Effective Raman theory for a three-level atom in the Λ configuration

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It is shown that a three-level atom in the Λ configuration with arbitrary detunings can be exactly reduced to a two-level system with an effective Raman coupling, which depends nonlinearly on the intensity of the two radiation fields. This is done by exactly evaluating the unitary transformation introduced by Alexanian and Bose [Phys. Rev. A **52**, 2218 (1995)] for a three-level atom coupled to two modes of the radiation field. We obtain an exact transformed Hamiltonian in which one of the three levels is decoupled for all values of the detunings. In particular, our result is then valid for any ratios of the coupling constants to detunings, even for zero detuning, in contrast to earlier work which requires that these ratios be small. We find the the eigenvalues of the exact transformed Hamiltonian and study its population dynamics. [S1050-2947(96)08808-7]

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I. INTRODUCTION

Quantum-optical interactions involving one atom with a few energy levels and one or more near-resonant modes of the quantized electromagnatic fields have been extensively studied by means of the Jaynes-Cummings model and its various extensions. Such models are shown to exhibit interesting nonclassical effects, such as the collapse and revival of the Rabi oscillations of the atomic inversion, antibunched light, and squeezing [1–8]. Among these extensions are those models that include two-photon couplings in single-mode and multimode cases [1–8]. Models describing Raman processes [4,5] belong to the latter. In dealing with two-photon transitions in Raman-type processes, one can consider a three-level system of energies E_1, E_2, and E_3 in the Λ configuration interacting with a pump ω_1, and a Stokes mode ω_2 as shown in Fig. 1 [4–6].The Hamiltonian of the system is written as [4,6]

$$H=\sum_{i=1}^{3} E_i\sigma_{ii}+\hbar\omega_1 a_1^\dagger a_1+\hbar\omega_2 a_2^\dagger a_2+\hbar g_1(a_1\sigma_{31}+a_1^\dagger\sigma_{13}) +\hbar g_2(a_2\sigma_{32}+a_2^\dagger\sigma_{23}), \tag{1}$$

where symbols a_j $(j=1,2)$ represent the field operators of modes 1 and 2, $\sigma_{ii}=|i\rangle\langle i|$ are the level occupation numbers, and $\sigma_{ij}=|i\rangle\langle j|$ $(i\neq j)$ are the transition operators from levels j to i. Levels 3 and 1 (2) are coupled by a dipole-coupling constant g_1 (g_2). There is no direct coupling between levels 1 and 2. The quantities Δ_1 and Δ_2 in Fig. 1 denote detunings given by $\Delta_j=(E_3-E_j)/\hbar-\omega_j$, $j=1,2$. Note that we have changed some notation with respect to the previous literature and, in particular, have interchanged the numbering of levels 2 and 3. Several authors have considered the Hamiltonian with $\Delta_1=\Delta_2=0$ [7,8]. Under the assumption that $\Delta_1=\Delta_2=\Delta$, and that level 3 is far off resonance (large detuning Δ) and, hence, can be adiabatically eliminated, Gerry and Eberly [4] and Cardimona *et al.* [5] have obtained an effective two-level Raman interaction Hamiltonian. This effective Raman Hamiltonian has the form of the usual Jaynes-Cummings model but with the single-mode field operators replaced by products of an annihilation operator of one mode and a creation operator of the other, i.e., it has a term of the form $\lambda(a_1^\dagger a_2\sigma_{12}+a_2^\dagger a_1\sigma_{21})$ with $\lambda\sim-\hbar g_1 g_2/\Delta$. Instead of adiabatic elimination, Alexanian and Bose [6] have recently proposed an interesting alternative method—unitary transformation to obtain a similar Hamiltonian for which one level decouples. Their unitary transformation is [6]

$$X'=\exp(S)X\exp(-S), \tag{2}$$

where X' denotes the transformed atomic and photon variables, and

$$S=\alpha(a_1\sigma_{31}-a_1^\dagger\sigma_{13})+\beta(a_2\sigma_{32}-a_2^\dagger\sigma_{23}), \tag{3}$$

where α and β are transformation parameters to be specified later (again, note that our notation is different from that in Ref. [6]). Claiming that the exact transformation cannot be found, Ref. [6] has evaluated it perturbatively and kept contributions to the second order in coupling constants in the transformed Hamiltonian. Except for containing intensity-dependent Stark shift terms, the approximate transformed Hamiltonian turns out to be the same as the one obtained by Gerry and Eberly [4] and Cardimona *et al.* [5]. With the

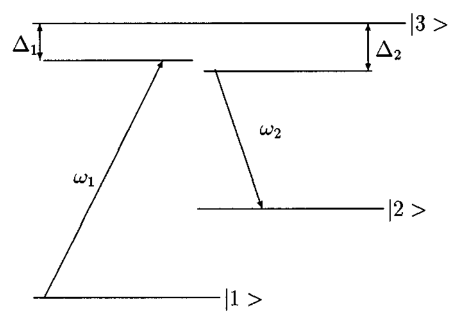

FIG. 1. Three-level atom in the Λ configuration.

