Right-unitary transformation theory and applications

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We develop a transformation theory in quantum physics, where the transformation operators, defined in the infinite-dimensional Hilbert space, have right-unitary inverses only. Through several theorems, we discuss the properties of state space of such operators. As one application of the right-unitary transformation (RUT), we show that using the RUT method, we can solve exactly various interactions of many-level atoms with quantized radiation fields, where the energy of atoms can be two levels, three levels in Λ, V, and $\equiv$ configurations, and up to higher (>3) levels. These interactions have wide applications in atomic physics, quantum optics, and quantum electronics. In this paper, we focus on two typical systems: one is a two-level generalized Jaynes-Cummings model, where the cavity field varies with the external source; the other one is the interaction of a three-level atom with quantized radiation fields, where the atoms have Λ-configuration energy levels, and the radiation fields are one-mode or two-mode cavities. [S1050-2947(96)06307-X]

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I. INTRODUCTION

Not only an important method, but also an intrinsic description of symmetry for the physical systems, transformation theory is always an interesting topic in physics that has acquired many studies. The transformations preserving the measurements are called *physical* transformations. We know that classical mechanics is founded on the symplectic manifold, where the transformations of momentum and position preserving the symplectomorphism are the physical transformations, which form a group with composition, called the symplectic group. These transformations are also called *canonical* transformations, because they leave both the canonical equations of motion and the Poisson brackets invariant.

Along the jump from classical theory to quantum theory, the Poisson brackets are replaced by the bosonic commutation relations. The linear transformations of bosons that leave the bosonic commutation relations invariant were first introduced by Bogoliubov [1]. These bosonic transformations were found to form the same symplectic group as that in classical mechanics, even though these two kinds of transformations have different physical meanings essentially [2]. The quantum canonical transformations of fermions were first introduced by Valatin [3] in his study of superconductivity. Since fermions are the purely quantum objects without classical correspondence, the Lie group formed by the fermionic transformations is no longer the symplectic group, but a sub-Lie group isomorphic with the *rotation* group [4]. As linear quantum transformations, the Bogoliubov-Valatin transformations have wide applications for example in the Bardeen-Cooper-Schrieffer theory [3] of superconductivity and in the calculation of black-hole radiation [5]. Recently, the supersymmetric transformation that mixes boson fields with fermion fields in a unified form advanced the transformation theory to a big stage [6]. However, some interesting physical predictions raised by the supersymmetric theory are still awaiting experiment to test. In quantum gauge theory, the Becchi-Rouet-Stora-Tyutin transformations of the gauge fields and the ghost fields are in the well-known supersymmetric forms [7], which play an important role in the renormalization proof of the standard model. In sum, the quantum transformation theory is still under development for various purposes. We notice that the above transformations are *unitary* transformations. Since quantum theory is based on Hilbert space, and the duality of the Hilbert space is defined through the Hermitian conjugate, it has been recognized that only unitary transformations do not change the Hilbert space. Nonunitary transformations cannot preserve the Hermitian duality, and they thus break the realities of physical observables and probabilities. That is why nonunitary transformations are always rejected in quantum theory [8].

In a previous paper [9], we introduced an alternative method, right-unitary transformation (RUT), to deal with the two-level Jaynes-Cummings (JC) model [10], which is a basis of the fully quantum description of radiation-matter interaction, and widely used in quantum optics, quantum electronics, etc. It was defined in Ref. [9] that if an operator U satisfies the conditions $UU^{\dagger}=1, U^{\dagger}U\neq 1$, it belongs to RUT. In a strict sense, U is a special nonunitary operator that has a right-unitary inverse only. In the matrix representation, U can only be the matrix in an infinite-dimensional space. Such an operator has been recognized as an *irregular* operator by mathematicians. However, in Ref. [9], we found that various JC models can be solved exactly by the RUT method. This method not only shows its own merits such as simplicity and general applicability, but also leads to a deep understanding of the JC models. This work further implies that the transformations applicable to quantum systems should not be restricted to unitary transformations. Instead, some nonunitary transformations which are regarded as irregular objects might have their particular utilities in physics.

We know that the unitary transformations do not change the measurements of a quantum system. But the situation is quite different when using the nonunitary transformations. In this paper, we attempt to develop a theory of the right-

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unitary transformation in mathematics, and discuss its applications in physics.

This paper is arranged as follows. In Sec. II, through six theorems, we reveal some common properties of the right-unitary transformation, and conclude a general procedure on how to apply RUT to quantum systems. Section II is the foundation of the whole work, it includes three subsections. In Sec. II A, we reveal that the state space of any operator $U \in \mathrm{RUT}$ is composed of two independent subspaces $\{|\Psi^0\rangle\}$ and $\{|\Psi^1\rangle\}$, where $\{|\Psi^0\rangle\}$ is called the *kernel* of U, which usually has a finite number of basic elements, and satisfies the equation $U\{|\Psi^0\rangle\}=0$. On the other hand, in the subspace $\{|\Psi^1\rangle\}$, U acts as a unitary operator. Similar to the unitary transformations, all the operator $U \in \mathrm{RUT}$ in the same Hilbert space form a semigroup, which is called a right-unitary semigroup. For a certain system with Hamiltonian H, through a theorem we show that if the kernel of U is checked to be isomorphic with a subset of the eigenkets of H, U can be used as a unitary transformation to the remaining subspace of H without changing the spectrum. In Sec. II B, we briefly discuss the application of RUT to the nonstationary system. In order to construct the RUT conveniently in the Fock space, we employ the recently introduced inverses $(b,b^\dagger)$ of boson creation and annihilation operators $(a^\dagger,a)$ [11] in Sec. II C, and show that the well-known quantum phase operators [12] constructed by a and $a^\dagger$ (or b and $b^\dagger$) form an Abelian subgroup of the right-unitary semigroup.

Section III attributes to one of the applications of RUT in physics, where we use the RUT method to treat the systems of many-level atoms interacting with the quantized radiation fields, where the RUT are constructed by the quantum phase operators. This section contains four subsections. In Sec. III A, we approach such a model that the atoms have two energy levels, and the radiation field is designed to be a one-mode cavity that varies with time. Through this model we hope to achieve the goal of controlling the effect such as atomic inversion of the system via the external source. This model can be regarded as the nonstationary JC model, which is found to exhibit such a property that there exists a particular relation between the atomic inversion and the energy exchange of the atomic system with the external source. Therefore, by measuring the energy exchange, we can understand the dependence of quantities such as the atomic inversion and the mean photon number on time (on the external source).

In recent years, the case of three-level atoms interacting with quantized electromagnetic field have obtained extensive studies. The model where the atoms have a Λ-configuration energy level has been applied to a number of different contexts such as the coherent population trapping, laser cooling [13], lasing without inversion [14], and electromagnetically induced transparency (EIT) [15]. In Sec. III B, we use the RUT method to solve exactly the atom-radiation interaction, where the atoms have Λ-configuration energy levels, and the radiation field is designed to be a one-mode cavity field. The case of a two-mode cavity is treated in Sec. III C, which is further divided into two parts: In Sec. III C 1, we consider such a situation that the detunings of the system are enough large, then we are able to construct a unitary transformation to separate the upper level from the system, and the remaining two low levels are still treated by the RUT method. In Sec. III C 2, we use the RUT method to solve the system exactly. In Sec. III D, we briefly show how to construct a RUT to solve the matter-radiation interactions where the atoms can be three levels in a V or $\equiv$ configuration, and up to four levels. Section IV is the conclusion.

In the present paper, we focus on the RUT method; the analyses of the quantities related to the interesting phenomena such as lasing without inversion and electromagnetically induced transparency are not given. Since there are many formulas and quantities in the paper, the meanings of the symbols are independent in each section.

II. RIGHT-UNITARY TRANSFORMATION

A. Stationary case

Consider a Hermitian operator Λ with a discrete spectrum,

$$\Lambda|\psi_n\rangle=\lambda_n|\psi_n\rangle, \quad n=1,2,\ldots,k, \tag{1}$$

the eigenkets $|\psi_n\rangle$ are orthogonal mutually, $\langle\psi_m|\psi_n\rangle=\delta_{mn}$. Let us choose an operator U to transform Λ into another frame Λ'. If Λ' is still a Hermitian quantity and has the same spectrum as Λ, we call U an *applicable transformation* to Λ. There raise two questions subsequently: How to transform Λ into Λ' by using U? Since we require that Λ' still be a Hermitian quantity, the transformation from Λ to Λ' is usually chosen as $\Lambda'=U\Lambda U^\dagger$. Then, what is the basic requirement to U? This question is indeed the crux of the transformation theory, and does not have a complete answer yet. Certainly, as we have pointed out in the Introduction, the unitary transformation is the *applicable transformation.* Here we would like to see how far we can approach beyond the unitary transformation. As it required that Λ' should have the same spectrum as Λ, it is proper to choose an extreme case, $\Lambda=1$, as an invariant in the transformation. This will result in such a requirement to U that $UU^\dagger=I$, in order to maintain the case of unity: $\Lambda=\Lambda'=1$. We know that in the infinite-dimensional Hilbert space, $UU^\dagger=I$ does not mean that U is a unitary operator. In fact, there exists the nonunitary transformation

$$\begin{aligned} UU^\dagger&=I,\\ U^\dagger U&=W\neq I. \end{aligned} \tag{2}$$

The aim of the present work is to understand how this kind of nonunitary transformation works in quantum theory. We here call the operator U satisfying Eq. (2) the right-unitary transformation, in order to distinguish it from the other nonunitary transformations. We know that RUT are the operators in the infinite-dimensional space, they can only be applied to the system having the same infinite-dimensional state space as RUT, i.e., k in Eq. (1) goes to infinity.

Before carrying a theoretical study on RUT, we first give a simple example of it: the quantum phase operators [12],

$$F_1=\frac{1}{\sqrt{aa^\dagger}}a, \quad F_1^\dagger=a^\dagger\frac{1}{\sqrt{aa^\dagger}}, \tag{3}$$

where a and $a^\dagger$ are bosonic annihilation and creation operators, respectively. It is easily calculated that

$$F_1F_1^\dagger = I, \qquad F_1^\dagger F_1 = I - |0\rangle\langle 0|, \tag{4}$$

which indicate that F_1 belongs to the right-unitary operators. This concrete example will be helpful to understand the following mathematical approach. The properties of the quantum phase operators will be carefully discussed in Sec. II C.

We now study the properties of RUT through the following several theorems.

Theorem I. For any operator $U \in \mathrm{RUT}$, $W \equiv U^\dagger U$ has a complete set of eigenkets $\{|\Psi\rangle\}$, where $\{|\Psi\rangle\}$ is constituted by two independent subsets $\{|\Psi^0\rangle\}$ and $\{|\Psi^1\rangle\}$. For the set $\{|\Psi^0\rangle\}$, U acts as an annihilation operator,

$$U|\Psi_i^0\rangle = 0, \quad |\Psi_i^0\rangle \in \{|\Psi^0\rangle\}. \tag{5}$$

For the other set $\{|\Psi^1\rangle\}$, U acts as a unitary operator, namely,

$$W|\Psi_j^1\rangle = |\Psi_j^1\rangle, \quad |\Psi_j^1\rangle \in \{|\Psi^1\rangle\}. \tag{6}$$

Proof. For any $U \in \mathrm{RUT}$, it follows from Eq. (2) that

$$U(1-W) = 0. \tag{7}$$

Equation (7) is further left-multiplied by $U^\dagger$; we have

$$W - W^2 = 0. \tag{8}$$

Equation (8) indicates that the Hermitian operator W has eigenvalues 0 and 1 only, and the corresponding eigenstates, denoted by $\{|\Psi^0\rangle\}$ and $\{|\Psi^1\rangle\}$, form a complete set, where the brackets are used to represent the case of degeneracy if it exists. For any $|\Psi_i^0\rangle \in \{|\Psi^0\rangle\}$, using Eq. (7), we obtain $U(1-W)|\Psi_i^0\rangle = U|\Psi_i^0\rangle = 0$, which proves Eq. (5). On the other hand, for any $|\Psi_j^1\rangle \in \{|\Psi^1\rangle\}$, we have $U^\dagger U|\Psi_j^1\rangle = W|\Psi_j^1\rangle = |\Psi_j^1\rangle$, namely, U acts as a unitary operator in the subspace $\{|\Psi^1\rangle\}$. QED.

As the starting point of our work, the above theorem reveals the general structure of the state space of RUT. Applying this theorem to the phase operator F_1, we have $\{|\Psi^0\rangle\} = \{|0\rangle\}$ and $\{|\Psi^1\rangle\} = \{|k\rangle, k = 1,2, \ldots, \infty\}$.

The following two corollaries are directly associated with Theorem I. First, from Eq. (5), we know the following.

Corollary I.a. For any operator $U \in \mathrm{RUT}$, there exists at least one vector $|\psi\rangle \neq 0$ that satisfies the equation $U|\psi\rangle = 0$.

Second, for an arbitrary vector $|\theta\rangle$, from

$$U^\dagger U|\theta\rangle = U^\dagger U[|\Psi^0\rangle\langle\Psi^0|\theta\rangle + |\Psi^1\rangle\langle\Psi^1|\theta\rangle] = \langle\Psi^1|\theta\rangle|\Psi^1\rangle, \tag{9}$$

(the brackets are omitted here) we obtain the following.

Corollary I.b. For $U \in \mathrm{RUT}$, $U^\dagger U$ is a projection operator that maps an arbitrary vector into the unitary subspace of U.

It is well known that all the unitary operators in the same Hilbert space form a unitary group. Similarly, we have the following group theorem for the RUT.

Theorem II. All the operators satisfying Eq. (2) in the same Hilbert space form a semigroup.

