Scalar curvature factor in the Schrödinger equation and scattering on a curved surface

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The scattering of free particles constrained to move on a cylindrically symmetric curved surface is studied. The nontrivial geometry of the space contributes to the scattering cross section through the kinetic as well as a possible scalar curvature term in the quantum Hamiltonian. The coefficient of the latter term is known to be related to the factor-ordering problem in curved-space quantization. Hence, in principle, the scattering data may be used to provide an experimental resolution of the theoretical factor-ordering ambiguity. To demonstrate the sensitivity required of such an experimental setup, the effect of a localized magnetic field in the scattering process is also analyzed. [S1050-2947(96)09108-1]

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I. INTRODUCTION

One of the oldest and, in some cases, most severe problems of quantization of classical systems is the factor-ordering ambiguity. The ordinary operator quantization of nonrelativistic classical systems involves promotion of the coordinate (x^i) and momentum (p_i) variables to linear operators $\hat{x}^i$ and $\hat{p}_i$ acting on a Hilbert space $\mathcal{H}$ and satisfying the Weyl-Heisenberg algebra. The dynamics is then dictated by the Schrödinger equation, which involves the Hamiltonian operator $\hat{H}$. The quantum Hamiltonian $\hat{H}$ is constructed from the classical Hamiltonian H according to the following requirements: (1) $\hat{H}$ is a self-adjoint linear operator acting on $\mathcal{H}$; (2) in the classical limit $\hbar \to 0$, $\hat{H} \to H$.

In general, these two conditions do not determine $\hat{H}$ uniquely. Thus, a classical system may lead to different quantum systems. For instance, consider the motion of a free particle of mass m moving on a Riemannian manifold M. The classical Hamiltonian is given by

$$H = \frac{1}{2m} g^{ij}(x) p_i p_j , \tag{1}$$

where g^{ij} are components of the inverse of the metric tensor:

$$g = g_{ij}(x) dx^i \otimes dx^j. \tag{2}$$

Since g^{ij} depend explicitly on the coordinates $x := (x^1, \ldots, x^n)$, requirements (1) and (2) do not determine $\hat{H}$ uniquely. In this case, even the additional requirement of form invariance under coordinate transformations does not lead to a unique choice for $\hat{H}$ [1].

To quantize a classical Hamiltonian, therefore, one may need to make a choice of factor ordering. One way of achieving this is to appeal to alternative quantization schemes that make the choice of factor ordering implicitly. The best known example of this is the path-integral quantization schemes.

In view of the form-invariance requirement, different choices of factor ordering for the system of Eq. (1) differ by a multiple of $\hbar^2 R$, where R denotes the scalar Ricci curvature of M. That is, in general, one has

$$\hat{H} = \frac{1}{2m} \hat{g}^{-1/4}\, \hat{p}_i\, \hat{g}^{ij}\, \hat{g}^{1/2}\, \hat{p}_j\, \hat{g}^{-1/4} + \frac{\lambda \hbar^2}{m} \hat{R}, \tag{3}$$

where $g := \det(g_{ij})$ and λ is a real parameter reflecting the factor-ordering ambiguity in quantizing (1). One can express (3) in a local coordinate representation:

$$\langle x | \hat{H} = \left(\frac{-\hbar^2}{2m} \Delta + \frac{\lambda \hbar^2}{m} R(x) \right) \langle x |. \tag{4}$$

Here, $\Delta := g^{ij} \nabla_i \nabla_j$ denotes the Laplace-Beltrami operator and ∇_i stands for the covariant derivative corresponding to the Levi-Civita connection along $\partial/\partial x^i$. Choosing the Hilbert space to be $L^2(M)$, i.e., the state vectors $|\psi\rangle$ to be scalar square integrable functions on M, one has

$$\langle x | \hat{H} | \psi \rangle = \frac{\hbar^2}{2m} \left(-\frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j + 2\lambda R \right) \psi(x). \tag{5}$$

Note that in general the momentum operators are represented in the coordinate representation according to

$$\langle x | \hat{p}_i = \hbar \left(-i \frac{\partial}{\partial x^i} + \omega_i \right) \langle x |,$$

where ω_i are components of a closed one-form ω on M. For convenience, we assume M to be simply connected, so that ω is exact. In this case it may be gauged away.

