in analogy to the calculation of the linewidth $\Delta\nu_a$ of the active mode and found these to be identical

$$\Delta\nu_b = \Delta\nu_a. \tag{4.7}$$

Therefore, as for the active mode the linewidth $\Delta\nu_b$ scales as $\Delta\nu_b \sim 1/N^2$.

V. QUANTUM FLUCTUATIONS OF THE PUMP

Several authors have previously taken into account the quantum statistics of an external driving field in its effect on the output from the so driven system [15–20]. Marte, Ritsch, and Walls in Ref. [18] consider coherent pumping of a laser by broadband squeezed light. They use the fact obtained earlier by Gardiner [21] that the interaction of a two-level atom with a squeezed vacuum (instead of a normal one) modifies the decay rates of atomic polarization. Ritsch and Zoller in Ref. [19] drop the assumption of white noise for the incoming squeezed vacuum and consider the driving of a two-level atom by finite-bandwidth squeezed light. Zakrzewski, Lewenstein, and Mossberg in Ref. [20] consider pump depletion.

It is worth pointing out here some difficulties arising when one tries to describe interaction of nonclassical driving fields (like squeezed or sub-Poissonian light) with atoms. First, nonclassical light cannot be described by a fluctuating c number in a density-matrix equation. Second, for nonclassical light with finite bandwidth the nonlinear response of the atomic system to the light field must be expected to be non-Markovian in character. Third, usually one has to deal with a situation where the pumped atomic medium does not react back on the source of the pump light. Thus, one has to describe the interaction between the light source and the atomic medium in a nonsymmetric way [22–24].

Our present goal is less ambitious than looking for further noise suppression by feeding squeezed light from one source into a second active medium. We just want to make sure that not too much of the squeezing is lost when our collection of three-level atoms is driven by a fluctuating pump. For that purpose it suffices to add broadband vacuum fluctuations to the c number amplitude of the pump wave. As an admittedly crude model we elevate the Rabi frequency Ω, introduced in Sec. II, to an operator

$$\Omega(t) = \Omega + \chi(t) \tag{5.1}$$

where Ω is a stationary real classical pump amplitude as before and $\chi(t)$ is a noise operator with the properties

$$[\chi(t),\chi^\dagger(t')] = \langle\chi(t)\chi^\dagger(t')\rangle = \frac{\Gamma}{2}\,\delta(t-t'),$$

$$\langle\chi(t)\rangle = \langle\chi^\dagger(t)\chi(t')\rangle = 0, \tag{5.2}$$

with an effective bandwidth Γ. We thus treat the pump field not as a dynamical variable but as an externaly imposed quantity with prescribed quantum statistics, incapable of experiencing back reaction from the atoms.

Our Hamiltonian H_0 from Eq. (2.1) is now extended to

$$H = H_0 + i\hbar(\chi S_{02} - \chi^\dagger S_{20}). \tag{5.3}$$

The new term in Eq. (5.3) entails additional multiplicative-noise forces in the evolution equations (2.6) of the atomic variables,

$$(\partial S_{02}/\partial t)_{\rm pf} = -\chi(S_{22} - S_{00}),$$

$$(\partial S_{12}/\partial t)_{\rm pf} = \chi S_{10},$$

$$(\partial S_{01}/\partial t)_{\rm pf} = -\chi S_{21}, \tag{5.4}$$

$$(\partial S_{00}/\partial t)_{\rm pf} = -(\partial S_{22}/\partial t)_{\rm pf} = -(\chi S_{20} + \chi^\dagger S_{02}),$$

$$(\partial S_{11}/\partial t)_{\rm pf} = 0.$$

Since these additional terms have zero means and since the operators C_1, C_2, C_3 from Eq. (2.7) are still conserved, the stationary solutions found above remain unchanged. Like the noise related to the collective damping the pump noise also leaves the stability analysis untouched. It is only the output fluctuations of the active and the passive mode that acquire additive corrections. These are given in the Appendix in their general form.