Proof. In this proof, the main task is to prove that for two arbitrary operators $U_1, U_2 \in \mathrm{RUT}$ in the same Hilbert space, that $U = U_1U_2$ should also belong to RUT. It is evident that $UU^\dagger = 1$. We now prove $U^\dagger U \neq 1$. Let $W_1 = U_1^\dagger U_1, W_2 = U_2^\dagger U_2$. Then $W \equiv U^\dagger U = U_2^\dagger W_1 U_2$. The Corollary I.a tells us that for U_2, there exists at least one ket vector $|\psi\rangle \neq 0$ to satisfy the equation $U_2|\psi\rangle = 0$. Which means $W|\psi\rangle = 0$, and $W \neq 1$. Therefore, $U \in$ RUT. The associative law is easily proved. Since all the RUT have not strict inverses, we conclude that all these RUT in the same Hilbert space form a right-unitary semigroup (RUSG). QED.

The above two theorems reveal some common properties of RUT, which are independent of the concrete structure of RUT. In order to study the structure of RUT in detail, we now introduce a useful quantity: the *kernel (K)* of the operator $U \in \mathrm{RUT}$:

$$K = \{|\Phi_i^0\rangle, i = 1,2, \ldots\}, \quad \text{for any} |\Phi_i^0\rangle \in K, \quad U|\Phi_i^0\rangle = 0. \tag{10}$$

Evidently, the subset $\{|\Psi^0\rangle\}$ in Theorem I and the arbitrary linear combinations of those elements in $\{|\Psi^0\rangle\}$ all belong to K. Therefore, K has infinite number of elements. Here we introduce another quantity, the *basic kernel*, which includes only the base ket vectors of K, namely, these ket vectors are orthogonal with each other. In most cases, the basic kernel has a finite number of elements. We sometimes simply call the basic kernel the kernel. For convenience, we usually denote $K = \{|\Phi^0\rangle\}$ or $K = |\Phi^0\rangle$ in the following presentation.

The kernel distinguishes clearly the RUT from the unitary transformation, since for any unitary transformation, K is an empty set. In a certain sense, K can be taken as a measure of the nonunitarity of the operator $U \in \mathrm{RUT}$: the fewer elements K has, the more nearly unitary is U. The latter discussion will show that K directly determines the applicability of RUT to a physical system.

Since RUT has been associated with a structure of a semigroup, we now prove the following theorem about the kernel of several RUT's product.

Theorem III. For those $U_1, U_2, \ldots, U_n, \ldots \in \mathrm{RUSG}$, the basic kernel of the operator

$$U(1,2, \ldots, n) = U_1U_2 \ldots U_n \tag{11}$$

is exactly

$$K = \{|\Phi^0(n)\rangle, U_n^\dagger|\Phi^0(n-1)\rangle, U_n^\dagger U_{n-1}^\dagger|\Phi^0(n-2)\rangle, U_n^\dagger \cdots U_3^\dagger U_2^\dagger|\Phi^0(1)\rangle\}, \tag{12}$$

where $|\Phi^0(m)\rangle$ is the basic kernel of U_m. (The bracket is omitted here.)

Proof. The proof includes three steps. First, let

$$W \equiv U^\dagger(1,2,\ldots,n)U(1,2,\ldots,n) = U_n^\dagger \cdots U_3^\dagger U_2^\dagger W_1 U_2 U_3 \cdots U_n, \quad (13)$$

using Theorem I, we obtain

$$W|\Phi^0(n)\rangle = W[U_n^\dagger|\Phi^0(n-1)\rangle] = \cdots = W[U_n^\dagger \cdots U_3^\dagger U_2^\dagger|\Phi^0(1)\rangle] = 0. \quad (14)$$

Therefore, all the elements

$$|\Phi^0(n)\rangle, U_n^\dagger|\Phi^0(n-1)\rangle, \ldots, U_n^\dagger \cdots U_3^\dagger U_2^\dagger|\Phi^0(1)\rangle$$

in Eq. (12) belong to the kernel of $U(1,2,\ldots,n)$.

For the second step, it is easily proved that all the elements in Eq. (12) are orthogonal with each other. We omit the proof here.

For the third step, we will show that an arbitrary element belonging to the kernel of U is uniquely determined by the elements in Eq. (12). Supposing an arbitrary vector $|\alpha\rangle$ that satisfies $W|\alpha\rangle = 0$, by Eq. (13), we have

$$W_1 U_2 U_3 \cdots U_n|\alpha\rangle = 0. \quad (15)$$

We conclude from the equation above that there are only two possible choices of $|\alpha\rangle$, as follows.

(i) $U_2 U_3 \cdots U_n|\alpha\rangle \in |\Phi^0(1)\rangle$. From Theorem I, we obtain that $U_3 \cdots U_n|\alpha\rangle = c_1 U_2^\dagger|\Phi^0(1)\rangle + c_2|\Phi^0(2)\rangle$, where c_i are the parameters commutative with all U_i. By this result, we further obtain $U_4 \cdots U_n|\alpha\rangle = c_1 U_3^\dagger U_2^\dagger|\Phi^0(1)\rangle + c_2 U_3^\dagger|\Phi^0(2)\rangle + c_3|\Phi^0(3)\rangle$. Following the same analysis, we eventually get to $|\alpha\rangle = c_1 U_n^\dagger \cdots U_3^\dagger U_2^\dagger|\Phi^0(1)\rangle + c_2 U_n^\dagger \cdots U_3^\dagger|\Phi^0(2)\rangle + \cdots + c_n|\Phi^0(n)\rangle$, which evidently belongs to K, where $\Sigma_{i=1}^n |c_i|^2 = 1$.

(ii) The other possibility is that $U_l \cdots U_n|\alpha\rangle = 0$, while $U_{l+1} \cdots U_n|\alpha\rangle = d_l|\Phi^0(l)\rangle \neq 0$, where $2 \leqslant l$. Using the same discussion as in (i), we obtain $|\alpha\rangle = d_l U_n^\dagger \cdots U_{l+1}^\dagger|\Phi^0(l)\rangle + \cdots + d_n|\Phi^0(n)\rangle$, which still belongs to K in Eq. (12).

The above analysis leads to a conclusion that the expression (12) uniquely determines the kernel of $U(1,2,\ldots,n)$. QED.

With the help of above several theorems, we now attempt to establish a connection between RUT and Λ in Eq. (1). It seems that the above properties of RUT are independent of Λ, even though RUT is initiated by its application to Λ. Theorem IV expresses a sufficient condition, under which we can apply RUT to Λ.

Theorem IV. If the kernel of U, $K = \{|\Phi_i\rangle\}$, is linearly isomorphic with the set $\{|\psi_l\rangle, l = 1,2,\ldots,s\}$, which is a subset of the total eigenkets $\{|\psi_n\rangle, n = 1,2,\ldots,\infty\}$ of the operator Λ in Eq. (1), then U can be taken as a unitary transformation to the remaining subspace of Λ without changing the spectrum.

Proof. We complete the proof by two steps. First, from the proposition we know that $|\psi_l\rangle = \Sigma_{i=1}^s d_{li}|\Phi_i\rangle$, $l \leqslant s$, by which we obtain further

$$U|\psi_l\rangle = 0, \quad l = 1,2,\ldots,s, \qquad U|\psi_m\rangle \neq 0, \quad m = s+1,\ldots,\infty. \quad (16)$$

The above equations indicate that the total state space of Λ can be divided into two parts, which are simply denoted as $\{|\psi_l\rangle\}$ and $\{|\psi_m\rangle\}$, respectively. The subspace $\{|\psi_l\rangle\}$ can be determined by both of the kernel of U and Λ, but the remaining subspace $\{|\psi_m\rangle\}$ is independent of K.

Second, we now deal with the remaining subspace $\{|\psi_m\rangle\}$. Let $|\psi'_m\rangle = U^\dagger U|\psi_m\rangle$. It follows from Eq. (2) that $U(|\psi'_m\rangle - |\psi_m\rangle) = 0$. Equation (14) implies

$$|\psi'_m\rangle = |\psi_m\rangle + \sum_{l=1}^s c_{ml}|\psi_l\rangle, \quad (17)$$

where c_{ml} are parameters to be determined. $|\psi'_m\rangle$ is further left-multiplied by a bra $\langle\psi_{l'}|$, $l' \leqslant s$:

$$\langle\psi_{l'}|U^\dagger U|\psi_m\rangle = \langle\psi_{l'}|\psi_m\rangle + \sum_{l=1}^s c_{ml}\langle\psi_{l'}|\psi_l\rangle, \quad (18)$$

from Eq. (16) and the orthogonality of the ket vectors $\{|\psi_n\rangle\}$ indicated in Eq. (1), we obtain $c_{ml} = 0$ in Eq. (17). Then, $|\psi'_m\rangle = U^\dagger U|\psi_m\rangle = |\psi_m\rangle$. This result evidently shows that in the subspace $\{|\psi_m\rangle\}$, U acts as a unitary operator.

It is easy to prove that $\Lambda' = U\Lambda U^\dagger$ has the same spectrum as Λ in the subspace $\{|\psi_m\rangle\}$. The new eigenkets of Λ', $|\Psi_m\rangle = U|\psi_m\rangle$, are also complete, since

$$\sum_{m=s+1}^\infty |\Psi_m\rangle\langle\Psi_m| = U\left\{1 - \sum_{l=1}^s |\psi_l\rangle\langle\psi_l|\right\}U^\dagger = UU^\dagger = 1. \quad (19)$$

Moreover, by $U^\dagger|\Psi_m\rangle = |\psi_m\rangle$, we obtain the eigenkets of Λ. QED.

The proposition in above Theorem IV that the kernel of U should be isomorphic with a subspace of Λ is very strong. How to weaken this proposition is still under investigation.

Notice that in the proof of Theorem IV, we have employed the eigenstates of Λ. Therefore, one may wonder how to use the RUT method to obtain the spectrum of Λ, provided we do not know these eigenstates at first. To answer this question, we should keep in mind that even when using the unitary transformation to solve a problem, there still is not a widely accepted rule on how to construct a unitary transformation, except some well-known problems in Ref. [2]. However, for the RUT method, based on Theorem IV, we conclude a general way on how to apply RUT to a concrete problem, there are several steps described as follows.

Consider a Hermitian operator such as Λ in Eq. (1). To solve the eigenvalue equation, we first attempt to construct a transformation U in the same Hilbert space as Λ to simplify it. For the purpose, U is a right-unitary operator, its kernel is easily obtained. Then we can directly check whether all the elements in the kernel (or the combinations of those elements

in the kernel) are the eigenstates of Λ. If they are, the kernel is called the *proper kernel*, which guarantees that we can directly follow Theorem IV to take U as a unitary transformation to transform Λ into $\Lambda' = U\Lambda U^{\dagger}$, without changing the spectrum.

We should notice that in the above steps of using a RUT, we need not know the eigenkets of Λ at first. The most important step is to construct a RUT with a *proper kernel*, as Theorem IV required. In practice, the number of the basic elements in the kernel should be as few as possible, because (i) with a smaller kernel, it is easier for us to check whether it is a proper kernel, and (ii) from Theorem I, we know that the smaller the kernel, the more nearly unitary is the RUT.

In Sec. III, we will follow the above steps to treat various matter-radiation interactions.

B. Nonstationary case

We now consider such a case that the Hamiltonian of a system represented by Λ varies with time. The evolution of the state obeys the Schrödinger equation ($\hbar = 1$)

$$\Lambda(t)|\psi(t)\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle, \quad |\psi(0)\rangle = |\psi_0\rangle. \tag{20}$$

We further assume that there exists another variable $\hat{R}$ in $\Lambda(t)$: $\Lambda(t) \equiv \Lambda(\hat{R},t)$, and the state space of $\hat{R}$ is an infinite-dimensional Hilbert space. Consider an operator $U \in$ RUT in the same Hilbert space as $\hat{R}$, and independent of time. U has a finite kernel: $K = \{|\Psi_i^0\rangle, i = 1,2, \ldots, s\}$, and a unitary subspace $\{|\Psi_i^1\rangle, i = s+1, s+2, \ldots, \infty\}$ as indicated in Theorem I. In order to apply U to the system $\Lambda(t)$, we present the following theorem, which is essentially a revision of Theorem IV.

Theorem V. For $U \in$ RUT, if all the elements or the linear combinations of those elements in the kernel of U are exactly the eigenstates of $\Lambda(t)$ in Eq. (20), i.e.,

$$\Lambda(t)|\xi_n\rangle = \lambda_n(t)|\xi_n\rangle, \quad m = 1,2, \ldots, s, \tag{21}$$

where $|\xi_n\rangle = \Sigma_{m=1}^{s} d_{nm}|\Psi_m^0\rangle$, the matrix (d_{nm}) has an inverse, and t in Eq. (21) is taken as a constant parameter, then U can be applied to the remaining subspace of $\Lambda(t)$ without changing the Schrödinger equation.