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inclusion of the Stark effects, Alexanian and Bose have found that the Rabi frequency and the collapse and revival times of the atomic inversion have strikingly different photon-intensity dependence than those found by Gerry and Eberly. However, all of these results [4–6] are suitable only when the ratios g_j/Δ_j $(j=1,2)$ are small. In fact, as will be seen later, these results may need modification in the presence of the intense pump or Stokes mode even if the ratios g_j/Δ_j $(j=1,2)$ are small. In view of the facts that the detunings are experimentally adjustable parameters that can be tuned to any value, and that Raman interactions are important when pump and/or Stokes modes are relatively strong, it seems worthwhile to attempt to find a general solution to this problem that describes the situation where coupling constants have arbitrary relation to the detunings. In fact, several authors have already investigated phenomona involving the opposite limit of zero detuning [7,8].

In this paper, we first show that all of the relevant unitary transformations introduced by Alexanian and Bose can be found exactly. Level 3 can be eliminated from the exact transformed Hamiltonian to obtain an effective two-level Raman interaction Hamiltonian with an intensity-dependent coupling between levels 1 and 2 which is valid for any magnitude of the ratios of coupling constants to detunings including zero detuning. The results reduce to the previous results [4–6] as the detunings become much greater than coupling constants. Next, we obtain the eigenvalues and the eigenstates of the exact transformed Hamiltonian. Because unitary transformations preserve the eigenvalues, these are in fact the exact eigenvalues of the Hamiltonian of Eq. (1). To our knowledge, this is the first time these exact eigenvalues have been calculated by any means. We then calculate the atomic inversion and Rabi freqency by means of the exact transformed Hamiltonian. The Rabi frequency is found to have a different intensity dependence than that obtained by Gerry and Eberly [4], Cardimona *et al.* [5], and Alexanian and Bose [6], although it reduces to the result of Alexanian and Bose [6] when detuning is much greater than coupling constants. It also agrees with the known result given by several other authors [7,8] for the zero-detuning case. Finally, we summarize our results and make concluding remarks.

II. THE EXACT TRANSFORMED HAMILTONIAN

Before calculating all the relevant unitary transformations given by Eqs. (2) and (3), let us rewrite the Hamiltonian of Eq. (1) in a more convenient and symmetric form as

$$H=\tfrac{1}{2}(E_1+E_2-\hbar\omega_1-\hbar\omega_2)+H_1+H_2, \tag{4}$$

with

$$H_1=\hbar\omega_1 N_1+\tfrac{1}{2}\hbar\Delta_1\sigma_{1z}+\hbar g_1(a_1\sigma_{31}+a_1^\dagger\sigma_{13}),$$

$$H_2=\hbar\omega_2 N_2+\tfrac{1}{2}\hbar\Delta_2\sigma_{2z}+\hbar g_2(a_2\sigma_{32}+a_2^\dagger\sigma_{23}), \tag{5}$$

where $\sigma_{1z}=1-2\sigma_{11}$, $\sigma_{2z}=1-2\sigma_{22}$, and

$$N_1=a_1^\dagger a_1+\tfrac{1}{2}(\sigma_{1z}+1),\quad N_2=a_2^\dagger a_2+\tfrac{1}{2}(\sigma_{2z}+1). \tag{6}$$

It is easily shown that N_1 and N_2 commute with the operator S given by Eq. (3) and with the Hamiltonian. Therefore, they are two invariant quantities under the unitary transformation (2) [i.e., $N_j=a_j^\dagger a_j+\frac{1}{2}(\sigma_{jz}+1)=N_j'=a_j'^\dagger a_j'+\frac{1}{2}(\sigma_{jz}'+1)$], and they are also two constants of motion. We only need to calculate the transformations of the four quantities σ_{1z}, σ_{2z}, $q_1\equiv(a_1\sigma_{31}+a_1^\dagger\sigma_{13})$, and $q_2\equiv(a_2\sigma_{32}+a_2^\dagger\sigma_{23})$ to obtain the transformed Hamiltonian. While doing so, the operators N_1 and N_2 can be treated as if they are c numbers, since they commute with the four quantities and the operator S.

We now calculate the transformations of the four quantities by the formula

$$X'=e^S X e^{-S}=\sum_{n=0}^{\infty}\frac{X^{(n)}}{n!}, \tag{7}$$

where S is given by Eq. (3), $X^{(n)}=[S,X^{(n-1)}]$, and $X^{(0)}\equiv X$ denotes σ_{1z}, σ_{2z}, q_1, and q_2. After some manipulations, we obtain

$$\sigma_{1z}^{(1)}=-2\alpha q_1;\quad \sigma_{2z}^{(1)}=-2\beta q_2, \tag{8}$$

$$\sigma_{1z}^{(2)}=-4\overline{\alpha^2}\sigma_{1z}(1-\sigma_{22})+2\alpha\beta q,$$

$$\sigma_{2z}^{(2)}=-4\overline{\beta^2}\sigma_{2z}(1-\sigma_{11})+2\alpha\beta q, \tag{9}$$

$$\sigma_{1z}^{(n+2)}=-(\overline{\beta^2}+4\overline{\alpha^2})\sigma_{1z}^{(n)}-3\overline{\alpha^2}\sigma_{2z}^{(n)},$$

$$\sigma_{2z}^{(n+2)}=-(4\overline{\beta^2}+\overline{\alpha^2})\sigma_{2z}^{(n)}-3\overline{\beta^2}\sigma_{1z}^{(n)}, \tag{10}$$

where $n=1,2,3,\ldots$, $\overline{\alpha}=\alpha\sqrt{N_1}$, $\overline{\beta}=\beta\sqrt{N_2}$, and