In the long history of the problem of curved-space quantization [1], there have been different arguments offered in support of the choices: $\lambda = 0$ [2,3], $\lambda = \frac{1}{12}$ [4,5] (see also [1]), $\lambda = \frac{1}{8}$ [6]. To the author's best knowledge, none of these arguments is based on solid factual grounds for the case of general Riemannian manifolds.

For Lie group manifolds and homogeneous spaces, there exist group-theoretical quantization methods (see [1] for ref-

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erences). But for these cases, R is constant and the ambiguity in λ is physically irrelevant. The situation is probably best described by Marinov [1], who says: ''I wonder, whether it is possible to decide at present between the two variants'' [$\lambda=0,\frac{1}{12}$] ''of quantizing the Hamiltonian as long as no nontrivial (i.e., $R\neq$ const) solvable examples are known.''

The same problem may be addressed for the supersymmetric extensions of (1). Reference [7] provides a thorough analysis of a supersymmetric quantum-mechanical system based on an arbitrary spin manifold. In this case, the quantum Hamiltonian is given as the square of the generator of the supersymmetry [8]. The quantization of the latter is free of factor-ordering ambiguity. Therefore, the Hamiltonian operator is unambiguous. Indeed, it is given by Eq. (3) with the choice $\lambda=\frac{1}{8}$. Furthermore, this choice for λ is shown to be consistent with the path-integral quantization scheme [7]. Note, however, that one may not conclude from the knowledge of a supersymmetric extension of (1) that $\lambda=\frac{1}{8}$ for the original purely bosonic theory. Although the form of the Hamiltonian is analogous, the Hilbert spaces are different. For example, for the system considered in [7] and [8] the Hilbert space is the space of spinors on M and the $\hat{p}_i$'s involve components of the spin connection.

The purpose of the present article is to seek physical consequences of the possible existence of the scalar curvature factor in the Hamiltonian. The only known physical effect that causes curvature in three-dimensional space is gravity. Thus one might be tempted to address the question by studying the effect of gravity on nonrelativistic quantum systems. It is clear that for the experimentally available quantum systems, the effect of the scalar curvature factor due to gravity must be extremely weak. This is because in addition to being a two-loop ($\hbar^2$) order effect, the scalar curvature factor is proportional to the Ricci scalar curvature. For such systems, the latter is caused by gravity and therefore it is extremely small. An alternative approach to the problem is to study effectively two-dimensional systems that are constrained to have dynamics in a curved surface. The curvature may, in principle, be maintained mechanically and thus made considerably large. In particular, we shall study a simple scattering problem in two dimensions. The idea is to pave the way for a potential experimental resolution of this sort of factor-ordering ambiguity.

Consider a free particle moving on a two-dimensional surface M embedded in $\mathbb{R}^3$. For simplicity, suppose that M is asymptotically flat and has a cylindrically symmetric geometry and trivial ($\mathbb{R}^2$) topology. Then the scattering problem may be formulated as in the case of potential scattering. The scattering data, however, reflect the nontrivial geometry of M. In particular, this involves the contribution of the scalar curvature factor.

In Sec. II, we review the basic formalism used in the study of the scattering problem in two dimensions. Section III exhibits the treatment of the specific problem of scattering due to geometry. Section IV treats the effect of a localized magnetic field in the scattering of charged particles moving on the curved surface of interest. The result is used to provide an order-of-magnitude estimate for the sensitivity required for observing the contribution of the scalar curvature factor to the scattering cross section. Section V includes the concluding remarks.