Once more all explicit results are significantly simplified in the good-cavity limit, $\xi \gg 1$. In this case we obtain for the amplitude quadrature fluctuation spectrum of the active-mode field outside the cavity

$$\langle \delta u_{\rm out}(\omega) \delta u_{\rm out}(\omega') \rangle = \frac{1}{4} \delta(\omega + \omega') \left\{ 1 - \frac{S(p,c,f)}{1 + \omega^2 \tau^2} \right\}. \tag{5.5}$$

The new squeezing function $S(p,c,f)$ is related to the previous $S_0(p,c)$ from (3.13) as

$$S(p,c,f) = S_0(p,c) - \frac{f}{4(1+c)} \frac{(1-2p)^2}{p(1-p)}, \tag{5.6}$$

i.e., diminished by an amount proportional to a dimensionless fluctuation strength

$$f = \frac{\Gamma}{\gamma}. \tag{5.7}$$

In the most interesting situation $f \ll 1$ it is possible to find analytically the optimum values of the coupling strength c_0 and the pump strength p_0 which maximize the squeezing function $S(p,c,f)$ for fixed f. One easily finds $c_0 = 1$, $p_0 = f^{1/3}/2$, and the maximum squeezing

$$S_{\max} = S(p_0, c_0, f) = 1 - 3f^{2/3}/8. \tag{5.8}$$

A plot of $S(p,c,f)$ is presented in Fig. 5 for $f = 0.1$.

To investigate the influence of the quantum fluctuations of the pump field on the linewidth $\Delta\nu_a$ of the superradiant laser we have calculated the new low-frequency version of $\langle \delta v_{\rm out}(\omega) \delta v_{\rm out}(\omega') \rangle$. We find the linewidth of the superradiant laser not affected at all by the quantum fluctuations of the pump.

We have also calculated the effect of the pump fluctuations on the squeezing properties and on the linewidth of the passive mode. As before, the zero-frequency properties of the passive mode are identical to those of the active mode.

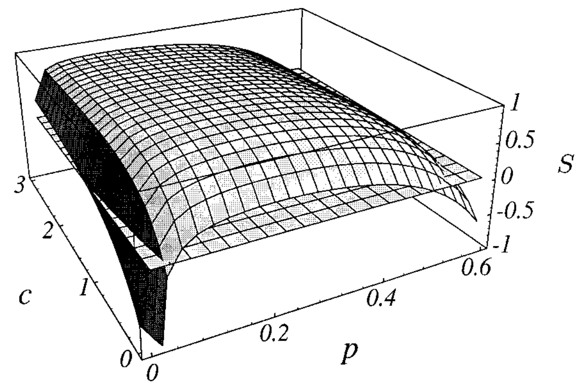

FIG. 5. Squeezing function $S(p,c,f)$ of the superradiant laser with quantum fluctuations of the pump of strength $f = 0.1$.

VI. SPONTANEOUS EMISSION

In this section we shall allow for spontaneous emission on the transition $2 \rightarrow 1$ and shall find out its influence on squeezing. It is already well known from the literature that spontaneous emission can almost entirely undo all squeezing. Spontaneous emission of the active medium in the linear optical amplifier destroys all squeezing of an external squeezed signal already at very low gain; such a device can therefore not be employed for amplification of nonclassical fields [25]. In incoherently pumped sub-Poissonian lasers spontaneous emission from the upper lasing level to the lower one (i.e., on the lasing transition) or to third levels also weakens the sub-Poissonian character of the laser radiation [26–29]. As we have shown in [4], when the collective relaxation $1 \rightarrow 0$ is replaced by usual spontaneous emission, thus bringing our scheme to an ordinary Raman laser, the maximum obtainable squeezing goes down from 100% to only 50%. This result was also found previously by Ritsch, Marte, and Zoller in [5].

In this section we keep the collective relaxation $1 \rightarrow 0$ but allow for spontaneous emission on the transition $2 \rightarrow 1$ (on this one only, for simplicity). Thus, we have a competition between collective and noncollective relaxation. Since we would like to separate the effects of spontaneous emission and of pump fluctuations, the latter are not considered in this section.

To describe a noncollective relaxation on the lasing transition, we introduce for each atom a coupling to its own separate reservoir according to the Hamiltonian

$$\delta H = i\hbar g_{12} \sum_{\mu=1}^{N} (d_\mu S_{21}^\mu - d_\mu^\dagger S_{12}^\mu), \tag{6.1}$$

where d_μ and $d_\mu^\dagger$ destroy and create quanta within the μth reservoir. We imagine all reservoirs eliminated by using the usual Born and Markov approximations and thus face the incremental time rates of change for the collective atomic variables

$$(\partial S_{02}/\partial t)_{\rm se} = -\gamma_s S_{02} + \sqrt{2\gamma_s}\sum_\mu S_{01}^\mu \eta_\mu ,$$