$$q\equiv(a_1^\dagger a_2\sigma_{12}+a_2^\dagger a_1\sigma_{21}). \tag{11}$$

Equations (7) and (10) can be written in a more concise form,

$$\begin{pmatrix}\sigma_{1z}'\\ \sigma_{2z}'\end{pmatrix}=\begin{pmatrix}\sigma_{1z}\\ \sigma_{2z}\end{pmatrix}+A\begin{pmatrix}\sigma_{1z}^{(1)}\\ \sigma_{2z}^{(1)}\end{pmatrix}+B\begin{pmatrix}\sigma_{1z}^{(2)}\\ \sigma_{2z}^{(2)}\end{pmatrix}, \tag{12}$$

where A and B are two matrices given by

$$A=\sum_{n=0}^{\infty}\frac{P^n}{(2n+1)!},\quad B=\sum_{n=1}^{\infty}\frac{P^{n-1}}{(2n)!}, \tag{13}$$

and the matrix P as well as its diagonal forms are

$$P=-\begin{pmatrix} \overline{\beta^2}+4\overline{\alpha^2}, & 3\overline{\alpha^2} \\ 3\overline{\beta^2}, & 4\overline{\beta^2}+\overline{\alpha^2} \end{pmatrix}=G^{-1}\begin{pmatrix} (i\xi)^2, & 0 \\ 0, & (2i\xi)^2 \end{pmatrix}G, \tag{14}$$

where

$$G=\frac{1}{\xi^2}\begin{pmatrix} \sqrt{2}\overline{\beta^2}, & -\sqrt{2}\overline{\alpha^2} \\ \rho, & \rho \end{pmatrix};$$

$$G^{-1}=\begin{pmatrix} (\sqrt{2})^{-1}, & \overline{\alpha^2}\rho^{-1} \\ -(\sqrt{2})^{-1}, & \overline{\beta^2}\rho^{-1} \end{pmatrix}, \tag{15}$$

with $\xi=\sqrt{\overline{\alpha^2}+\overline{\beta^2}}$ and $\rho=\sqrt{\overline{\alpha^4}+\overline{\beta^4}}$. The matrices A and B, then, become

$$A=G^{-1}\begin{pmatrix} \dfrac{\sin(\xi)}{\xi}, & 0 \\ 0, & \dfrac{\sin(2\xi)}{2\xi} \end{pmatrix}G;$$

$$B=G^{-1}\begin{pmatrix} \dfrac{1-\cos(\xi)}{\xi}, & 0 \\ 0, & \dfrac{1-\cos(2\xi)}{2\xi} \end{pmatrix}G. \tag{16}$$

Substituting Eq. (16) into Eq. (12) and using Eqs. (8) and (9) as well as the expressions of matrices G and G^{-1}, we obtain, after some manipulations, the exact transformation results of the operators σ'_{1z} and σ'_{2z} as follows:

$$\begin{aligned}
\sigma'_{1z}&=\sigma_{1z}-2\alpha\frac{\sin(\xi)}{\xi^3}(\overline{\alpha^2}\cos\xi+\overline{\beta^2})q_1+2\overline{\alpha^2}\beta(1-\cos\xi)\frac{\sin(\xi)}{\xi^3}q_2+\frac{2\alpha\beta}{\xi^4}(1-\cos\xi)(\overline{\alpha^2}\cos\xi+\overline{\beta^2})q \\
&\quad+\frac{2\overline{\alpha^2}\overline{\beta^2}}{\xi^4}(1-\cos\xi)^2\sigma_{2z}(1-\sigma_{11})-\frac{2\overline{\alpha^2}}{\xi^4}(1-\cos\xi)[\overline{\alpha^2}(1+\cos\xi)+2\overline{\beta^2}]\sigma_{1z}(1-\sigma_{22}), \\
\sigma'_{2z}&=\sigma_{2z}-2\beta\frac{\sin(\xi)}{\xi^3}(\overline{\beta^2}\cos\xi+\overline{\alpha^2})q_2+2\overline{\beta^2}\alpha(1-\cos\xi)\frac{\sin(\xi)}{\xi^3}q_1+\frac{2\alpha\beta}{\xi^4}(1-\cos\xi)(\overline{\beta^2}\cos\xi+\overline{\alpha^2})q \\
&\quad+\frac{2\overline{\alpha^2}\overline{\beta^2}}{\xi^4}(1-\cos\xi)^2\sigma_{1z}(1-\sigma_{22})-\frac{2\overline{\beta^2}}{\xi^4}(1-\cos\xi)[\overline{\beta^2}(1+\cos\xi)+2\overline{\alpha^2}]\sigma_{2z}(1-\sigma_{11}).
\end{aligned} \tag{17}$$

The transformed quantities q'_1 and q'_2 can easily be obtained by the differentiations of the transformed operators σ'_{1z} and σ'_{2z} since one can show

$$q'_1=-\frac{1}{2\alpha}\left[\frac{d}{dt}(e^{St}\sigma_{1z}e^{-St})\right]_{t=1},\quad q'_2=-\frac{1}{2\beta}\left[\frac{d}{dt}(e^{St}\sigma_{2z}e^{-St})\right]_{t=1}, \tag{18}$$

which can equivalently be evaluated by

$$q'_1=-\frac{1}{2\alpha}\left[\frac{d}{dt}\sigma'_{1z}(\alpha\to\alpha t;\beta\to\beta t)\right]_{t=1},\quad q'_2=-\frac{1}{2\beta}\left[\frac{d}{dt}\sigma'_{2z}(\alpha\to\alpha t;\beta\to\beta t)\right]_{t=1}. \tag{19}$$