Proof. Since the matrix (d_{nm}) has an inverse, two subspaces $\{|\xi_n\rangle, n = 1,2, \ldots, s\}$ and $\{|\Psi_n^0\rangle, n = 1,2, \ldots, s\}$ are isomorphic with each other. Theorem I indicates that there is a complete set of ket vectors $\{|\xi_n\rangle, |\Psi_i^1\rangle, n = 1,2, \ldots, s; i = s+1, s+2, \ldots, \infty\}$ for $W \equiv U^{\dagger}U$, where the ket vectors are assumed to be orthogonal with each other. Then the wave function of $\Lambda(t)$ can be expressed as

$$|\psi(t)\rangle = \sum_{m=1}^{s} f_m(t)|\xi_m\rangle + \sum_{n=s+1}^{\infty} g_n(t)|\Psi_n^1\rangle, \tag{22}$$

where the time-dependent parameters $f_m(t)$ and $g_n(t)$ are to be determined. Taking this expression (22) into Eq. (20), by Eq. (21) and the orthogonality of those ket vectors, we obtain that the parameters $f_m(t)$ satisfy the equation

$$\frac{\partial}{\partial t} f_m(t) = -i\lambda_m(t) f_m(t), \quad m = 1,2, \ldots, s, \tag{23}$$

which is solved as

$$f_m(t) = f_m(0)\exp\left[-i\int_0^t \lambda_m(t')dt'\right], \quad m = 1,2, \ldots, s. \tag{24}$$

In the remaining subspace $\{|\Psi_i^1\rangle, i = s+1, s+2, \ldots, \infty\}$, since U acts as a unitary transformation, the Schrödinger equation (20) can be turned into

$$\Lambda'(t)|\psi(t)\rangle' = i\frac{\partial}{\partial t}|\psi(t)\rangle', \tag{25}$$

where $\Lambda'(t) = U\Lambda(t)U^{\dagger}$, and $|\psi(t)\rangle' = \Sigma_{l=s+1}^{\infty} g_l(t)U|\Psi_l^1\rangle$. In the transformed frame $\Lambda'(t)$, the parameters $g_n(t)$ are obtained to satisfy the equation

$$\frac{\partial}{\partial t} g_n(t) = -i\sum_{l=s+1}^{\infty} g_l(t)\langle\Psi_n^1{}'|\Lambda'(t)|\Psi_l^1\rangle', \quad n = s+1, s+2, \ldots, \infty, \tag{26}$$

where $|\Psi_l^1\rangle' = U|\Psi_l^1\rangle$. Equations (25) and (26) evidently show that U can be applied to the remaining subspace of $\Lambda(t)$. QED.

The condition (21) is the key point of this theorem, by which the total state space of $\Lambda(t)$ can be divided into two parts: one part is determined by the kernel of U; the other part can be treated by U as a unitary transformation. In the above proof, the subspace $\{|\Psi_i^1\rangle, i = s+1, s+2, \ldots, \infty\}$ has been used to expand the wave function. In fact, for a concrete problem, we do not know this subspace at first. Instead, we can directly apply U to $\Lambda(t)$ as shown in Eq. (25) to obtain this subspace, provided the condition Eq. (21) is satisfied already.

The above $U \in$ RUT is assumed to be independent of time, since $\hat{R}$ is a time-independent variable in $\Lambda(t)$. One can extend Theorem V into the case of many variables without difficulty.

C. Inverses of bosonic operators and examples of RUT

As one of the infinite-dimensional Hilbert spaces, the Fock space has been widely used in quantum physics. In order to construct RUT conveniently in the Fock space, we employ the recently introduced inverses of boson creation and annihilation operators $a^{\dagger}$ and a, $[a, a^{\dagger}] = 1$. Mehta *et al.* [11] found that $a^{\dagger}$ has a left inverse b, and a has a right inverse $b^{\dagger}$,

$$ba^{\dagger} = ab^{\dagger} = 1. \tag{27}$$

The properties of $b, b^{\dagger}$ and their applications to squeezed states, and the Möbius transformation, have obtained detailed studies in Ref. [12]. In Ref. [9], we showed that b and $b^{\dagger}$ can be formally expressed in terms of a and $a^{\dagger}$,

$$b=\left(\frac{1}{aa^{\dagger}}\right)a, \quad b^{\dagger}=a^{\dagger}\left(\frac{1}{aa^{\dagger}}\right). \tag{28}$$

From Eq. (28), we further obtain $bb^{\dagger}=1/aa^{\dagger}$. Using this relation, we arrive at an interesting result: a and $a^{\dagger}$ can be formally expressed in terms of b and $b^{\dagger}$, too,

$$a=\left(\frac{1}{bb^{\dagger}}\right)b, \quad a^{\dagger}=b^{\dagger}\left(\frac{1}{bb^{\dagger}}\right). \tag{29}$$

We know in the scheme of second quantization that an arbitrary operator can be expanded by a and $a^{\dagger}$, namely, a and $a^{\dagger}$ form a *complete operator set.* Equation (29) implies that any operator can also be expanded by b and $b^{\dagger}$. Therefore, b and $b^{\dagger}$ form a *complete operator set,* too, which is connected to that of a and $a^{\dagger}$ by the nonlinear transformations (28) or (29).

Equation (28) significantly simplifies the calculation in the representation of b and $b^{\dagger}$. Various results in Ref. [12] such as the analytic studies of b and $b^{\dagger}$ in the representation of Bargmann space are easily obtained by using Eqs. (28) and (29). For example, using Eq. (28), one can prove the following useful formulas:

$$b^{k}b^{\dagger k}=\frac{1}{a^{k}a^{\dagger k}}=\frac{1}{(N+1)(N+2)\cdots(N+k)}\equiv\frac{N!}{(N+k)!}, \tag{30}$$

$$b^{\dagger k}a^{k}=\sum_{n=0}^{\infty}b^{\dagger k}|n\rangle\langle n|a^{k}=\sum_{n=0}^{\infty}|n+k\rangle\langle n+k|$$
$$=1-\sum_{n=0}^{k-1}|n\rangle\langle n|, \tag{31}$$

where $N=a^{\dagger}a$.

We now look at the well-known operator, namely, the phase operator (phasor) F_1, as shown in Eq. (3), which is initiated from the quantization of the phase factor in quantum mechanics [12]. In this paper we will use only the photon-lowering property of these phase operators, and that the phase properties of the electromagnetic field are not calculated. The higher-order phase operators are defined as

$$F_k\equiv(F_1)^k=\frac{1}{\sqrt{a^{k}a^{\dagger k}}}a^{k}, \quad F_k^{\dagger}\equiv(F_1^{\dagger})^k=a^{\dagger k}\frac{1}{\sqrt{a^{k}a^{\dagger k}}}. \tag{32}$$

In the representation of b and $b^{\dagger}$, F_1 and $F_1^{\dagger}$ are found to have the same forms as Eq. (3),

$$F_1=\frac{1}{\sqrt{bb^{\dagger}}}b, \quad F_1^{\dagger}=b^{\dagger}\frac{1}{\sqrt{bb^{\dagger}}}. \tag{33}$$

By Eq. (30), we have

$$F_k=\frac{1}{\sqrt{b^{k}b^{\dagger k}}}b^{k}, \quad F_k^{\dagger}=b^{\dagger k}\frac{1}{\sqrt{b^{k}b^{\dagger k}}}. \tag{34}$$

With these preparations, we now prove the following theorem.

Theorem VI. All the phase operators defined by Eqs. (3) and (32) form a subgroup of the right-unitary semigroup: $\{1,F_k,k=1,2,\ldots\}\subset$ RUSG.

Proof. It is clear that $F_kF_k^{\dagger}=I$. On the other hand, using Eq. (31), we obtain

$$F_k^{\dagger}F_k=I-\sum_{n=0}^{k-1}|n\rangle\langle n|\neq I. \tag{35}$$

These mean that $F_k\in$ RUT. Moreover, $F_kF_l=F_{k+l}\in$ RUT. We therefore conclude that $\{F_k,k=1,2,\ldots,\infty\}$ form an Abelian subgroup of RUSG. The kernel of the element F_k is directly obtained as $K(F_k)=\{|0\rangle,|1\rangle,\ldots,|k-1\rangle\}$. QED.

Using this explicit example of RUT, one can easily check the theorems presented above. We are not going to dig deeper into the theory of RUT here. It is important to see how to apply RUT to the physical system.

III. APPLICATION OF RUT TO THE SYSTEM OF MANY-LEVEL ATOMS INTERACTING WITH QUANTIZED RADIATION FIELDS

Under the rotating-wave approximation, the interaction of two-level atoms with the quantized radiation field is described by the Jaynes-Cummings model [10], which has been extensively applied in quantum optics, quantum electronics, etc. Various modifications and generalizations to the original JC model have been made to approach quantum effects such as quantum collapses and revivals of atomic coherence [16], squeezing phenomenon [17], and so on. These JC models have the common attractive property that they all can be solved exactly. Since the supersymmetric structure was found to exist for the JC model, the JC model is viewed as a generalization of the supersymmetric harmonic oscillator system, and its solvability may be interpreted in terms of supersymmetric breaking [18]. Using a deformed oscillator algebra, Bonatsos *et al.* [19] gave a unified solvable formulation of various JC Hamiltonians. Yu *et al.* [20] pointed out further that there embeds an unusual su(2) algebraic structure in these JC Hamiltonians.

The methods mentioned above are valid for the two-level stationary JC models only. It is difficult to extend these methods to the time-dependent JC models, or to the case where the atoms have higher (>2) energy levels. In a previous paper [9], we employed a right-unitary transformation to solve exactly a generalized two-level JC model and pointed out that there exists a geometric phase in the model.

In this section, we will show that the RUT method developed above can be generally applied to various forms of atom-radiation interactions, where the atoms can have two, three, and higher energy levels in different configurations, and the radiation field can be a one-mode or two-mode cavity. This section includes four subsections. In Sec. III A, by constructing a RUT, we follow Theorem V to solve such a model that the two-level atoms interact with a radiation field which is dependent on time. This model is called the nonstationary JC model. The importance of three-level matter-radiation interactions has been pointed out in the Introduction. However, some interesting characteristics such as the supersymmetric structure, and the su(2) structure are not embedded in the three-level models. There still is not a gener-

ally accepted method to treat these models, according to our knowledge. In Secs. III B–III D, following Theorem IV, we will show that various interactions of many-level atoms with one- or two-mode cavities can be unitedly treated by the RUT method, where the energy of atoms can be two-level and three-level with Λ, V, and $\equiv$-configurations. The procedures of treatment are as simple as those in the two-level JC model.

A. Two-level nonstationary Jaynes-Cummings model

The approach of the stationary two-level JC models by using the RUT method can be found in Ref. [9]. In this section, we consider such a case that the radiation field interacting with the two-level atoms varies with the external source. Through this model, we hope to achieve the goal of controlling the effects such as atomic collapse and revival, and the statistics of photon number through the external source. Under the rotating-wave approximation, the Hamiltonian of the system with density $\rho(N)$-dependent multiphoton (l) interaction has the following general form ($\hbar=1$):

$$H(t)=\omega a^{\dagger}a+\frac{1}{2}\omega_0\sigma_3+a^{\dagger l}\rho(N)\gamma^{*}(t)\sigma_{-}+\rho(N)a^{l}\gamma(t)\sigma_{+}, \tag{36}$$

where ω and ω_0 are the field and atomic transition frequencies, respectively. $\sigma_{\pm}=(\sigma_1\pm i\sigma_2)/2$, where σ_1, σ_2, and σ_3 are three Pauli matrices. $\gamma(t)$ represents the change of radiation field with the external source, $\gamma(t)\neq 0$. The detuning $\Delta=\omega_0-l\omega$ should satisfy the condition $|\Delta|\ll\omega_0,\omega$, in order to preserve the reliability of the rotating-wave approximation. The evolution of the system with time is described by the Schrödinger equation

$$H(t)|\Psi(t)\rangle=i\frac{\partial}{\partial t}|\Psi(t)\rangle,\quad |\Psi(0)\rangle=|\Psi_0\rangle. \tag{37}$$

To solve this equation, we construct the following operator U:

$$U=\begin{pmatrix}1 & 0\\ 0 & F_l\end{pmatrix}, \tag{38}$$

where $F_l=(1/\sqrt{b^l b^{\dagger l}})b^l$ is the phase operator given by Eq. (34). It should be mentioned that this U is expressed in the two-dimensional representation of the Pauli matrices, where the parameters are phase operators. From Theorem VI, we know that U has the properties

$$UU^{\dagger}=I,\qquad U^{\dagger}U=\begin{pmatrix}1, & 0\\ 0, & 1-\sum_{k=0}^{l-1}|k\rangle\langle k|\end{pmatrix}, \tag{39}$$

which indicate that U belongs to RUT, and the kernel of U is

$$K=\left\{|\psi_k^0\rangle=\begin{pmatrix}0\\ |k\rangle\end{pmatrix},\quad k=0,1,\ldots,l-1\right\}. \tag{40}$$

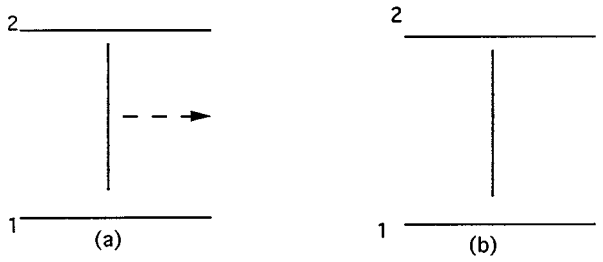

FIG. 1. (a) shows that in the frame H, the transition from the atomic state 2 to state 1 is achieved by creating photons. (b) shows that in the transformed frame H', there is no photon created or annihilated in the transition from 2 to 1.