$$(\partial S_{12}/\partial t)_{\rm se} = -\gamma_s S_{12} - \sqrt{2\gamma_s}\sum_\mu (S_{22}^\mu - S_{11}^\mu)\eta_\mu ,$$

$$(\partial S_{01}/\partial t)_{\rm se} = -\sqrt{2\gamma_s}\sum_\mu \eta_\mu^\dagger S_{02}^\mu , \qquad (6.2)$$

$$(\partial S_{00}/\partial t)_{\rm se} = 0,$$

$$(\partial S_{11}/\partial t)_{\rm se} = -(\partial S_{22}/\partial t)_{\rm se}$$
$$= 2\gamma_s S_{22} - \sqrt{2\gamma_s}\sum_\mu (\eta_\mu S_{21}^\mu + \eta_\mu^\dagger S_{12}^\mu).$$

Here $\gamma_s \propto g_{12}^2$ is a constant of noncollective atomic relaxation for the transition $2 \to 1$ and $\eta_\mu(t)$, $\mu = 1,2,\ldots,N$ are corresponding Langevin forces, independent for different atoms,

$$[\eta_\mu(t), \eta_\nu^\dagger(t')] = \langle \eta_\mu(t)\eta_\nu^\dagger(t')\rangle = \delta_{\mu\nu}\delta(t-t'),$$

$$\langle \eta_\mu(t)\rangle = \langle \eta_\mu^\dagger(t)\eta_\nu(t')\rangle = \langle \eta_\mu(t)\eta_\nu(t')\rangle = 0. \qquad (6.3)$$

There is an important scaling difference between the collective relaxation terms in Eqs. (2.6) and noncollective ones in Eqs. (6.2). The first ones scale as N^2 while the latter are proportional only to N. Therefore, for the semiclassical approximation, $N \gg 1$, noncollective relaxation terms are very small compared to the collective ones. We may therefore eventually confine ourselves to the limit $\gamma_s \to 0$ without incurring more than an error of order $1/N$, i.e., one inherent in the semiclassical approximation anyway. Of course, the limit $\gamma_s \to 0$ must not be taken by blindly setting $\gamma_s = 0$ in Eq. (6.2). We must first realize that the incoherent terms (6.2) break the collectivity of the dynamics, i.e., destroy the conservation of C_2 and C_3 in Eq. (2.7). Needless to say the conservation of the number of atoms is retained. A stationary solution of the Heisenberg-Langevin equations (2.6) with the incoherent terms (6.2) will therefore not connect continuously with the previously obtained solution (3.3) for full cooperativity.

To appreciate the point just made, a closer look at the construction of the semiclassical stationary solution is indicated. As previously, we (i) drop the time derivatives and the noise in Eq. (2.6), (ii) degrade operators to c numbers, (iii) fix an arbitrary phase to make all stationary means real, and (iv) employ the pump parameter p and the coupling strength c [cf. Eqs. (3.1)]. From $\dot a = 0$, $\dot S_{00} = \dot S_{11} = 0$ we then express $\overline{a}$, $\overline{S_{12}}$, and $\overline{S_{20}}$ in terms of $\overline{S_{01}} \equiv xN$ and $\overline{S_{22}}$, obtaining

$$\overline{S_{12}} = \pm (N/\sqrt{c})\sqrt{x^2 - \gamma_s \overline{S_{22}}/\gamma N^2}. \qquad (6.4)$$

Note that the incoherent correction in the radicand is of order $1/N$. We should also mention that in contrast to the fully cooperative case without spontaneous emission we must now keep track of the $\pm$ alternative. In the strict absence of spontaneous emission this alternative also formally arises; however, the minus sign leads to a physically unacceptable solution in conflict with the Schwartz inequality $|\overline{S_{ij}}|^2 \leq \overline{S_{ii}}\,\overline{S_{jj}}$. In Sec. III we had not met that alternative since the Schwinger representation $S_{ij} = z_i^\dagger z_j$ there employed enforces $|\overline{S_{ij}}|^2 = \overline{S_{ii}}\,\overline{S_{jj}}$ to begin with. Here, in the presence of spontaneous emission, both signs lead to solutions in agreement with the Schwartz inequality.