The results are

$$\begin{aligned}
q'_1&=\frac{[\overline{\beta^2}\cos\xi+\overline{\alpha^2}\cos(2\xi)]}{\xi^2}q_1-\frac{\overline{\alpha^2}\beta[\cos\xi-\cos(2\xi)]}{\alpha\xi^2}q_2-\frac{\beta\sin\xi}{\xi^3}(2\overline{\alpha^2}\cos\xi+\overline{\beta^2}-\overline{\alpha^2})q \\
&\quad+\frac{2\overline{\alpha^2}\sin\xi}{\alpha\xi^3}(\overline{\beta^2}+\overline{\alpha^2}\cos\xi)\sigma_{1z}(1-\sigma_{22})-\frac{2\overline{\beta^2}\overline{\alpha^2}\sin\xi}{\alpha\xi^3}(1-\cos\xi)\sigma_{2z}(1-\sigma_{11}), \\
q'_2&=\frac{[\overline{\alpha^2}\cos\xi+\overline{\beta^2}\cos(2\xi)]}{\xi^2}q_2-\frac{\overline{\beta^2}\alpha[\cos\xi-\cos(2\xi)]}{\beta\xi^2}q_1-\frac{\alpha\sin\xi}{\xi^3}(2\overline{\beta^2}\cos\xi+\overline{\alpha^2}-\overline{\beta^2})q \\
&\quad+\frac{2\overline{\beta^2}\sin\xi}{\beta\xi^3}(\overline{\alpha^2}+\overline{\beta^2}\cos\xi)\sigma_{2z}(1-\sigma_{11})-\frac{2\overline{\beta^2}\overline{\alpha^2}\sin\xi}{\beta\xi^3}(1-\cos\xi)\sigma_{1z}(1-\sigma_{22}).
\end{aligned} \tag{20}$$

Equations (17)–(20) are the exact unitary transformation results for all the relevant quantities needed to find the exact transformed Hamiltonian. Noting that $H'=\frac{1}{2}(E_1+E_2-\hbar\omega_1-\hbar\omega_2)+H'_1+H'_2$, and

$$2H_1' = \hbar\omega_1 N_1 + \tfrac{1}{2}\hbar\Delta_1\sigma_{1z}' + \hbar g_1 q_1',$$

$$H_2' = \hbar\omega_2 N_2 + \tfrac{1}{2}\hbar\Delta_2\sigma_{2z}' + \hbar g_2 q_2', \tag{21}$$

we then get the exact transformed Hamiltonian

$$H' = E_0 + \hbar\omega_1 N_1 + \hbar\omega_2 N_2 + \tfrac{1}{2}\hbar\eta\sigma_{33} + \hbar\lambda(a_1^\dagger a_2\sigma_{12} + a_2^\dagger a_1\sigma_{21}) + \tfrac{1}{2}\hbar\omega(\sigma_{22} - \sigma_{11}) + \hbar\gamma_1(a_1\sigma_{31} + a_1^\dagger\sigma_{13}) + \hbar\gamma_2(a_2\sigma_{32} + a_2^\dagger\sigma_{23}), \tag{22}$$

where N_1 and N_2 are still given by Eq. (6) and the other parameters are functions of N_1 and N_2 and are given by

$$\omega = (\Delta_1 - \Delta_2)\left[1 + \frac{\overline{\beta^2}\,\overline{\alpha^2}}{2\xi^4}(8\cos\xi - 7)\right] + \frac{4\overline{\beta^2}\,\overline{\alpha^2}}{\xi^4}\left(\frac{g_1}{\alpha} - \frac{g_2}{\beta}\right)\xi\sin\xi + \frac{\Delta_2\overline{\beta^4} - \Delta_1\overline{\alpha^4}}{2\xi^4} - \frac{\overline{\beta^2} - \overline{\alpha^2}}{2\xi^4}\left[(\Delta_2\overline{\beta^2} + \Delta_1\overline{\alpha^2})\cos(2\xi) + 2\left(\frac{g_1\overline{\alpha^2}}{\alpha} + \frac{g_2\overline{\beta^2}}{\beta}\right)\xi\sin(2\xi)\right], \tag{23}$$

$$\lambda = \frac{\alpha\beta}{\xi^4}\left\{[\Delta_1\overline{\beta^2} + \Delta_2\overline{\alpha^2} + (\Delta_2\overline{\beta^2} + \Delta_1\overline{\alpha^2})\cos\xi](1 - \cos\xi) - \left[\left(\frac{g_2}{\beta} - \frac{g_1}{\alpha}\right)(\overline{\alpha^2} - \overline{\beta^2}) + 2\left(\frac{g_1\overline{\alpha^2}}{\alpha} + \frac{g_2\overline{\beta^2}}{\beta}\right)\cos\xi\right]\xi\sin\xi\right\}, \tag{24}$$

$$\eta = (\Delta_1 + \Delta_2)\left(1 - \frac{3\overline{\alpha^2}\,\overline{\beta^2}}{2\xi^4}\right) - \frac{3(\Delta_1\overline{\alpha^4} + \Delta_2\overline{\beta^4})}{2\xi^4} + \frac{3(\Delta_1\overline{\alpha^2} + \Delta_2\overline{\beta^2})}{2\xi^2}\cos(2\xi) + \left(\frac{g_1\overline{\alpha^2}}{\alpha} + \frac{g_2\overline{\beta^2}}{\beta}\right)\frac{3\sin(2\xi)}{\xi}, \tag{25}$$