One can check that the vectors $|\psi_k^0\rangle$, $k=0,1,\ldots,l-1$, in K are exactly the eigenstates of $H(t)$, where the eigenvalues

$$E_k^0=\omega k-\omega_0/2. \tag{41}$$

These results evidently show that U is covered by Theorem V. We now follow Theorem V to solve Eq. (37). The wave function $|\Psi(t)\rangle$ can be divided into two parts:

$$|\Psi(t)\rangle=|\Psi^0(t)\rangle+|\Psi^1(t)\rangle, \tag{42}$$

where $|\Psi^0(t)\rangle$ is related to the kernel and given by Eq. (24) as

$$|\Psi^0(t)\rangle=\sum_{k=0}^{l-1}f_k e^{-i(\omega k-\omega_0/2)t}|\psi_k^0\rangle. \tag{43}$$

The other part, $|\Psi^1(t)\rangle$, is determined by the unitary subspace of U. To obtain $|\Psi^1(t)\rangle$, we make a transformation: $H'(t)=UH(t)U^{\dagger}$, that is,

$$H'(t)=\begin{pmatrix}\omega N+\frac{1}{2}\omega_0 & g_l(N)\gamma(t)\\ g_l(N)\gamma^{*}(t) & \omega(N+l)-\frac{1}{2}\omega_0\end{pmatrix}, \tag{44}$$

where $g_l(N)=\rho(N)[(N+l)!/N!]^{\frac{1}{2}}$. The Hamiltonian now turns out to be a function of the photon number N, where the creation and annihilation of photon have been erased by the transformation U.

We know that in the original frame $H(t)$, the transition from the atomic state 2 to the state 1 is induced by the dipole term $a^{\dagger l}\rho(N)\gamma^{*}(t)$, and the transition is always achieved by creating l photons. On the other hand, the transition from state 1 to state 2 is achieved by annihilating the same l photons. However, in the new frame $H'(t)$, all of these transitions are caused not by the creating or annihilating photon. The transition induced by dipole can only happen between those states having the same number of photons, as shown in Fig. 1. Therefore, the states with different photon numbers can be treated independently.

$H'(t)$ is further rearranged into the compact form

$$H'(t)=B_0-\vec{\mu}\cdot\vec{B}(t), \tag{45}$$

where $B_0=\omega(N+l/2)$, $B_1(t)=\mathrm{Re}[\gamma(t)]g_l(N)$, $B_2(t)=-\mathrm{Im}[\gamma(t)]g_l(N)$, and $B_3=(\omega_0-l)/2$, $\vec{\mu}=-\vec{\sigma}$.

The above result [Eq. (45)] reveals that in the transformed frame, the Hamiltonian $H'(t)$ of the nonstationary JC model describes the motion of a spin-$\frac{1}{2}$ atom under an effective magnetic field $\vec{B}(t)$, which is a function of the photon number N and time t.

The evolution of subspace $|\Psi^1(t)\rangle'$ is described by the Schrödinger equation (23),

$$H'(t)|\Psi^1(t)\rangle' = i\frac{\partial}{\partial t}|\Psi^1(t)\rangle', \tag{46}$$

where

$$|\Psi^1(t)\rangle' = \sum_{k=0}^{\infty}\begin{pmatrix}\alpha_k(t)|k\rangle\\ \beta_k(t)|k\rangle\end{pmatrix},$$

the coefficients $\alpha_k(t)$ and $\beta_k(t)$ satisfy the equations

$$\begin{aligned}\dot{\alpha}_k(t) &= -i\alpha_k(t)(\omega k+\omega_0/2) - i\beta_k(t)g_l(k)\gamma(t),\\ \dot{\beta}_k(t) &= -i\beta_k(t)[\omega(k+l)+\omega_0/2] - i\alpha_k(t)g_l(k)\gamma^*(t),\end{aligned} \tag{47}$$

which are further arranged into

$$\begin{aligned}\ddot{\alpha}_k(t)+\dot{\alpha}_k(t)A'_k[l,\gamma(t)]+\alpha_k(t)B'_k[l,\gamma(t)] &= 0,\\ \ddot{\beta}_k(t)+\dot{\beta}_k(t)A'_k[l,\gamma^*(t)]+\beta_k(t)B'_k[l,\gamma^*(t)] &= 0,\end{aligned} \tag{48}$$

where

$$\begin{aligned}A'_k[l,\gamma(t)] &= i\omega(2k+l)-\dot{\gamma}(t)/\gamma(t),\\ B'_k[l,\gamma(t)] &= g_l^2(k)|\gamma(t)|^2 - i(\omega k+\omega_0/2)\dot{\gamma}(t)/\gamma(t)\\ &\quad -\omega^2(k^2+lk)-\omega\omega_0/2+\omega_0^2/4.\end{aligned}$$

For a given $\gamma(t)$, by solving the above two equations, one can obtain the evolution of the system with time. For instance, we choose a special case,

$$\gamma(t) = e^{i\lambda t}, \tag{49}$$

to approach the dependence of some physical properties in this model on λ. With this particular $\gamma(t)$, $\alpha_k(t)$ and $\beta_k(t)$ are solved to be

$$\begin{aligned}\alpha_k(t) &= \alpha_k^+ e^{i\omega_k^+ t}+\alpha_k^- e^{i\omega_k^- t},\\ \beta_k(t) &= -h_k\alpha_k^+ e^{i(\omega_k^+-\lambda)t}+h_k^{-1}\alpha_k^- e^{i(\omega_k^--\lambda)t},\end{aligned} \tag{50}$$

where

$$\begin{aligned}\omega_k^{\pm} &= -\frac{1}{2}[\omega(2k+l)-\lambda]\pm\sqrt{\frac{1}{4}(\omega_0-\omega l+\lambda)^2+[g_l(k)]^2},\\ h_k &= \left(\omega k+\frac{\omega_0}{2}+\omega_k^+\right)\Big/ g_l(k),\end{aligned} \tag{51}$$

and the coefficients $\alpha_k^{\pm}$ and $\beta_k^{\pm}$ are determined by the initial state in the transformed frame:

$$|\Psi^1(0)\rangle' = \sum_{k=0}^{\infty}\begin{pmatrix}(\alpha_k^+ + \alpha_k^-)|k\rangle\\ (-h_k\alpha_k^+ + h_k^{-1}\alpha_k^-)|k\rangle\end{pmatrix}. \tag{52}$$

If the initial state in the transformed frame is chosen as

$$|\Psi^1(0)\rangle' = \sum_{k=0}^{\infty}\begin{pmatrix}\mu_k|k\rangle\\ \nu_k|k\rangle\end{pmatrix}, \tag{53}$$

then the coefficients $\alpha_k^{\pm}$ and μ_k, ν_k are related with each other by the equations

$$\begin{aligned}\alpha_k^+ &= \frac{1}{1+h_k^2}(\mu_k - h_k\nu_k),\\ \alpha_k^- &= \frac{h_k}{1+h_k^2}(h_k\mu_k+\nu_k).\end{aligned} \tag{54}$$

For convenience, we still use the initial state as Eq. (52) in the later calculations.

Since the initial state is arbitrarily chosen, Eq. (50) is in fact the general solution of the nonstationary JC model in the transformed frame. From Theorem V, we know that the solution can be directly turned into the *untransformed* frame by

$$|\Psi^1(t)\rangle = U^{\dagger}|\Psi^1(t)\rangle'. \tag{55}$$

Combining this solution and the kernel part in Eq. (43), we obtain the exact solution of the nonstationary JC model for an arbitrary initial state as

$$\begin{aligned}|\Psi(t)\rangle = &\sum_{n=0}^{l-1} f_n e^{-i(\omega n-\omega_0/2)t}\begin{pmatrix}0\\ |n\rangle\end{pmatrix}\\ &+\sum_{k=0}^{\infty}\begin{pmatrix}[\alpha_k^+ e^{i\omega_k^+ t}+\alpha_k^- e^{i\omega_k^- t}]|k\rangle\\ [-h_k\alpha_k^+ e^{i\omega_k^+ t}+h_k^{-1}\alpha_k^- e^{i\omega_k^- t}]e^{-i\lambda t}|k+l\rangle\end{pmatrix}.\end{aligned} \tag{56}$$

Using the normalization condition $\langle\Psi(t)|\Psi(t)\rangle = 1$, we know that all the coefficients should satisfy the equation

$$\sum_{n=0}^{l-1}|f_n|^2+\sum_{k=0}^{\infty}[(1+h_k^2)|\alpha_k^+|^2+(1+h_k^{-2})|\alpha_k^-|^2] = 1. \tag{57}$$

The above solution (56) shows that the kernel term is independent of $\gamma(t)$, thereby a trivial term. Our interest is in the second term in the solution (56), because this term is not only related to $\gamma(t)$, but also determined by the unitary subspace of U. We now evaluate the following physical quantities associated with the nonstationary JC model by using RUT.

1. Statistics of the photon number

Let $\overline{n^{\pm}}(t) = \langle\Psi(t)|(\sigma_0\pm\sigma_3)N/2|\Psi(t)\rangle$, where σ_0 is a 2×2 unit matrix, and σ_3 is the third Pauli matrix. For $\overline{n^+}$, since $U[(\sigma_0+\sigma_3)N/2]U^{\dagger} = (\sigma_0+\sigma_3)N/2$, we can treat it by RUT in the transformed frame,

$$\overline{n^+}(t)=[\langle\Psi^1(t)|U^\dagger](\sigma_0+\sigma_3)N/2[U|\Psi^1(t)\rangle]$$
$$=\sum_{k=0}^{\infty} k[|\alpha_k^+|^2+|\alpha_k^-|^2+\xi_k(t)], \quad (58)$$

where

$$\xi_k(t)=\alpha_k^+{}^*\alpha_k^- e^{-i(\omega_k^+-\omega_k^-)t}+\alpha_k^+\alpha_k^-{}^* e^{i(\omega_k^+-\omega_k^-)t}. \quad (59)$$

By the same way, we have

$$\overline{n^-}(t)=\sum_{n=0}^{l-1} n|f_n|^2+\sum_{k=0}^{\infty}(k+l)[|\alpha_k^+|^2h_k^2$$
$$+|\alpha_k^-|^2h_k^{-2}-\xi_k(t)]. \quad (60)$$

It is proper to refer to $\overline{n^+}(t)$ and $\overline{n^-}(t)$ as the mean photon number of the field for the cases where the atom is found in the excited and ground states, respectively. $\xi_k(t)$ is an important quantity in this model, it measures the collapses and revivals of the atomic inversion in a single mode with the frequency

$$\Delta_k\equiv\omega_k^+-\omega_k^-=\sqrt{(\omega_0-\omega l+\lambda)^2+4[g_l(k)]^2}. \quad (61)$$

Δ_k is usually called Rabi frequency, where the parameter λ obviously affects the Rabi frequency.

2. Atomic inversion

The atomic inversion in the JC model is illustrated by the quantity $\overline{\sigma_3}=\langle\Psi(t)|\sigma_3|\Psi(t)\rangle$. If we take the two-level atoms as the neutro particles with spin $\frac{1}{2}$, then $\overline{\sigma_3}/2$ measures the mean spin value which varies with time. Since σ_3 is invariant under U, namely, $U\sigma_3U^\dagger=\sigma_3$, we can directly calculate this quantity in the transformed frame. Using the normalization condition Eq. (57), we obtain

$$\overline{\sigma_3}=2\sum_{k=0}^{\infty}[|\alpha_k^+|^2+|\alpha_k^-|^2+\xi_k(t)]-1. \quad (62)$$

This result shows that the atomic inversion is measured by the same $\xi_k(t)$ as that appears in the statistics of photon number. In order to evaluate explicitly the effect of λ in the nonstationary JC model, we choose the following special initial state without the kernel part for an example:

$$|\Psi(0)\rangle=\sum_{k=0}^{\infty} e^{-|z|^2/2}\begin{pmatrix}\frac{z^k}{\sqrt{2k!}}|k\rangle\\ \frac{z^k}{\sqrt{2k!}}|k+l\rangle\end{pmatrix}. \quad (63)$$

For this initial state, we have (see Appendix)

$$\xi_k(t)=-\left(\frac{x^ke^{-x}}{k!}\right)\frac{(\Delta+\lambda)g_l(k)}{(\Delta+\lambda)^2+4g_l^2(k)}$$
$$\times\cos[\sqrt{(\Delta+\lambda)^2+4g_l^2(k)}t], \quad (64)$$

$$|\alpha_k^+|^2+|\alpha_k^-|^2=\frac{x^ke^{-x}}{k!}\left[\frac{1}{2}+\frac{(\Delta+\lambda)g_l(k)}{(\Delta+\lambda)^2+4g_l^2(k)}\right], \quad (65)$$

where the detuning $\Delta=\omega_0-\omega l$, $x=|z|^2$. With these results, $\overline{\sigma_3}$ is obtained to be

$$\overline{\sigma_3}=\sum_{k=0}^{\infty}\left(\frac{x^ke^{-x}}{k!}\right)\frac{4(\Delta+\lambda)g_l(k)}{(\Delta+\lambda)^2+4g_l^2(k)}$$
$$\times\sin^2\left[\frac{1}{2}\sqrt{(\Delta+\lambda)^2+4g_l^2(k)}t\right]. \quad (66)$$

Equation (66) expresses exactly the collapse and revival of the atomic coherence in this nonstationary model, where λ plays the same role as the detuning Δ. By changing λ, we can effectively control the atomic inversion of the system, thus meet our original purpose of studying the properties of the atomic system via the external source. If it is designed: $\lambda\approx-\Delta$, the atomic inversion disappears for the initial state (63). On the other hand, even if $\Delta=0$, from Eq. (66) we know that the atomic inversion still exists for the external source $\lambda\neq0$,

$$\overline{\sigma_3}=\sum_{k=0}^{\infty}\left(\frac{x^ke^{-x}}{k!}\right)\frac{4\lambda g_l(k)}{\lambda^2+4g_l^2(k)}\sin^2\left[\frac{1}{2}\sqrt{\lambda^2+4g_l^2(k)}t\right]. \quad (67)$$

3. Energy

The mean energy of the system is $\overline{E(t)}=\langle\Psi(t)|H(t)|\Psi(t)\rangle$. For the state (56), we divide $\overline{E(t)}$ into three terms,

$$\overline{E(t)}=\langle\Psi^0(t)|H(t)|\Psi^0(t)\rangle+[\langle\Psi^1(t)|H(t)|\Psi^0(t)\rangle+\text{H.c.}]$$
$$+\langle\Psi^1(t)|H(t)|\Psi^1(t)\rangle, \quad (68)$$

where the first term in $\overline{E(t)}$ is easily obtained to be

$$\overline{E^0}(t)=\sum_{n=0}^{l-1}(\omega n-\omega_0/2)|f_n|^2. \quad (69)$$

From Theorem I, we know that the kernel are orthogonal to the unitary subspace, which implies that the second term in above $\overline{E(t)}$ is zero. The third term can be treated by RUT as

$$\overline{E^1}(t)=[\langle\Psi^1(t)|U^\dagger][UH(t)U^\dagger][U|\Psi^1(t)\rangle]. \quad (70)$$

A simple calculation gives

$$\overline{E^1}(t)=\sum_{k=0}^{\infty}\{|\alpha_k^+|^2[h_k^2\lambda-(1+h_k^2)\omega_k^+]$$
$$+|\alpha_k^-|^2[\lambda h_k^{-2}-(1+h_k^{-2})\omega_k^-]-\lambda\xi_k(t)\}. \quad (71)$$

The variation of the mean energy $\overline{E(t)}$ with time is still dominated by $\xi_k(t)$, which measures the energy exchange of the radiation-matter system with the external source.