Next, we formally consider $\dot S_{ij} = 0$ as linear inhomogeneous equations for the population differences $S_{22} - S_{11}$, $S_{11} - S_{00}$, and $S_{00} - S_{22}$ which we formally solve. For self-consistency, the population differences thus found must sum up to zero; that latter requirement reads, after a little rearrangement,

$$\gamma_s\left[p\sqrt{c}(p^2 - x^2)\overline{S_{22}}/N \pm (cx^2 + p^2)\sqrt{x^2 - \gamma_s \overline{S_{22}}/\gamma N^2}\right] = 0. \qquad (6.5)$$

Obviously, in the fully cooperative case $\gamma_s = 0$ this consistency condition is trivially fulfilled; in the present case the square bracket must vanish, and that equation now takes the role previously played by the conservation of C_2 in nailing down all stationary means. At this point we may and should indeed drop the $1/N$ correction $\gamma_s \overline{S_{22}}/\gamma N^2$ in the square bracket, thus implementing the formal limit $\gamma_s \to 0$. Clearly, that limit is not equivalent to setting $\gamma_s = 0$ from the outset.

The inequivalence of $\gamma_s = 0$ and $\gamma_s \to 0$ is somewhat reminiscent of the symmetry breaking in second-order phase transitions. For instance, a Heisenberg magnet below the Curie temperature is assigned, by the mean-field equations (which correspond to our semiclassical approximation), a vanishing magnetization in the strict absence of a symmetry breaking magnetic field h; however, if $h \neq 0$ is allowed, the limit $h \to 0$ leaves a spontaneous magnetization.

To finally establish the stationary solution in the limit $\gamma_s \to 0$ we employ the population conservation (2.11) and arrive at

$$\overline{S_{01}}/N = x, \quad \overline{S_{02}}/N = -x^2/(\sqrt{c}p), \quad \overline{S_{12}}/N = \pm x/\sqrt{c},$$

$$\overline{S_{00}}/N = (1/3)\left[1 \pm \frac{x^2}{cp}(1 + cp^2/x^2 - 2c)\right],$$

$$\overline{S_{11}}/N = (1/3)\left[1 \pm \frac{x^2}{cp}(1 - 2cp^2/x^2 + c)\right], \qquad (6.6)$$

$$\overline{S_{22}}/N = (1/3)\left[1 \pm \frac{x^2}{cp}(-2 + cp^2/x^2 + c)\right],$$

$$\overline{a} = \mp \frac{N\gamma}{g_{12}}\sqrt{c}x,$$

with x defined by the following equation:

$$2(1+c)x^4 + cp(cp \mp 1)x^2 + c^2p^3(p \pm 1) = 0. \qquad (6.7)$$

This is to be confronted with the first of Eqs. (2.12). Here we have, in fact, two different equations corresponding to two different signs, upper and lower, which stem, in turn, from two different signs in the mean polarization $\overline{S_{12}}$ in (6.4).

Solving (6.7) with the lower sign, we obtain the following result for x which we shall call the ‘‘gamma’’ solution:

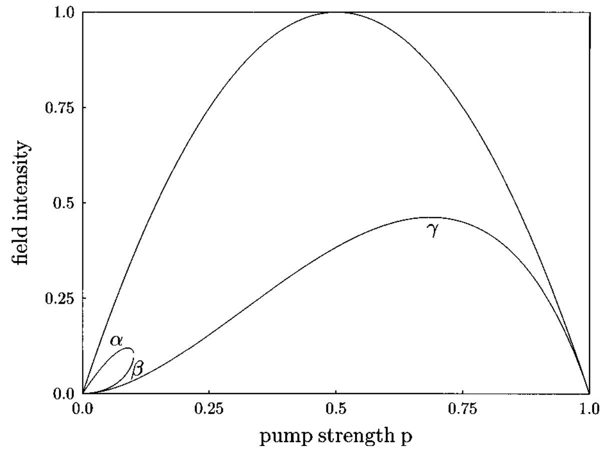

FIG. 6. Stationary intracavity intensity of the superradiant laser vs pump strength p without (uppermost curve) and with (α, β and γ) spontaneous emission for the coupling strength $c=0.1$.

$$x_\gamma^2=\frac{cp(1+cp)}{4(1+c)}\left\{\left(1+\frac{8(1+c)p(1-p)}{(1+cp)^2}\right)^{1/2}-1\right\}. \tag{6.8}$$