$$E_0 = \tfrac{1}{2}(E_1 + E_2 - \hbar\omega_1 - \hbar\omega_2) + \frac{\hbar(\Delta_1 + \Delta_2)}{6} - \frac{\hbar\eta}{6}, \tag{26}$$

$$\gamma_1 = (\Delta_2 - \Delta_1)\frac{\alpha\overline{\beta^2}}{\xi^3}\sin\xi + \frac{\alpha\overline{\beta^2}}{\xi^2}\left(\frac{g_1}{\alpha} - \frac{g_2}{\beta}\right)\cos\xi - \frac{\alpha}{2\xi^3}(\Delta_2\overline{\beta^2} + \Delta_1\overline{\alpha^2})\sin(2\xi) + \frac{\alpha}{\xi^2}\left(\frac{g_1\overline{\alpha^2}}{\alpha} + \frac{g_2\overline{\beta^2}}{\beta}\right)\cos(2\xi),$$

$$\gamma_2 = (\Delta_1 - \Delta_2)\frac{\beta\overline{\alpha^2}}{\xi^3}\sin\xi + \frac{\beta\overline{\alpha^2}}{\xi^2}\left(\frac{g_2}{\beta} - \frac{g_1}{\alpha}\right)\cos\xi - \frac{\beta}{2\xi^3}(\Delta_2\overline{\beta^2} + \Delta_1\overline{\alpha^2})\sin(2\xi) + \frac{\beta}{\xi^2}\left(\frac{g_1\overline{\alpha^2}}{\alpha} + \frac{g_2\overline{\beta^2}}{\beta}\right)\cos(2\xi), \tag{27}$$

where $\overline{\alpha} = \alpha\sqrt{N_1}$, $\overline{\beta} = \beta\sqrt{N_2}$, and $\xi = \sqrt{\overline{\alpha^2} + \overline{\beta^2}}$.

These equations describe an infinite set of unitary transformations. We may now choose the transformation parameters α and β so as to provide the simplest transformed Hamiltonian and, in particular, so as to decouple level 3 from the other two levels; i.e., they are chosen such that

$$\gamma_1 = \gamma_2 = 0. \tag{28}$$

Then, the Hamiltonian has the form

$$H' = E_0 + \hbar\omega_1 N_1 + \hbar\omega_2 N_2 + \tfrac{1}{2}\hbar\eta\sigma_{33} + \hbar\lambda(a_1^\dagger a_2\sigma_{12} + a_2^\dagger a_1\sigma_{21}) + \tfrac{1}{2}\hbar\omega(\sigma_{22} - \sigma_{11}), \tag{29}$$

where the parameters ω, λ, η, and E_0 are still given, respectively, by Eqs. (23)–(26). The exact transformed Hamiltonian in Eq. (29) with α, β determined by Eq. (28) holds for any magnitudes of the ratios g_1/Δ_1 and g_2/Δ_2 and is easily shown to reduce to the previous results [4–6], when both the ratios are very small. Obviously, the λ and ω terms in the Hamiltonian given by Eq. (29) only produce transitions between levels 1 and 2, while the other terms do not cause any transitions among the three levels. This means that as long as values of α and β can be found that satisfy Eq. (28), level three can be exactly decoupled and does not contribute to the population dynamics. We know from previous results that the approximate solution $\alpha \approx g_1/\Delta_1, \beta \approx g_2/\Delta_2$ exists if both the ratios g_1/Δ_1 and g_2/Δ_2 are very small. In the following, we shall show that there exist solutions without any approximations for the case $\Delta_1 = \Delta_2 \equiv \Delta$ suitable for any values of the ratios g_1/Δ and g_2/Δ. Consequently, we can put $\sigma_{33} = 0$ in Eq. (29) to obtain an effective two-level Raman interaction Hamiltonian with levels 1 and 2 subject to an intensity-dependent coupling, i.e., the effective coupling parameter λ depends on photon numbers.

For the purpose of illustration, let us consider the case of $\Delta_1 = \Delta_2 \equiv \Delta$. Equation (28) is satisfied in this case if we choose

$$\alpha = \frac{g_1}{2\sqrt{g_1^2 + g_2^2}}\arctan\left(\frac{2\sqrt{g_1^2 + g_2^2}}{\Delta}\right),$$

$$\beta = \frac{g_2}{2\sqrt{\overline{g_1^2} + \overline{g_2^2}}} \arctan\left(\frac{2\sqrt{\overline{g_1^2} + \overline{g_2^2}}}{\Delta} \right), \tag{30}$$

where $\overline{g_j} = g_j \sqrt{N_j}$, $j = 1,2$. We find, after some manipulations, that complicated expressions for the parameters in the transformed Hamiltonian [Eq. (29)] are greatly simplified and have the forms

$$\omega = \frac{\overline{g_1^2} - \overline{g_2^2}}{\overline{g_1^2} + \overline{g_2^2}} \left[\sqrt{\left(\frac{\Delta}{2}\right)^2 + \overline{g_1^2} + \overline{g_2^2}} - \frac{\Delta}{2} \right],$$

$$\eta = 2\Delta + 3\left[\sqrt{\left(\frac{\Delta}{2}\right)^2 + \overline{g_1^2} + \overline{g_2^2}} - \frac{\Delta}{2} \right],$$