From Eqs. (62) and (71), we abstract the following interesting relation between the mean energy and the atomic inversion:

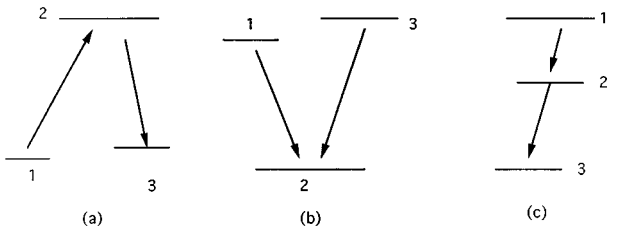

FIG. 2. (a) represents the transitions of the atoms with Λ configuration energy levels. (b) indicates the V configuration, and (c) the $\equiv$ configuration.

$$\frac{d\overline{E}}{dt}=-\frac{\lambda}{2}\frac{d\overline{\sigma_3}}{dt}. \tag{72}$$

This relation expresses such a distinct property of this model that by measuring the change of energy with time, we can obtain the information such as the dependence of the atomic inversion and the mean photon number on time. However, this property does not mean that the atomic inversion is caused by the external source. In fact, when $\lambda=0$, $\overline{E}$ turns out to be independent of time, but the atomic inversion shown in Eq. (67) still exists generally.

B. Three-level atoms interacting with one-mode cavity

We now concentrate on the three-level atom-radiation systems. The three-level atoms are classified by the configurations of their energy levels. Generally, there are three kinds of configurations, Λ, V, and $\equiv$, as shown in Fig. 2, where the atoms with Λ-configuration energy level have been widely used in the subjects such as the coherent population trapping, laser cooling [13], lasing without inversion [14], and electromagnetically induced transparency (EIT) [15].

We know that the exact solvability of various two-level JC models is attributed to a unified formulation of JC models by a deformed oscillator algebra [19], or by a su(2) structure [20]. However, there is not a unified solvable formulation to these three-level systems. Starting from this section, we will show that by the RUT method, we can solve the various three-level models mentioned above. To avoid making the paper too long, almost all the physical properties associated with the solutions are not discussed in this paper.

In this section, we consider a system of three-level atoms with energies ε_1, ε_2, and ε_3 in the Λ configuration, which interacts with a one-mode cavity field as shown in Fig. 2(a). The interaction is considered to be density $[\rho_1(N),\rho_2(N)]$-dependent, and in the multiphoton (l_1,l_2) form. Under the rotating-wave approximation, the Hamiltonian of the system is written as

$$H=\sum_{i=1}^{3}\varepsilon_i S_{ii}+\omega a^{\dagger}a+[a^{\dagger l_1}\rho_1(N)S_{12}+\rho_2(N)a^{l_2}S_{23}+\text{H.c.}], \tag{73}$$

where ω is the frequency of the radiation field, S_{ij} are the atomic operators given by $S_{ij}=|i\rangle\langle j|$, $i,j=1,2,3$. We notice that various three-level Hamiltonians with one-mode cavity in the literature are covered by this general one. To solve this Hamiltonian, we now introduce the following operator U:

$$U=\begin{pmatrix} F_{l_1} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & F_{l_2}\end{pmatrix}, \tag{74}$$

where F_{l_1} and F_{l_2} are two phase operators. From Theorem VI, we know that the operator U belongs to RUT. The kernel of U is

$$K=\left\{ |\psi_1^0(k_1)\rangle=\begin{pmatrix}|k_1\rangle\\0\\0\end{pmatrix},\quad |\psi_2^0(k_2\rangle=\begin{pmatrix}0\\0\\|k_2\rangle\end{pmatrix},\right.$$
$$\left. k_1<l_1,\quad k_2<l_2\right\}. \tag{75}$$

Theorem IV tells us that if this kernel is isomorphic to a subset of the eigenkets of H, then the operator U can be applied to H. It is easily checked that above $|\psi_1^0(k_1)\rangle$ and $|\psi_2^0(k_2)\rangle$ happen to be the eigenkets of H. The eigenvalues corresponding to these eigenkets are

$$\begin{aligned}E_1^0(k_1)&=\varepsilon_1+\omega k_1,\\ E_2^0(k_2)&=\varepsilon_3+\omega k_2,\end{aligned} \tag{76}$$

where we have assumed that the energy spectrum is nondegenerate between two sets $\{|\psi_1^0(k_1)\rangle\}$ and $\{|\psi_2^0(k_2)\rangle\}$. These results indicate that the operator U is covered by Theorem IV, and can be applied to the remaining eigenket set of H besides those in Eq. (75).

Let $H'=UHU^{\dagger}$; a direct calculation gives

$$H'=\frac{1}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3)+\omega\left(N+\frac{l_1+l_1}{3}\right)+H_1, \tag{77}$$

where H_1 is

$$H_1=\begin{pmatrix} h_1 & g_1(N) & 0\\ g_1(N) & -(h_1+h_2) & g_2(N)\\ 0 & g_2(N) & h_2\end{pmatrix}, \tag{78}$$

where h_1 and h_2 are two constants: $h_1=(2\varepsilon_1-\varepsilon_2-\varepsilon_3)/3+(2l_1-l_2)\omega/3$, $h_2=(2\varepsilon_3-\varepsilon_1-\varepsilon_2)/3+(2l_2-l_1)\omega/3$; and $g_1(N)=\rho_1(N)[(N+l_1)!/N!]^{1/2}$, $g_2(N)=\rho_2(N)[(N+l_2)!/N!]^{1/2}$. Notice that H_1 is constructed to be a traceless operator matrix; in this form it is easier to compute its eigenvalues.

The above results show that in the transformed frame, the Hamiltonian of the three-level matter-radiation interaction system becomes a 3×3 matrix which depends on the photon number N only. For the ket vector with a single photon number, this Hamiltonian simply describes the usual three-level stationary system, which can be solved exactly. We note that in the transformed frame, the transitions induced by the dipole occur only between the states containing the same photon number. This picture is similar that in the two-level system.

Assuming the eigenket of H_1 as

$$|\Psi(n)\rangle' = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \otimes |n\rangle, \tag{79}$$

where $\beta_i \equiv \beta_i(n)$, $i=1,2,3$. The eigenvalue λ of H_1 is determined by the equation $\det|H_1-\lambda|=0$, that is,

$$\lambda^3 + \lambda p + q = 0, \tag{80}$$

where

$$\begin{aligned} p &= -[g_1^2(n)+g_2^2(n)+h_1^2+h_2^2+h_1h_2], \\ q &= h_1 g_1^2(n) + h_2 g_1^2(n) + (h_1+h_2)h_1h_2. \end{aligned} \tag{81}$$

The solutions of Eq. (80) are given by [21]

$$\begin{aligned} \lambda_1(n) &= A_+ + A_-, \\ \lambda_{2,3}(n) &= \frac{1}{2}(A_+ + A_-) \pm \frac{\sqrt{3}i}{2}(A_+ - A_-), \end{aligned} \tag{82}$$

where $A_\pm \equiv A_\pm(n)$,

$$A_\pm = \sqrt[3]{\frac{1}{2}q \pm \left(\frac{1}{27}p^3 + \frac{1}{4}q^2\right)^{1/2}}. \tag{83}$$

Combining these λ_i with the diagonal term in H', we obtain the eigenvalues of the Hamiltonian as

$$\begin{aligned} E_1(n) &= \tfrac{1}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3) + \omega\left(n + \frac{l_1+l_1}{3}\right) + \lambda_1, \\ E_{2,3}(n) &= \tfrac{1}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3) + \omega\left(n + \frac{l_1+l_1}{3}\right) + \lambda_{2,3}. \end{aligned} \tag{84}$$

We assume that the energy spectrum is nondegenerate; then the eigenkets of H' corresponding to the above eigenvalues are orthogonal mutually, which are obtained to be

$$|\Psi_i(n)\rangle' = \frac{1}{\zeta_i(n)} \begin{pmatrix} \dfrac{g_1(n)}{\lambda_i - h_1} \\ 1 \\ \dfrac{g_2(n)}{\lambda_i - h_2} \end{pmatrix} \otimes |n\rangle, \quad i=1,2,3, \tag{85}$$

where $\zeta_i(n)$ are the normalization factors:

$$\zeta_i(n) = \left[1 + \left(\frac{g_1(n)}{\lambda_i - h_1}\right)^2 + \left(\frac{g_2(n)}{\lambda_i - h_2}\right)^2\right]^{1/2}.$$

The above solutions of the three-level system with the Λ configuration is in the transformed frame. From Theorem IV, we can directly obtain the eigenkets in the original frame H by the equation $|\Psi_i(n)\rangle = U^\dagger |\Psi_i(n)\rangle'$, that is,

$$|\Psi_i(n)\rangle = \frac{1}{\zeta_i(n)} \begin{pmatrix} \left[\dfrac{g_1(n)}{\lambda_i - h_1}\right] |n+l_1\rangle \\ |n\rangle \\ \left[\dfrac{g_2(n)}{\lambda_i - h_1}\right] |n+l_2\rangle \end{pmatrix}, \quad i=1,2,3. \tag{86}$$

For an arbitrary initial state $|\varphi(0)\rangle$, the evolution of the state with time becomes

$$\begin{aligned} |\varphi(t)\rangle = &\sum_{k=0}^{l_1-1} \sum_{i=1}^{2} C_i(k) e^{-iE_i^0(k)t} |\psi_i^0(k)\rangle \\ &+ \sum_{n=0}^{\infty} \sum_{j=1}^{3} D_j(n) e^{-iE_j(n)t} |\Psi_j(n)\rangle, \end{aligned} \tag{87}$$

where $C_i(k) = \langle \psi_i^0(k) | \varphi(0)\rangle$ and $D_j(n) = \langle \Psi_j(n) | \varphi(0)\rangle$.

Now we conclude that the total eigenstate set of the three-level atom-radiation system is constituted by two subsets as in Eqs. (75) and (86), where the subset in Eq. (75) is the kernel of U, but for the other subset Eq. (86), U acts as a unitary operator.

We recall that for the two-level JC model, there generally exists a relation $H' \sim \vec{\sigma}\cdot\vec{B}$. However, for the above three-level system, from the expression of H', we know that the system cannot be taken as a spin-1 particle interacting with an external magnetic field, that is, the relation such as $H' \sim \vec{S}\cdot\vec{B}$ does not exist here. Therefore, there is no concept of spin in the Λ-configuration system, and many methods developed for the two-level JC model do not fit for the three-level case. However, for the RUT method, one may notice that the procedure in solving the three-level model is exactly the same as the procedure presented in Ref. [9] in solving the two-level JC model.