Obviously, this solution exists for $0\leqslant p\leqslant 1$ as does the solution (3.3) for full cooperativity. Equation (6.7) with the upper sign has two solutions which we shall call ''alpha'' and ''beta,''

$$x_{\alpha,\beta}^2=\frac{cp(1-cp)}{4(1+c)}\left\{1\pm\left(1-\frac{8(1+c)p(1+p)}{(1-cp)^2}\right)^{1/2}\right\}, \tag{6.9}$$

where ''+'' corresponds to the alpha solution and the ''−'' to the beta solution. These two solutions exist for the pump strength p in the limits $0\leqslant p\leqslant p_m$ where p_m is equal to

$$p_m=\frac{1}{(5+2\sqrt{6})(1+c)-1}. \tag{6.10}$$

We can see from (6.10) that $p_m\leqslant 1$ for any value of the coupling strength c. When $c\to 0$, $p_m\to(\sqrt{6}-2)/4\approx 0.11$, and then decreases monotonically with growing c. Figure 6 shows the intensity of the mean intracavity field for all three solutions alpha, beta, and gamma, and also for the solution without spontaneous emission.

A full understanding of why several stationary solutions can arise and of their stability properties is possible only after a discussion of submaximal cooperativity at $\gamma_s=0$, when $c_2\leqslant 1$, $c_3\leqslant 1$. This will be treated in a forthcoming publication.

While in general multistability turns out to be possible there is one simple limit, that of weak pumping ($p\ll 1$) and a bad cavity ($\xi\ll 1$), where only the α solution is stable. For that case we have determined the squeezing spectrum. Figure 7 compares this squeezing spectrum with that of the solution without spontaneous emission. Obviously, spontaneous emission leads to the appearence of a new high narrow peak around zero frequency. That peak has a height $\propto N\gamma/\gamma_s$ and a width $\propto\gamma_s$, such that the area underneath remains constant in the limit $\gamma_s\to 0$, indicating an asymptotic δ-function peak,

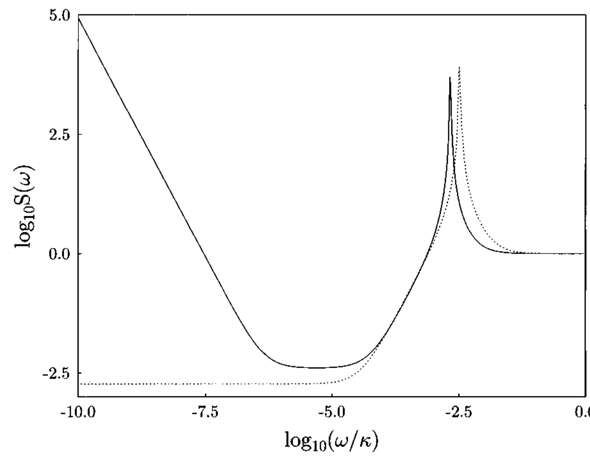

FIG. 7. Squeezing spectrum without (dots) and with spontaneous emission (solid curve) for α solution; $\gamma_s/N\gamma=5\times 10^{-10}$, $N\gamma/\kappa_a=0.01$, $c=0.9$, $p=0.03$.

$$\langle\delta u_{\rm out}(\omega)\delta u_{\rm out}(\omega')\rangle$$

$$\propto\delta(\omega+\omega')\frac{(\gamma_s/N\gamma)+n_1(\omega/N\gamma)^2+\dots}{(\gamma_s/N\gamma)^2+d_1(\omega/N\gamma)^2+d_2(\omega/N\gamma)^4+\cdots}, \tag{6.11}$$

with dimensionless coefficients $n_1,\dots$ in the numerator and $d_1,d_2,\dots$ in the denominator, all of which are independent of γ_s. One sees that the δ-function peak indeed arises as $\gamma_s\to 0$. However, for $\gamma_s=0$ one power of ω^2 cancels whereupon one comes back to the case of Sec. III. A second peak in the spectrum of Fig. 7 reflects relaxation oscillations typical of the bad-cavity limit to which the figure pertains. Most importantly, there is a broad intermediate frequency range where the squeezing is strong and not qualitatively changed from the one obtained for $\gamma_s=0$.

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APPENDIX: THE OUTPUT FLUCTUATIONS