$$E_0 = \tfrac{1}{2}(E_1 + E_2 - \hbar\omega_1 - \hbar\omega_2) - \frac{1}{2}\hbar\left[\sqrt{\left(\frac{\Delta}{2}\right)^2 + \overline{g_1^2} + \overline{g_2^2}} - \frac{\Delta}{2} \right],$$

$$\lambda = -\frac{g_1 g_2}{\overline{g_1^2} + \overline{g_2^2}} \left[\sqrt{\left(\frac{\Delta}{2}\right)^2 + \overline{g_1^2} + \overline{g_2^2}} - \frac{\Delta}{2} \right], \tag{31}$$

where $\overline{g_j} = g_j \sqrt{N_j}$, $j = 1,2$. Equation (29), together with Eq. (31), gives the exact transformed Hamiltonian for the case $\Delta_1 = \Delta_2 \equiv \Delta$, valid for any values of ratios g_1/Δ and g_2/Δ even for $\Delta = 0$. It is seen that the absolute value of the effective two-level coupling parameter λ is a monotonically decreasing function of the detuning, which means that the smaller the detuning, the stronger the effective coupling between the levels 1 and 2. This is a reasonable result that can be anticipated physically because the smaller the detuning, the stronger the direct couplings between levels 1 and 3 and between levels 2 and 3, and also the effective coupling between levels 1 and 2. Obviously, our results reduce to the previous ones [4–6] when $\Delta^2 \gg (\overline{g_1^2} + \overline{g_2^2})$, since, in this case,

$$\lambda \approx -\frac{g_1 g_2}{\Delta}, \quad \Delta^2 \gg (\overline{g_1^2} + \overline{g_2^2}), \tag{32}$$

as found in Refs. [4–6]. Note that this implies that the previous results [4–6] require *both* of the ratios g_1/Δ_1 and g_2/Δ_2 to be very small, and they need modifications if one of them is not small or when the pump or Stokes mode is relatively intense such that the quantity $(\overline{g_1^2} + \overline{g_2^2})$ is comparable to or greater than the square of the detuning Δ^2 even if both of the ratios g_1/Δ_1 and g_2/Δ_2 are themselves very small. Note that when $\Delta^2 \ll (\overline{g_1^2} + \overline{g_2^2})$, we find $\lambda \approx -g_1 g_2 / \sqrt{\overline{g_1^2} + \overline{g_2^2}}$, which remains finite as $\Delta \to 0$, unlike the previous result $\lambda = -g_1 g_2/\Delta$. Finally, it is pointed out that the intensity-dependent coupling occurs naturally in the transformed Hamiltonian while previous studies usually introduce it phenomenologically.

III. EIGENVALUES AND EIGENSTATES

In this section, we calculate the eigenvalues of the exact transformed Hamiltonian and express the corresponding eigenvectors in terms of the uncoupled basis $\{|j;n_1,n_2\rangle, j=1,2,3; n_1,n_2 = 0,1,2,\ldots\}$ where $|j;n_1,n_2\rangle = |j\rangle \otimes |n_1,n_2\rangle$ represents a state in which the atom is in the state $|j\rangle$, while the photonic state is represented by $|n_1,n_2\rangle$, with n_1,n_2 being the photon numbers in the two modes. For simplicity, we consider the case $\Delta_1 = \Delta_2 \equiv \Delta$. The Hamiltonian is rewritten as

$$H = H_{\rm d} + H_{\rm nd} \tag{33}$$

with

$$H_{\rm d} = E_0 + \hbar\omega_1 N_1 + \hbar\omega_2 N_2 + \tfrac{1}{2}\hbar\eta\sigma_{33},$$

$$H_{\rm nd} = \hbar\lambda(a_1^\dagger a_2 \sigma_{12} + a_2^\dagger a_1 \sigma_{21}) + \tfrac{1}{2}\hbar\omega(\sigma_{22} - \sigma_{11}), \tag{34}$$

where $N_1 = a_1^\dagger a_1 + 1 - \sigma_{11}$, $N_2 = a_2^\dagger a_2 + 1 - \sigma_{22}$, and the parameters ω, λ, η, and E_0 are given by Eq. (31). $H_{\rm d}$ and $H_{\rm nd}$ have, respectively, diagonal and nondiagonal forms in the $|j;n_1,n_2\rangle$ representation, hence the meanings of their subscripts ''d'' and ''nd.''

Using the facts that $N_j|3;n_1,n_2\rangle = (n_j+1)|3;n_1,n_2\rangle$, and $H_{nd}|3;n_1,n_2\rangle = 0$, one sees that $|3;n_1,n_2\rangle (n_2 = 0,1,2,\ldots)$ are eigenvectors of the transformed Hamiltonian, i.e., $H'|3;n_1,n_2\rangle = E_{n_1,n_2}|3;n_1,n_2\rangle$ and

$$E_{n_1,n_2} = E_1 + n_1\hbar\omega_1 + n_2\hbar\omega_2 + \hbar\left[\omega_1 + \frac{\Delta}{2} + \sqrt{\left(\frac{\Delta}{2}\right)^2 + g_1^2(n_1+1) + g_2^2(n_2+1)} \right]. \tag{35}$$