To investigate the dynamics of the above system, we usually use the method of density matrix. Supposing at $t=0$ the density operator as

$$\rho(0) = \sum_{k_1,k_2=0}^{\infty} |\varphi(k_1)\rangle\langle\varphi'(k_2)|. \tag{88}$$

Using Eq. (87), one can directly obtain the evolution of the density operator with time by the equation $\rho(t) = e^{-iHt}\rho(0)e^{iHt}$. With $\rho(t)$, one can calculate some quantities such as the atomic inversion, population trapping in this model. We now choose such a special initial state that the atoms are in the level $i=1$, and the photon state is represented by the coherent state. Then $\rho(0)$ is written into

$$\rho(0) = \sum_{k_1,k_2=0}^{\infty} \frac{e^{-|z|^2} z^{k_1} z^{*k_2}}{\sqrt{(k_1!k_2!)}} |k_1\rangle\langle k_2| S_{11}. \tag{89}$$

Making use of Eq. (87), we obtain

$$\rho(t) = |Z(t)\rangle\langle Z(t)|, \tag{90}$$

where

$$|Z(t)\rangle=\sum_{k=0}^{l_1-1}\frac{e^{-(|z|^2/2)}z^k}{\sqrt{k!}}e^{-iE_1^0(k)t}|\psi_1^0(k)\rangle + \sum_{n=0}^{\infty}\sum_{j=1}^{3}\frac{e^{-(|z|^2/2)}z^{n+l_1}g_1(n)e^{-iE_j(n)t}}{\sqrt{(n+l_1)!}\zeta_j(n)[\lambda_j(n)-h_1]}|\Psi_j(n)\rangle. \tag{91}$$

Using $\rho(t)$, we can obtain the probability of finding the atom in the state $i=2$ which is initially in the state $i=1$: $P_{1\to2}(t)=\Sigma_{k=0}^{\infty}\langle i=2,k|\rho(t)|k,i=2\rangle$,

$$P_{1\to2}(t)=\sum_{k=0}^{\infty}\frac{e^{-x}x^{k+l_1}}{(k+l_1)!}\left|\sum_{i=1}^{3}\frac{g_1(k)e^{-i\lambda_i(k)t}}{\zeta_i^2(k)[\lambda_i(k)-h_1]}\right|^2, \tag{92}$$

where $x=|z|^2$. We know that the above transition $(1\leftrightarrow2)$ is induced by dipole. A more interesting quantity is $P_{1\to3}(t)$, which is induced by two dipoles: $(1\leftrightarrow2)$ and $(2\leftrightarrow3)$,

$$P_{1\to3}(t)=\sum_{k}\frac{e^{-x}x^{k+l_1}}{(k+l_1)!}\times\left|\sum_{i=1}^{3}\frac{g_1(k)g_2(k)e^{-i\lambda_i(k)t}}{\zeta_i^2(k)[\lambda_i(k)-h_1][\lambda_i(k)-h_2]}\right|^2. \tag{93}$$

The statistics of the photon number at the state $i=3$ is obtained to be

$$\overline{n_3}(t)=\sum_{k}\frac{e^{-x}x^{k+l_1}}{(k+l_1)!}\left|\sum_{i=1}^{3}\frac{g_1(k)g_2(k)(k+l_2)e^{-i\lambda_i(k)t}}{\zeta_i^2(k)[\lambda_i(k)-h_1][\lambda_i(k)-h_2]}\right|^2. \tag{94}$$

The atomic inversion of the system under the special initial state is $\overline{S_3}(t)\equiv\langle S_3\rangle$, which is explicitly given by

$$\overline{S_3}(t)=\sum_{k}\frac{e^{-x}x^{k+l_1}}{(k+l_1)!}\left|\sum_{i=1}^{3}\frac{g_1^2(k)e^{-i\lambda_i(k)t}}{\zeta_i^2(k)[\lambda_i(k)-h_1]^2}\right|^2-P_{1\to3}(t). \tag{95}$$

C. Three-level atoms interacting with a two-mode cavity field

In this section, we consider a system of three-level atoms with energies ε_1, ε_2, and ε_3, which interacts with a two-mode cavity field: a pump mode $(a_1,a_1^\dagger)$ of frequency ω_1 and a Stokes mode $(a_2,a_2^\dagger)$ of frequency ω_2 as shown in Fig. 2(a). The interactions are generalized to be multiphoton (l_1,l_2) forms, and dependent on densities $\rho_1(N_1),\rho_2(N_2)$, respectively. Therefore, the Hamiltonian is

$$H=H_0+H_1, \tag{96}$$

where

$$H_0=\sum_{i=1}^{3}\varepsilon_iS_{ii}+\omega_1a_1^\dagger a_1+\omega_2a_2^\dagger a_2, \qquad H_1=a_1^{\dagger l_1}\rho_1(N_1)S_{12}+\rho_2(N_2)a_2^{l_2}S_{23}+\text{H.c.} \tag{97}$$

This Hamiltonian covers various special cases in the literature. In the above Sec. III B we have shown that the solution of three-level case is much more complicated than that of the two-level case. To avoid the complication, one usually treats H by the perturbation theory in the interaction picture. Especially in the case that the detunings of the two modes are very large, the upper level can be eliminated adiabatically from the three-level system; then the system is reduced into a simple two-level case with two quantized modes [22], where the Stark shift terms appear and give arise to some interesting physical effects [23].

We divide this section into two parts. In the first part, we consider the case that the detunings of the system are large enough. Then by constructing a unitary transformation, we separate the upper level from the system to the first-order approximation; the remaining two low levels are still treated by the RUT method. We further discuss the relation between the so-called ''dressed'' states and ''bare'' states from the viewpoint of transformation. In the second part, we use the same RUT method to solve the system exactly.

1. Approximate treatment in the case of large detunings

In the case that the detunings of two modes are very large, we accordingly introduce the following unitary transformation Q,

$$Q=\exp(X), \tag{98}$$

where

$$X=[-\delta_1a_1^{\dagger l_1}\rho_1(N_1)S_{12}+\delta_2\rho_2(N_2)a_2^{l_2}S_{23}]-\text{H.c.}, \tag{99}$$

where δ_1 and δ_2 are two parameters to be determined. Under Q, the Hamiltonian (96) is transformed into $H'=QHQ^\dagger$. Using the Baker-Hausdorff formula, H' becomes

$$H'=H_0+H_1+[X,H_0]+[X,H_1]+\frac{1}{2}[X,[X,H_0]] + \frac{1}{2}[X,[X,H_1]]+\cdots. \tag{100}$$

If we choose $\delta_1=\Delta_1^{-1}\equiv(\varepsilon_2-\varepsilon_1-\omega_1l_1)^{-1}$, $\delta_2=\Delta_2^{-1}\equiv(\varepsilon_2-\varepsilon_3-\omega_2l_2)^{-1}$, where Δ_1 and Δ_2 are two detunings, then

$$[X,H_0]=-H_1. \tag{101}$$

For the large Δ_i, $i=1,2$, we expand H' up to the first order of $1/\Delta_i$ as

$$H'\approx H_0+[X,H_1]+\frac{1}{2}[X,[X,H_0]]=H_0+\frac{1}{2}[X,H_1], \tag{102}$$

that is,

$$H'=\sum_{i=1}^{2}\omega_i a_i^{\dagger}a_i+\left[\varepsilon_1-\frac{1}{\Delta_1}\rho_1^2(N_1-l_1)a_1^{\dagger l_1}a_1^{l_1}\right]S_{11}+\left[\varepsilon_3-\frac{1}{\Delta_2}\rho_2^2(N_2-l_2)a_2^{\dagger l_2}a_2^{l_2}\right]S_{33}$$
$$-\left\{\left[\frac{1}{2}\left(\frac{1}{\Delta_1}+\frac{1}{\Delta_2}\right)\rho_1(N_1-l_1)\rho_2(N_2)a_1^{\dagger l_1}a_2^{l_2}\right]S_{13}+\text{H.c.}\right\}+\left[\varepsilon_2+\frac{1}{\Delta_1}\rho_1^2(N_1)a_1^{l_1}a_1^{\dagger l_1}+\frac{1}{\Delta_2}\rho_2^2(N_2)a_2^{l_2}a_2^{\dagger l_2}\right]S_{22}. \tag{103}$$

We have two comments on the above results.

(i) The Hamiltonian H' indicates that if the atomic states are in the form

$$|\varphi\rangle'=\begin{pmatrix}|\varphi_1\rangle\\0\\|\varphi_3\rangle\end{pmatrix}, \tag{104}$$

the term about S_{22} in H' does not contribute. Then the system is reduced into a two-level atomic system. The two terms $-(1/\Delta_1)\rho_1^2(N_1-l_1)a_1^{\dagger l_1}a_1^{l_1}S_{11}$ and $-(1/\Delta_2)\rho_2^2(N_2-l_2)a_2^{\dagger l_2}a_2^{l_2}S_{33}$ appearing in the Hamiltonian are called Stark effect terms, which are in terms of the photon numbers.

(ii) We regard the operators in the original frame H as ''bare'' operators, and the atomic states as ''bare'' states; then the operators in the transformed frame H' are ''dressed'' operators, and the atomic states are ''dressed'' states. The state (104) is in fact a dressed state. We know that all the states prepared in experiment are bare states. Therefore, turned back to experiment, the bare state corresponding to Eq. (104) is

$$|\varphi\rangle=e^{-X}|\varphi\rangle'\approx\begin{pmatrix}|\varphi_1\rangle\\\dfrac{\rho_1(N_1)}{\Delta_1}a_1^{l_1}|\varphi_1\rangle+\dfrac{\rho_2(N_2)}{\Delta_2}a_2^{l_2}|\varphi_3\rangle\\|\psi_3\rangle\end{pmatrix}. \tag{105}$$

This result shows that the atomic state $i=2$ still contributes in the original system.

We now solve the Hamiltonian H' by the RUT method developed above. Consider an operator V,

$$V=\begin{pmatrix}F_{l_1}(1)&0&0\\0&1&0\\0&0&F_{l_2}(2)\end{pmatrix}, \tag{106}$$

where $F_{l_1}(1)=[a_1^{l_1}a_1^{\dagger l_1}]^{-1/2}a_1^{l_1}$, $F_{l_2}(2)=[a_2^{l_2}a_2^{\dagger l_2}]^{-1/2}a_2^{l_2}$ are two phase operators. Thus V belongs to RUT, and its kernel is

$$K=\left\{|\psi_1^0(k_1,k_2)\rangle=\begin{pmatrix}|k_1,k_2\rangle\\0\\0\end{pmatrix},\quad |\psi_2^0(k_1',k_2')\rangle=\begin{pmatrix}0\\0\\|k_1',k_2'\rangle\end{pmatrix},\quad k_1<l_1,k_2'<l_2\right\}. \tag{107}$$

We obtain that $|\psi_1^0(k_1,k_2)\rangle$ are the eigenkets of H'; the eigenvalues are

$$E_1^0(k_1,k_2)=\varepsilon_1+\omega_1k_1+\omega_2k_2; \tag{108}$$

$|\psi_2(k_1',k_2')\rangle$ are the eigenkets of H' with the eigenvalues

$$E_2^0(k_1',k_2')=\varepsilon_3+\omega_1k_1'+\omega_2k_2', \tag{109}$$

where we have assumed that there is no degeneracy between $E_1^0(k_1,k_2)$ and $E_2^0(k_1',k_2')$. These results indicate that V is covered by Theorem IV, and can be applied to H' to obtain the remaining subspace besides Eq. (107). We make the transformation

$$H''=VH'V^{\dagger}, \tag{110}$$

that is,

$$H''=B_0(N_1,N_2)(S_{11}+S_{33})+B_3(N_1,N_2)(S_{11}-S_{33})$$
$$+B_1(N_1,N_2)(S_{13}+S_{31})+B_4(N_1,N_2)S_{22}, \tag{111}$$

where $B_i(N_1,N_2)$, $i=0,1,3,4$, are given by

$$B_0(N_1,N_2)=\frac{1}{2}(\varepsilon_1+\varepsilon_3)+\sum_{i=1}^{2}\left[\omega_i\left(N_i+\frac{l_i}{2}\right)-\frac{1}{2\Delta_i}g_i^2(N_i)\right], \tag{112}$$

$$B_1(N_1,N_2)=-\frac{1}{2}\left(\frac{1}{\Delta_1}+\frac{1}{\Delta_2}\right)g_1(N_1)g_2(N_2),$$

$$B_3(N_1,N_2)=\frac{1}{2}(\varepsilon_1-\varepsilon_3)$$
$$+\frac{1}{2}\left[\omega_1l_1-\omega_2l_2+\frac{g_1^2(N_1)}{\Delta_1}-\frac{g_2^2(N_2)}{\Delta_2}\right],$$

$$B_4(N_1,N_2)=\varepsilon_2+\sum_{i=1}^{2}\left[\omega_i N_i+\frac{1}{\Delta_i}g_i^2(N_i)\right],$$

where $g_i(N_i)=\rho_i(N_i)[(N_i+l_i)!/N_i!]^{1/2}$, $i=1,2$. The Hamiltonian H'' is a function of N_1 and N_2, and composed of two independent parts:

$$H_1''\equiv B_0(N_1,N_2)(S_{11}+S_{33})+B_3(N_1,N_2)(S_{11}-S_{33})+B_1(N_1,N_2)(S_{13}+S_{31})$$

is a two-level system containing a two-mode cavity; the other part is $H_2''\equiv B_4(N_1,N_2)S_{22}$. The eigenkets of H_1'' are directly written as

$$|\Psi^+(k_1,k_2)\rangle''=\begin{pmatrix}\cos\frac{\theta}{2}|k_1,k_2\rangle\\0\\\sin\frac{\theta}{2}|k_1,k_2\rangle\end{pmatrix},$$
$$|\Psi^-(k_1,k_2)\rangle''=\begin{pmatrix}-\sin\frac{\theta}{2}|k_1,k_2\rangle\\0\\\cos\frac{\theta}{2}|k_1,k_2\rangle\end{pmatrix}, \tag{113}$$

where

$$\theta\equiv\theta(k_1,k_2)=\cos^{-1}\{B_3(k_1,k_2)/[B_1^2(k_1,k_2)+B_3^2(k_1,k_2)]^{1/2}\}.$$

The eigenvalues corresponding to these two kets are, respectively,

$$E^{\pm}(k_1,k_2)=B_0(k_1,k_2)\pm[B_1^2(k_1,k_2)+B_3^2(k_1,k_2)]^{1/2}. \tag{114}$$

The third eigenket set of H'' is determined by H_2'',

$$|\Psi_3(k_1,k_2)\rangle''=\begin{pmatrix}0\\|k_1,k_2\rangle\\0\end{pmatrix}. \tag{115}$$

The eigenvalues are simply

$$E_3(k_1,k_2)=B_4(k_1,k_2). \tag{116}$$

Using $V^\dagger$, we can turn these ket vectors to the original frame H'. $|\Psi_3(k_1,k_2)\rangle''$ are invariant under $V^\dagger$. But $|\Psi^{\pm}(k_1,k_2)\rangle''$ turn out to be

$$|\Psi^+(k_1,k_2)\rangle'=\begin{pmatrix}\cos\frac{\theta}{2}|k_1+l_1,k_2\rangle\\0\\\sin\frac{\theta}{2}|k_1,k_2+l_2\rangle\end{pmatrix}, \tag{117}$$

$$|\Psi^-(k_1,k_2)\rangle'=\begin{pmatrix}-\sin\frac{\theta}{2}|k_1+l_1,k_2\rangle\\0\\\cos\frac{\theta}{2}|k_1,k_2+l_2\rangle\end{pmatrix}.$$