We here list the output fluctuations of the active and the passive mode. There are three contributions to each of these fluctuations resulting from the vacuum fluctuating forces of the active mode $\eta_a=\Sigma_a+i\Delta_a$, the vacuum fluctuating forces of the passive mode $\eta_b=\Sigma_b+i\Delta_b$ and the pump fluctuations $\chi=\sqrt{\gamma}(\Sigma_p+i\Delta_p)$. In order to save space we here use the abbreviation $x=\omega/\gamma N$. The output fluctuations of the active mode read as

$$\delta u_{\rm out}(\omega)=\frac{N_u^a(x)\Sigma_a(\omega)+N_u^b(x)\Sigma_b(\omega)+N_u^p(x)\Sigma_p(\omega)}{D_u(x)},$$

$$\delta v_{\rm out}(\omega)=\frac{N_v^a(x)\Delta_a(\omega)+N_v^b(x)\Delta_b(\omega)+N_v^p(x)\Delta_p(\omega)}{D_v(x)},$$

where

$$D_u(x)=(ix)^3-(ix)^2\left(\frac{p(1+2c)-c}{1+c}-\frac{1}{\xi}\right)+ix\left(\frac{pc(3+c-2p)}{1+c}-\frac{p}{\xi}(1-c)\right)+\frac{4pc(1-p)}{\xi},$$

$$N_u^a(x)=-D_u(x)+\frac{2}{\xi}\left((ix)^2-ix\frac{p(1+2c)-c}{1+c}+\frac{pc(3+c-2p)}{1+c}\right),$$

$$N_u^b(x)=-\left(\frac{2c}{\xi}\right)\frac{ix(1-p)-p(c-1+2p)}{1+c},$$

$$N_u^p(x)=-\left(\frac{2p(1-p)}{1+c}\right)^{1/2}\left(\frac{c}{\xi}\right)(2-4p+ix),$$

$$D_v(x)=ix\left[(ix)^2-ix\left(\frac{p(1+2c)-c}{1+c}-\frac{1}{\xi}\right)+\frac{pc(c-1+2p)}{1+c}-\frac{p}{\xi}(1-c)\right],$$

$$N_v^a(x)=-D_v(x)+\frac{2}{\xi}\left(-(ix)^2+ix\frac{p(1+2c)-c}{1+c}-\frac{pc(c-1+2p)}{1+c}\right),$$

$$N_v^b(x)=-\left(\frac{2c}{\xi}\right)\frac{ix(1-p)-p(1-c+2pc)}{1+c},$$

$$N_v^p(x)=\left(\frac{2p(1-p)}{1+c}\right)^{1/2}\left(\frac{c}{\xi}\right)(ix)^2.$$

The polynomials $D_u(x)$, $D_v(x)$ are $P_u(ix)$ and $P_v(ix)$ from Eq. (3.8), divided by ix.

The output fluctuations of the passive mode are

$$\delta\widetilde{u}_{\text{out}}(\omega)=\frac{N_{\tilde{u}}^a(x)\Sigma_a(\omega)+N_{\tilde{u}}^b(x)\Sigma_b(\omega)+N_{\tilde{u}}^p(x)\Sigma_p(\omega)}{D_u(x)},$$

$$\delta\widetilde{v}_{\text{out}}(\omega)=\frac{N_{\tilde{v}}^a(x)\Delta_a(\omega)+N_{\tilde{v}}^b(x)\Delta_b(\omega)+N_{\tilde{v}}^p(x)\Delta_p(\omega)}{D_v(x)},$$

with

$$N_{\tilde{u}}^a(x)=\left(\frac{2c}{\xi}\right)\frac{ix(1-p)+p(1-c+2cp)}{1+c},$$

$$N_{\tilde{u}}^b(x)=D_u(x)+2(ix)^2\frac{p(1+2c)-c}{1+c}-2ix\left(\frac{2cp(1-p)}{1+c}-\frac{p(1+2c)-c}{\xi(1+c)}\right)+\frac{2c}{\xi}\frac{p(4p+2cp-c-3)}{1+c},$$

$$N_{\tilde{u}}^p(x)=\left(\frac{2p(1-p)}{1+c}\right)^{1/2}\left[(ix)^2-ix\left(c(1-2p)-\frac{1}{\xi}\right)-\frac{2c}{\xi}(1-2p)\right],$$

$$N_{\tilde{v}}^a(x)=-\left(\frac{2c}{\xi}\right)\frac{ix(1-p)-p(1-c+2p)}{1+c},$$

$$N_{\tilde{v}}^b(x)=D_v(x)+2(ix)^2\frac{p(1+2c)-c}{1+c}+2ix\left(\frac{2pc(1-p)}{1+c}+\frac{p(1+2c)-c}{\xi(1+c)}\right)+2\frac{pc(1-c+2cp)}{\xi(1+c)},$$

$$N_{\tilde{v}}^p(x)=\left(\frac{2p(1-p)}{1+c}\right)^{1/2}(ix)^2\left(ix+\frac{c(1-c-2p)}{1+c}+\frac{1}{\xi}\right).$$

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