The remaining eigenvectors can be expressed as a linear combination of the vectors $|1;n_1,n_2\rangle$ and $|2;n_1-1,n_2+1\rangle$, which are the eigenvectors of N_1 with eigenvalue n_1 and of N_2 with eigenvalue (n_2+1), and they are also eigenvectors of H_d. One can therefore easily show

$$H'|\Psi^\pm_{n_1,n_2}\rangle = E^\pm_{n_1,n_2}|\Psi^\pm_{n_1,n_2}\rangle, \tag{36}$$

with

$$|\Psi^+_{n_1,n_2}\rangle = -\sin\theta|1;n_1,n_2\rangle + \cos\theta|2;n_1-1,n_2+1\rangle,$$

$$|\Psi^-_{n_1,n_2}\rangle = \cos\theta|1;n_1,n_2\rangle + \sin\theta|2;n_1-1,n_2+1\rangle, \tag{37}$$

$$E^+_{n_1,n_2} = E_1 + n_1\hbar\omega_1 + n_2\hbar\omega_2$$

$$E^-_{n_1,n_2} = E_1 + n_1\hbar\omega_1 + n_2\hbar\omega_2 - \hbar\left[\sqrt{\left(\frac{\Delta}{2}\right)^2 + g_1^2 n_1 + g_2^2(n_2+1)} - \frac{\Delta}{2} \right], \tag{38}$$

where

$$\sin\theta = \frac{g_2\sqrt{n_2+1}}{\sqrt{g_1^2 n_1 + g_2^2(n_2+1)}},$$

$$\cos\theta = \frac{g_1\sqrt{n_1}}{\sqrt{g_1^2 n_1 + g_2^2(n_2+1)}}. \tag{39}$$

It is interesting to note that the eigenstates [Eq. (37)] of the exact transformed Hamiltonian turn out to be the same as those of the approximate one given by Alexanian and Bose [6] [see their Eq. (20), noting that we have different notation]. However, the eigenvalues given by us differ, of course, from theirs because $E^{\pm}_{n_1,n_2}$ in Eq. (38) are the eigenvalues of the exact transformed Hamiltonian and they are valid for all possible ratios of g_1/Δ and g_2/Δ including zero detuning $\Delta=0$. $E^{\pm}_{n_1,n_2}$ in Eq. (38) reduces to their corresponding results as $\Delta^2 \gg [g_1^2 n_1 + g_2^2(n_2+1)]$. As noted above, the eigenvalues obtained here are also the exact eigenvalues of the original Hamiltonian given by Eq. (1) (for $\Delta_1=\Delta_2$) while theirs are not, since their unitary transformation is not exact.

IV. ATOMIC INVERSION AND RABI FREQUENCY

Since we have obtained the exact transformed Hamiltonian different from previous results, it is worthwhile to analyze the corresponding population dynamics. Again, we consider the case $\Delta_1=\Delta_2$ for simplicity. The density operator $\rho(t)$ evolves according to the formula

$$\rho(t)=\exp\left(-i\frac{H't}{\hbar}\right)\rho(0)\exp\left(i\frac{H't}{\hbar}\right), \tag{40}$$

where H' is given by Eq. (29) [or Eq. (31)], and the initial density operator $\rho(0)$ is assumed to have the form [6]

$$\rho(0)=\sum_{m_1 m_2 n_1 n_2} C_{m_1 m_2; n_1 n_2} |1;m_1,m_2\rangle\langle 1;n_1,n_2|. \tag{41}$$

The atomic inversion $W(t)$ is given by

$$W(t)=\rho^A_{22}(t)-\rho^A_{11}(t), \tag{42}$$

where ρ_{jj} are the diagonal elements of reduced density operators of the atom, i.e.,

$$\rho^A_{jj}(t)=\sum_{l_1,l_2=0}^{\infty}\langle j;l_1,l_2|\rho(t)|j;l_1,l_2\rangle, \quad j=1,2,3. \tag{43}$$

It is obvious that $\rho^A_{11}(t)+\rho^A_{22}(t)+\rho^A_{33}(t)=1$ and $\rho^A_{33}(t)=0$ because of the particular choice of the operator $\rho(0)$ in Eq. (41) and the fact that H' does not produce a transition between level 3 and the other two levels. Therefore, the atomic inversion becomes

$$W(t)=-1+2\rho^A_{22}(t). \tag{44}$$

Noting from Eq. (33) that $H'=H_{\rm d}+H_{\rm nd}$, $[H_{\rm d},H_{\rm nd}]=0$ and $|2;l_1,l_2\rangle$ is the eigenvector of $H_{\rm d}$ with eigenvalue $E_{\rm d}$ [the explicit form of $E_{\rm d}$ is not needed for calculating $\rho^A_{22}(t)$], one obtains

$$\exp\left(i\frac{H't}{\hbar}\right)|2;l_1,l_2\rangle=\exp\left(i\frac{E_{\rm d}t}{\hbar}\right)\exp\left(i\frac{H_{\rm nd}t}{\hbar}\right)|2;l_1,l_2\rangle \tag{45}$$

and hence

$$\rho^A_{22}(t)=\sum_{l_1,l_2=0}^{\infty}\langle 2;l_1,l_2|\exp\left(-i\frac{H_{\rm nd}t}{\hbar}\right)\times\rho(0)\exp\left(i\frac{H_{\rm nd}t}{\hbar}\right)|2;l_1,l_2\rangle. \tag{46}$$

Using Eq. (34), one can show

$$\exp\left(i\frac{H_{\rm nd}t}{\hbar}\right)=(1-\sigma_{33})\left(\cos(\Omega t)+i\frac{H_{\rm nd}}{\hbar}\frac{\sin(\Omega t)}{\Omega}\right), \tag{47}$$

where

$$\Omega\equiv\Omega(N_1,N_2)=\frac{1}{2}\left[\sqrt{\left(\frac{\Delta}{2}\right)^2+g_1^2N_1+g_2^2N_2}-\frac{\Delta}{2}\right]. \tag{48}$$

The quantity Ω can be called the Rabi operator, since its eigenvalues give the usual Rabi frequencies.