The results (114) and (117) indicate the existence of Rabi frequency

$$\omega(k_1,k_2)=[B_1^2(k_1,k_2)+B_3^2(k_1,k_2)]^{1/2} \tag{118}$$

in the system H'. Therefore, some interesting effects appearing in the usual two-level system can also appear here. However, the above comment (ii) has indicated that these ket vectors (117) are in fact the dressed ket vectors, induced by the unitary transformation Q as Eq. (98). For the viewpoint of experiment, it is meaningful to find these ket vectors in the bare form. Up to the first-order approximation, we can use Eq. (105), and find that $|\Psi_3(k_1,k_2)\rangle''$ maintain the same form, but the vectors $|\Psi^{\pm}(k_1,k_2)\rangle'$ turn out to be

$$|\Psi^+(k_1,k_2)\rangle=\begin{pmatrix}\cos\frac{\theta}{2}|k_1+l_1,k_2\rangle\\\chi_1|k_1,k_2\rangle\\\sin\frac{\theta}{2}|k_1,k_2+l_2\rangle\end{pmatrix},$$
$$|\Psi^-(k_1,k_2)\rangle=\begin{pmatrix}-\sin\frac{\theta}{2}|k_1+l_1,k_2\rangle\\\chi_2|k_1,k_2\rangle\\\cos\frac{\theta}{2}|k_1,k_2+l_2\rangle\end{pmatrix}, \tag{119}$$

where

$$\chi_1=\frac{\rho_1(k_1)}{\Delta_1}\sqrt{\frac{(k_1+l_1)!}{k_1}}\cos\frac{\theta}{2}+\frac{\rho_1(k_2)}{\Delta_2}\sqrt{\frac{(k_2+l_2)!}{k_2}}\sin\frac{\theta}{2},$$
$$\chi_2=-\frac{\rho_1(k_1)}{\Delta_1}\sqrt{\frac{(k_1+l_1)!}{k_1}}\sin\frac{\theta}{2}+\frac{\rho_1(k_2)}{\Delta_2}\sqrt{\frac{(k_2+l_2)!}{k_2}}\cos\frac{\theta}{2}. \tag{120}$$

The normalization factors are not included in the above ket vectors. Now we can conclude that only for the ket vectors as Eq. (119), the Rabi oscillation can appear in experiment.

Recall that H' results from the first-order approximation of H under the condition that $\langle g_1^2(N_1)/\Delta_1\rangle,\langle g_2^2(N_2)/\Delta_2\rangle\ll\Delta_1,\Delta_2$. We now consider the following two special cases to reduce the above eigenvalues Eq. (114).

First, for the case that is often used in the literature,

$$\Delta_1=\Delta_2=\Delta, \tag{121}$$

these $E^{\pm}(k_1,k_2)$ are reduced to

$$E^{\pm}(k_1,k_2)=\frac{1}{2}(\varepsilon_1+\varepsilon_3)+\sum_{i=1}^{2}\left[\omega_i\left(k_i+\frac{l_i}{2}\right)-\frac{1}{2\Delta}g_i^2(k_i)\right]\pm\left|\frac{g_1^2(k_1)-g_2^2(k_2)}{2\Delta}\right|. \tag{122}$$

The second case is contrary to Eq. (121) such that

$$\left\langle\frac{g_1^2(N_1)}{\Delta_1}\right\rangle,\left\langle\frac{g_2^2(N_2)}{\Delta_2}\right\rangle\ll|\Delta_1-\Delta_2|; \tag{123}$$

we may accordingly reduce $E^{\pm}(k_1,k_2)$ into the form

$$E^{\pm}(k_1,k_2)=\frac{1}{2}(\varepsilon_1+\varepsilon_3)+\sum_{i=1}^{2}\left[\omega_i\left(k_i+\frac{l_i}{2}\right)-\frac{1}{2\Delta_i}g_i^2(k_i)\right]\pm\frac{1}{2}\left[\Delta_2-\Delta_1+\frac{g_1^2(k_1)}{\Delta_1}-\frac{g_2^2(k_2)}{\Delta_2}+\left(\frac{1}{\Delta_1}+\frac{1}{\Delta_2}\right)\frac{g_1(k_1)g_2(k_2)}{2(\Delta_2-\Delta_1)}\right]. \tag{124}$$

2. Exact solution

We know that the approximate solution to H in the above section is based on the condition

$$\left\langle\frac{g_1^2(N_1)}{\Delta_1}\right\rangle,\left\langle\frac{g_2^2(N_2)}{\Delta_2}\right\rangle\ll\Delta_1,\Delta_2. \tag{125}$$

However, the quantities $g_i(N_i)\equiv\rho_i(N_i)[(N_i+l_i)!/N_i!]^{1/2}$ $i=1,2$, increase with the photon numbers N_i for certain densities, which means that the condition (125) does not hold for the states of large photon numbers. In this section, we will solve the Hamiltonian (96) exactly.

We employ again the right-unitary operator V in Eq. (106) as a transformation. It is easily checked that the kernel of V given by Eq. (107) is still a set of the eigenkets of H. Since the kernel is invariant under the unitary transformation Q [Eq. (98)], the eigenvalues of H corresponding to these eigenkets are still the same as those of H' [Eq. (100)]. Based on these results, we can use V as a unitary transformation to H to obtain its remaining subspace. Let

$$\overline{H}=VHV^{\dagger}. \tag{126}$$

A direct calculation gives

$$\overline{H}=\overline{H_0}+\overline{H_1}, \tag{127}$$

where

$$\overline{H_0}=\sum_{i=1}^{3}\varepsilon_i/3+\sum_{j=1}^{2}\omega_j(N_j+l_j/3), \tag{128}$$

and $\overline{H_1}$ is a traceless matrix,

$$\overline{H_1}=\begin{pmatrix} f_1 & g_1(N_1) & 0\\ g_1(N_1) & -(f_1+f_2) & g_2(N_2)\\ 0 & g_2(N_2) & f_2\end{pmatrix}, \tag{129}$$

where $g_1(N_1)$ and $g_2(N_2)$ are given in Eq. (112), and

$$f_1=\tfrac{1}{3}(2\varepsilon_1-\varepsilon_2-\varepsilon_3+2\omega_1l_1-\omega_2l_2)=\tfrac{1}{3}(\Delta_2-2\Delta_1),\qquad f_2=\tfrac{1}{3}(2\varepsilon_3-\varepsilon_1-\varepsilon_2+2\omega_2l_2-\omega_1l_1)=\tfrac{1}{3}(\Delta_1-2\Delta_2). \tag{130}$$

The Hamiltonian now becomes a function of photon numbers N_1 and N_2, where the creation and annihilation of the photon in the transitions between the atomic states are erased by the transformation V. Therefore, in the new frame, the transitions can only happen between the atomic states having the same photon number N_1 (and N_2). For a ket vector with photon numbers k_1 and k_2

$$|\Psi(k_1,k_2)\rangle'=\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\otimes|k_1,k_2\rangle, \tag{131}$$

where $\beta_i\equiv\beta_i(k_1,k_2)$, $i=1,2,3$, $\overline{H}$ becomes the usual stationary three-level system, where the eigenvalues of $\overline{H_1}$ are determined by the equation $\det|\overline{H_1}-\lambda|=0$, namely,

$$\lambda^3+\lambda p_1+q_1=0, \tag{132}$$

where

$$p_1=-[g_1^2(k_1)+g_2^2(k_2)+f_1^2+f_2^2+f_1f_2],\qquad q_1=f_1g_2^2(k_2)+f_2g_1^2(k_1)+(f_1+f_2)f_1f_2. \tag{133}$$

The solutions of this equation are [21]

$$\lambda_1\equiv\lambda_1(k_1,k_2)=C_++C_-,\qquad \lambda_{2,3}\equiv\lambda_{2,3}(k_1,k_2)=\frac{1}{2}(C_++C_-)\pm\frac{\sqrt{3}i}{2}(C_+-C_-), \tag{134}$$

where $C_{\pm}\equiv C_{\pm}(k_1,k_2)$ are given by

$$C_{\pm}=\sqrt[3]{\frac{1}{2}q_1\pm\left(\frac{1}{27}p_1^3+\frac{1}{4}q_1^2\right)^{1/2}}. \tag{135}$$

Combining the above λ_i with the term $\overline{H_0}$ in $\overline{H}$, we obtain the exact eigenvalues of the three-level system,

$$E_1(k_1,k_2)=\sum_{i=1}^{3}\varepsilon_i/3+\sum_{j=1}^{2}\omega_j(k_j+l_j/3)+\lambda_1,\qquad E_{2,3}(k_1,k_2)=\sum_{i=1}^{3}\varepsilon_i/3+\sum_{j=1}^{2}\omega_j(k_j+l_j/3)+\lambda_{2,3}. \tag{136}$$

We assume that there is no degeneracy in the energy spectrum. Then, the eigenstates of H corresponding to the above eigenvalues are orthogonal mutually, which are given by

$$|\Psi_i(k_1,k_2)\rangle' = \frac{1}{\xi_i}\begin{pmatrix} \frac{g_1}{\lambda_i - f_1} \\ 1 \\ \frac{g_2}{\lambda_i - f_2} \end{pmatrix} \otimes |k_1,k_2\rangle, \quad i=1,2,3, \tag{137}$$

where ξ_i are the normalization factors:

$$\xi_i = \left[1 + \left(\frac{g_1}{\lambda_i - f_1}\right)^2 + \left(\frac{g_2}{\lambda_i - f_2}\right)^2\right]^{1/2}.$$

Using Theorem IV, we obtain the eigenkets in the original frame as

$$|\Psi_i(k_1,k_2)\rangle = \frac{1}{\xi_i}\begin{pmatrix} \left(\frac{g_1}{\lambda_i - f_1}\right)|k_1+l_1,k_2\rangle \\ |k_1,k_2\rangle \\ \left(\frac{g_2}{\lambda_i - f_2}\right)|k_1,k_2+l_2\rangle \end{pmatrix}, \quad i=1,2,3. \tag{138}$$

Up to now, we have solved exactly the system of Λ-configuration atoms interacting with a two-mode cavity field by the RUT method. One may notice that the above procedure is almost the same as that in Sec. III B; this means that to the RUT method, that the system contains one-mode cavity or two-mode cavity has no difference, even though the physical meanings of solutions are quite different. Using the expression of f_1 and f_2, one can expand the above solution according to the condition of large detunings, and compare the result with the approximate solution obtained in preceding section. Based on the above solution, one can further study the physical effects in this model.

In the literature, one studies the three-level system where the cavity fields change with time, i.e., the nonstationary case. We should point out that one can follow Theorem V to treat this nonstationary system, and the procedure is similar to that in Sec. III A.

D. Application of RUT to the atoms with other configurations

The above approach concentrates on the system where the atoms have Λ-configuration energy levels only. Recently, V-configuration atoms were found to exhibit some interesting effects [24]. In principle, under the condition that the detunings are very large, a system of V-configuration atoms interacting with cavity fields can be treated by the same way as the above approximate treatment to Λ-configuration atoms. Unfortunately, for the $\equiv$-configuration atomic system, this treatment becomes invalid.

However, as we have pointed out, the detunings are not always very large for various atomic systems. Moreover, $g_i(N_i)$ in Eq. (112) always increase with the photon number N_i, but the detunings are invariant with N_i. Therefore, the perturbation treatment introduced above is valid only for the situation of large detunings and low photon number state. One may notice that the exact treatment of three-level atomic system by the RUT method is evidently simpler than the perturbation treatment, and suitable to various situations. In this section, we briefly show how to apply the RUT method to the systems of V- and $\equiv$-configuration atoms.

We first look at a system of V-configuration atoms interacting with a one-mode cavity field, shown in Fig. 2(b). The generalized Hamiltonian is

$$H = \sum_{i=1}^{3} \varepsilon_i S_{ii} + \omega a^\dagger a + [\rho_1(N) a^{l_1} S_{12} + a^{\dagger l_2}\rho_2(N) S_{23} + \text{H.c.}], \tag{139}$$

where we assume $l_1 \leqslant l_2$. To solve this Hamiltonian, we here construct an operator U,

$$U = \begin{pmatrix} F_{l_2-l_1} & 0 & 0 \\ 0 & F_{l_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{140}$$

where $F_{l_2-l_1}$ and F_{l_2} are two phaser operators. U belongs to RUT, and the kernel of U is

$$K = \left\{ |\psi_1^0(k_1)\rangle = \begin{pmatrix} |k_1\rangle \\ 0 \\ 0 \end{pmatrix}, \quad |\psi_2^0(k_2)\rangle = \begin{pmatrix} 0 \\ |k_2\rangle \\ 0 \end{pmatrix}, \right.$$
$$\left. k_1 < l_2 - l_1, \quad k_2 < l_2 \right\}. \tag{141}$$

We can take $|\Psi\rangle = \chi_1|\psi_1^0(k_1)\rangle + \chi_2|\psi_2^0(k_2)\rangle$ to check whether $|\Psi\rangle$ is the eigenket of H. Without difficulty, we find that the following sets of vectors,

$$|\psi_1(k_1)\rangle = \begin{pmatrix} \cos\frac{\theta_{k_1}}{2}|k_1\rangle \\ \sin\frac{\theta_{k_1}}{2}|k_1+l_1\rangle \\ 0 \end{pmatrix},$$

$$|\psi_2(k_1)\rangle = \begin{pmatrix} -\sin\frac{\theta_{k_1}}{2}|k_1\rangle \\ \cos\frac{\theta_{k_1}}{2}|k_1+l_1\rangle \\ 0 \end{pmatrix},$$

$$|\psi_3(k_2)\rangle = \begin{pmatrix} 0 \\ |k_2\rangle \\ 0 \end{pmatrix}, \tag{142}$$

are the eigenkets of H, where $k_1 < l_2 - l_1, k_2 < l_1$, $\theta_{k_1} = \tan^{-1}[2g_{l_1}(k_1)/(\varepsilon_1 - \varepsilon_2 - \omega l_1)]$, and $g_{l_1}(k_1) = \rho_1(k_1)[(k_1+l_1)!/k_1!]^{1/2}$. The eigenvalues corresponding to above eigenkets are obtained to be

$$E_1^0(k_1)=\tfrac{1}{2}(\varepsilon_1+\varepsilon_2+\omega l_1)+\omega k_1 + \sqrt{\tfrac{1}{4}(\varepsilon_1-\varepsilon_2-\omega l_1)^2+g_{l_1}(k_1)^2},$$
$$E_2^0(k_1)=\tfrac{1}{2}(\varepsilon_1+\varepsilon_2+\omega l_1)+\omega k_1 - \sqrt{\tfrac{1}{4}(\varepsilon_1-\varepsilon_2-\omega l_1)^2+g_{l_1}(k_1)^2},$$
$$E_3^0(k_2)=\varepsilon_2+\omega k_2 . \tag{143}$$

On the other hand, one can check that each element in the kernel can be written as a linear combination of eigenkets in Eq. (141). These results indicate that U can be applied to H to obtain its remaining subspace. A simple calculation gives

$$H'=UHU^\dagger = \begin{pmatrix} \varepsilon_1+\omega(N+l_2-l_1) & g_{l_1}(N+l_2-l_1) & 0 \\ g_{l_1}(N+l_2-l_1) & \varepsilon_2+\omega(N+l_2) & g_{l_2}(N) \\ 0 & g_{l_2}(N) & \varepsilon_3+\omega N \end{pmatrix}. \tag{144}$$

The Hamiltonian now becomes a matrix function of photon number N. One can follow the procedure in the above sections to obtain the eigenvalues and eigenkets of H'. We omit these here.

We now look at a system of three-level atoms with a $\equiv$-configuration energy level, which interact with a one-mode cavity as shown in Fig. 2(c). The Hamiltonian is

$$H=\sum_{i=1}^{3}\varepsilon_i S_{ii}+\omega a^\dagger a+[\rho_1(N)a^{l_1}S_{12}+\rho_2(N)a^{l_2}S_{23}+\text{H.c.}]. \tag{145}$$

To solve the Hamiltonian, we introduce an operator matrix V as

$$V=\begin{pmatrix} 1 & 0 & 0 \\ 0 & F_{l_1} & 0 \\ 0 & 0 & F_{l_1+l_2} \end{pmatrix}, \tag{146}$$

V evidently belongs to RUT. The kernel of V is

$$K=\left\{ |\psi_1^0(k_1)\rangle=\begin{pmatrix} 0 \\ |k_1\rangle \\ 0 \end{pmatrix}, \quad |\psi_2^0(k_2)\rangle=\begin{pmatrix} 0 \\ 0 \\ |k_2\rangle \end{pmatrix}, \right. \left. k_1\leqslant l_1, \quad k_2\leqslant l_1+l_2 \right\}. \tag{147}$$

Using the same method as in the above case of the V configuration, we obtain that this kernel is isomorphic with to subset of the eigenkets of H, where the eigenkets are

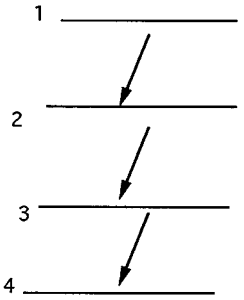

FIG. 3. Four-level atoms with ladder-configuration energies.

$$|\psi_1(k_1)\rangle=\begin{pmatrix} 0 \\ \cos\frac{\theta_{k_1}}{2}|k_1\rangle \\ \sin\frac{\theta_{k_1}}{2}|k_1+l_2\rangle \end{pmatrix},$$
$$|\psi_2(k_1)\rangle=\begin{pmatrix} 0 \\ -\sin\frac{\theta_{k_1}}{2}|k_1\rangle \\ \cos\frac{\theta_{k_1}}{2}|k_1+l_2\rangle \end{pmatrix}, |\psi_3(k_2)\rangle=\begin{pmatrix} 0 \\ 0 \\ |k_2\rangle \end{pmatrix}, \tag{148}$$

where $k_1<l_1$, $k_2<l_2$, and $\theta_{k_1}=\tan^{-1}[2g_{l_2}(k_1)/(\varepsilon_2-\varepsilon_3-\omega l_2)]$. The eigenvalues corresponding to above eigenkets are

$$E_1^0(k_1)=\tfrac{1}{2}(\varepsilon_2+\varepsilon_3+\omega l_2)+\omega k_1 + \sqrt{\tfrac{1}{4}(\varepsilon_2-\varepsilon_3-\omega l_2)^2+g_{l_2}(k_1)^2},$$
$$E_2^0(k_1)=\tfrac{1}{2}(\varepsilon_2+\varepsilon_3+\omega l_2)+\omega k_1 - \sqrt{\tfrac{1}{4}(\varepsilon_2-\varepsilon_3-\omega l_2)^2+g_{l_2}(k_1)^2},$$
$$E_3^0(k_2)=\varepsilon_3+\omega k_2 . \tag{149}$$

One can prove that an arbitrary element in the kernel K can be expressed by the above $|\psi_1(k)\rangle$, $|\psi_2(k)\rangle$, and $|\psi_3(k)\rangle$ in linear form. This means that K is isomorphic with one set of the eigenkets of H. Therefore, the operator U satisfies Theorem IV, and can be applied to the remaining eigenket set of H. We obtain

$$H'=VHV^\dagger = \begin{pmatrix} \varepsilon_1+\omega N & g_{l_1}(N) & 0 \\ g_{l_1}(N) & \varepsilon_2+\omega(N+l_1) & g_{l_2}(N+l_1) \\ 0 & g_{l_2}(N+l_1) & \varepsilon_3+\omega(N+l_1+l_2) \end{pmatrix}. \tag{150}$$

H' is a matrix whose matrix elements are functions of photon number N. H' can be solved by the regular way. We omit these here.

The above approach shows that all the Λ-, V-, and $\equiv$-configuration atoms interacting with quantized cavity fields can be treated unitedly by RUT method, where the right-unitary operators are simply diagonal matrices constructed by phaser operators.

In the last part of this section, we would like to show how to apply the RUT method to the atom-radiation interaction system where the atoms have a higher (>3) energy level. For an example, we here choose a simple case: four-level atoms with the configuration shown in Fig. 3. Under the rotating-wave approximation, the generalized Hamiltonian of the atoms interacting with one-mode cavity field is

$$H=\sum_{i=1}^{4}\varepsilon_i S_{ii}+\omega a^{\dagger}a + [\rho_1 a^{l_1}S_{12}+\rho_2 a^{l_2}S_{23}+\rho_3 a^{l_1}S_{34}+\text{H.c.}]. \quad (151)$$

For simplicity, we assume ρ_i, $i=1,2,3$, be constants. To solve this Hamiltonian, we construct a right unitary operator as

$$V_4=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & F_{l_1} & 0 & 0 \\ 0 & 0 & F_{l_1+l_2} & 0 \\ 0 & 0 & 0 & F_{l_1+l_2+l_3} \end{pmatrix}. \quad (152)$$

The kernel of V_4 is

$$K=\left\{ |\psi_1^0(k_1)\rangle=\begin{pmatrix} 0 \\ |k_1\rangle \\ 0 \\ 0 \end{pmatrix}, \quad |\psi_2^0(k_2)\rangle=\begin{pmatrix} 0 \\ 0 \\ |k_2\rangle \\ 0 \end{pmatrix}, \quad |\psi_3^0(k_3)\rangle=\begin{pmatrix} 0 \\ 0 \\ 0 \\ |k_3\rangle \end{pmatrix} \right\}, \quad (153)$$

where $k_1 \leqslant l_1, k_2 \leqslant l_1+l_2, k_3 \leqslant l_1+l_2+l_3$. Within these ket vectors, the four-level system is reduced into a three-level case. One can further check that these ket vectors are isomorphic with a subset of the eigenkets of H. Which means that we can use V_4 as a unitary operator to the other subspace of H. Let $H'=V_4 H V_4^{\dagger}$; then

$$H'=\begin{pmatrix} \varepsilon_1+\omega N & g_1(N) & 0 & 0 \\ g_1(N) & \varepsilon_2+\omega(N+l_1) & g_2(N) & 0 \\ 0 & g_2(N) & \varepsilon_3+\omega(N+l_1+l_2) & g_3(N) \\ 0 & 0 & g_3(N) & \varepsilon_4+\omega(N+l_1+l_2+l_3) \end{pmatrix}, \quad (154)$$

where

$$g_1(N)=\rho_1[(N+l_1)!/N!]^{1/2},$$

$$g_2(N)=\rho_2[(N+l_1+l_2)!/(N+l_1)!]^{1/2},$$

$$g_3(N)=\rho_2[(N+l_1+l_2+l_3)!/(N+l_1+l_2)!]^{1/2}.$$

For the ket with a fixed photon number N, H' is simply a 4×4 constant matrix. Thus, its eigenkets and eigenvalues are easily obtained.

One can further follow this method to solve other configurations of four-level systems. These are omitted here.

IV. CONCLUSION

In conclusion, we have developed the right-unitary transformation theory, and initially discussed its applications in physics. The first part of this paper discusses to the theory. We found that the state space of any operator $U \in \text{RUT}$ (precisely W) is composed of two independent parts, $\{|\Psi^0\rangle\}$ and $\{|\Psi^1\rangle\}$, where $\{|\Psi^0\rangle\}$ is called the *kernel* of U, which satisfies $U\{|\Psi^0\rangle\}=0$. On the other hand, in the subspace $\{|\Psi^1\rangle\}$, U acts as a unitary operator. The properties of RUT such as semigroup, kernel, etc., were discussed through several theorems. Based on these properties, we concluded a general way on how to apply the RUT to a physical system. For a physical quantity such as the Hamiltonian H, suppose its eigenstates as $S=\{|\Phi_i\rangle, i=1,2,\ldots,\infty\}$; then

$$H=\sum_{i=1}^{\infty} E_i |\Phi_i\rangle\langle\Phi_i|, \quad (155)$$

where E_i are eigenvalues. If a subset of S: $S_1=\{|\Phi_i\rangle, i=1,2,\ldots,s\}$ is checked to be isomorphic with the kernel of U, then the supplement set of S_1 is evidently isomorphic with the unitary subspace of U. Therefore,

$$H'=UHU^{\dagger}=\sum_{i=s+1}^{\infty} E_i |\Phi_i'\rangle\langle\Phi_i'|, \quad (156)$$

where $|\Phi_i'\rangle=U|\Phi_i\rangle$. The state space $S'=\{|\Phi_i'\rangle, i=s+1,\ldots,\infty\}$ is proved to be complete. Equation (156) shows that the frame H' has the same spectrum (precisely a subset of the spectrum) as H.

Based on the above results, in the second part of this paper we used the RUT method to deal with the systems of

many-level atoms interacting with the quantized radiation fields, where the RUT are the matrices constructed by the well-known phase operators. We have studied two typical systems: One involves the Jaynes-Cummings models, which were found to exhibit some interesting effects, and have obtained much study in recent years. We solved a nonstationary generalized JC model, and found that atomic inversion of the system can be controlled through the external source. Another system carefully studied is the interaction of the three-level atoms with one- or two-mode cavity field. This system has been widely applied in various contexts of quantum optics such as lasing without inversion, electromagnetically induced transparency, etc. This paper provides a unified method for these topics.

We would like to point out that the RUT method can be applied to some simplified quantum electrodynamic (QED) system, such as the photon-electron and phonon-electron interactions. It will be discussed in a forthcoming presentation.

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APPENDIX

For the initial state, Eq. (63),

$$\mu_k=\nu_k=e^{-|z|^2/2}\frac{z^k}{\sqrt{2k!}}\equiv f(z). \tag{A1}$$

Then,

$$\alpha_k^+=\left(\frac{1-h_k}{1+h_k^2}\right)f(z),\quad \alpha_k^-=\left(\frac{h_k+h_k^2}{1+h_k^2}\right)f(z). \tag{A2}$$

From Eq. (51), we have

$$h_k=\frac{\omega_k^+ +\omega k+\frac{\omega_0}{2}}{g_l(k)}=\frac{-g_l(k)}{\omega_k^- +\omega k+\frac{\omega_0}{2}} \tag{A3}$$

and

$$h_k^2=-\frac{\omega_k^+ +\omega k+\frac{\omega_0}{2}}{\omega_k^- +\omega k+\frac{\omega_0}{2}}. \tag{A4}$$

Using these results, we rearrange $\alpha_k^{\pm}$ as

$$\alpha_k^+=\left(\frac{\omega_k^- +\omega k+\frac{\omega_0}{2}+g_l(k)}{\omega_k^- -\omega_k^+}\right)f(z),$$

$$\alpha_k^-=-\left(\frac{\omega_k^+ +\omega k+\frac{\omega_0}{2}+g_l(k)}{\omega_k^- -\omega_k^+}\right)f(z). \tag{A5}$$

With these expressions, we directly obtain Eqs. (64) and (65).

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