Substituting Eq. (47) into Eq. (46), and using the expression (41) for $\rho(0)$ and the expression of $H_{\rm nd}$ in Eq. (34) together with the parameters ω and λ given by Eq. (31), we finally obtain the expression of the atomic inversion as follows:

$$W(t)=-1+8\sum_{n_1,n_2=0}^{\infty}C_{n_1n_2;n_1n_2}\times\frac{n_1(n_2+1)g_1^2g_2^2}{[g_1^2n_1+g_2^2(n_2+1)]^2}\sin^2(t\Omega_{n_1n_2}), \tag{49}$$

where the Rabi frequency $\Omega_{n_1n_2}=\Omega(N_1=n_1,N_2=n_2+1)$, i.e.,

$$\Omega_{n_1n_2}=\frac{1}{2}\left[\sqrt{\left(\frac{\Delta}{2}\right)^2+g_1^2n_1+g_2^2(n_2+1)}-\frac{\Delta}{2}\right]. \tag{50}$$

It is perhaps not too surprising to note that the expression (49) for the atomic inversion has the same form as the one obtained by Alexanian and Bose using their approximate transformed Hamiltonian [6] except for different Rabi frequencies. The Rabi frequency we find here has an different intensity dependence than that obtained in Ref. [6], although it reduces to it correctly in the appropriate limit. However we do not obtain the $[n_1(n_2+1)]$ dependence found in the adiabatic elimination approach [4,5] in any limit. The ‘‘collapse’’ and ‘‘revival’’ times of the atomic inversion would, therefore, differ from the previous results [4–6]. The situation where the atomic inversion is a periodic function of t for particular values of coupling constants g_1 and g_2 discussed in Ref. [6] does not show up here, since the Rabi frequency here does not have a linear dependence on field intensities as does the one in Ref. [6]. Of course, the Rabi frequency here is easily seen to reduce to the one given by Alexanian and Bose [6] when $\Delta^2\gg[g_1^2n_1+g_2^2(n_2+1)]$, as it should be, and it becomes the one given in Ref. [8] when $\Delta=0$ (note that the different references have different definitions of Rabi frequency that differ from each other by a factor 2).

V. CONCLUSIONS

In this paper, we have shown that the unitary transformation introduced by Alexanian and Bose can be found without any approximation. We have obtained the exact transformed Hamiltonian and shown that one of the three levels (level 3) can be made to decouple from the other two levels and hence can be eliminated from the exact transformed Hamiltonian to obtain an effective two-level Raman interaction Hamiltonian with an intensity-dependent coupling between levels 1 and 2. The Raman interaction Hamiltonian is, within the framework of the original Hamiltonian proposed by Gerry and Eberly [4], valid for any magnitudes of the ratios of the coupling constants to detunings including zero detuning, and reduces to the previous results [4–6], when detunings are much greater than coupling constants. In addition, the Rabi frequency obtained by us becomes the previous ones as the detuning is zero [7,8] and very large compared with coupling constants [6]. It is valid, when the detuning is in between them and hence bridges the gap between these two limiting cases. We have also obtained the eigenvalues of the exact transformed Hamiltonian and thus of the original Hamiltonian given by Eq. (1). The fact that all of the relevant unitary transformations can be obtained exactly suggests that the Heisenberg equations of the corresponding operators may be calculated without any approximations. In fact, this was proven for the special case of the zero-detuning $\Delta=0$ (Ref. [8]) and is currently under investigation for the general case of nonzero detuning. Our results are a generalization of the previous ones [4–6], and might find their applications in investigating the dynamics and statistics of atomic and field quantities in situations of strong couplings (large g), small detunings, and intense field. Our results also show that the unitary transformation method proposed by Alexanian and Bose [6] not only provides an interesting alternative for the adiabatic elimination when both are suitable, but also may succeed when the adiabatic elimination approach ceases to work. It is, in our view, surprising and intriguing that there in fact exists a unitary transformation such that one level (level 3) can be totally eliminated regardless of whether or not there is far off resonance and regardless of how large the field intensities are. The fact that level 3 can be eliminated is a consequence of the exact transformation results but is hard to understand physically at first, since it seems to mean that there exist only two-photon processes or processes involving even photons while a single-photon process is totally absent. After much thought, we find out that the conclusion that one level can be made to decouple from the other two levels is, in fact, the result in dressed-state description. In other words, the level 3 which can be eliminated is the dressed level, not the original bare level 3. The single-photon processes are implicitly used to ‘‘dress’’ levels and hence only processes involving photons of even numbers are apparent and highlighted in such a description. This raises an interesting question as to why the effective two-level Hamiltonian of the dressed-state description reduces to that of the adiabatic approximation under large detuning conditions while the adiabatic elimination procedure does not seem to introduce any transformation from a bare-state description to a dressed-state one. This question will be thoroughly discussed elsewhere.

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