II. SCATTERING IN TWO DIMENSIONS

Consider the ordinary time-independent potential scattering problem in two dimensions. The Lippmann-Schwinger equation for the Hamiltonian

$$\hat{H}=\hat{H}_0+\hat{V} \tag{6}$$

is given by

$$|\psi^{(\pm)}\rangle=|\phi\rangle+\frac{1}{E-\hat{H}_0\pm i\epsilon}\hat{V}|\psi^{(\pm)}\rangle, \tag{7}$$

where we follow the notation of Sakurai [9]. In Eqs. (6) and (7),

$$\hat{H}_0=\frac{1}{2m}\,\delta^{ij}\hat{p}_i\hat{p}_j \tag{8}$$

is the free Hamiltonian and $\hat{V}$ is the interaction potential.

In the coordinate representation, the second term in (7) can be written in the form

$$\left\langle x\left|\frac{1}{E-\hat{H}_0\pm i\epsilon}\hat{V}\right|\psi^{(\pm)}\right\rangle =\frac{m}{2\pi^2\hbar^2}\int d^2x'\,I^{(\pm)}(x,x',k)\langle x'|\hat{V}|\psi^{(\pm)}\rangle, \tag{9}$$

where

$$I^{(\pm)}:=\int d^2q\frac{e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}}{k^2-q^2\pm i\epsilon},\quad k:=|\vec{k}|=\frac{\sqrt{2mE}}{\hbar}. \tag{10}$$

As in the three-dimensional case, one may switch to the polar coordinates to evaluate the integrals (10). Performing the angular integration and consulting [10], one has

$$I^{(\pm)}=2\pi\int_0^\infty dq\,\frac{qJ_0(q|\vec{x}-\vec{x}'|)}{k^2-q^2\pm i\epsilon} =-i\pi^2H_0^{(1)}(\pm k|\vec{x}-\vec{x}'|+i\tilde{\epsilon}),$$

where J_0 and $H_0^{(1)}$ are Bessel and Hankel functions and $\tilde{\epsilon}$ is another infinitesimal positive parameter.

For the scattering problem, one considers the large $r:=|\vec{x}|$ limit. Using the asymptotic properties of the Hankel function [10], for $r\gg|\vec{x}'|$ one finds

$$\left\langle x\left|\frac{1}{E-\hat{H}_0\pm i\epsilon}\hat{V}\right|\psi^{(+)}\right\rangle =\left(\frac{-ime^{-i\pi/4}}{\sqrt{2\pi}\hbar^2}\right)\frac{e^{ikr}}{\sqrt{kr}}\int d^2x'\,e^{-i\vec{k}'\cdot\vec{x}'}\langle x'|\hat{V}|\psi^{(+)}\rangle,$$

where $\vec{k}':=k\hat{x}$. The iterative solution of (6) is then carried out by setting $\langle x|\phi\rangle=\langle x|k\rangle=e^{i\vec{k}\cdot\vec{x}}/2\pi$. One then obtains

$$\langle x|\psi^{(+)}\rangle=\frac{1}{2\pi}\left(e^{i\vec{k}\cdot\vec{x}}+\frac{e^{ikr}}{\sqrt{r}}f(\vec{k}',\vec{k})\right). \tag{11}$$

In the first Born approximation,

$$f(\vec{k}',\vec{k})\approx f^{(1)}(\vec{k}',\vec{k}) = \frac{-i\sqrt{2\pi}me^{-i\pi/4}}{\sqrt{k}\hbar^2}\int dx'^2 e^{-i\vec{k}'\cdot\vec{x}'}\langle x'|\hat{V}|k\rangle, \tag{12}$$

where

$$\langle x'|\hat{V}|k\rangle = \frac{1}{2\pi}\int d^2x'' e^{i\vec{k}\cdot\vec{x}''}\langle x'|\hat{V}|x''\rangle. \tag{13}$$

The scattering cross section is related to its amplitude f by the relation

$$\frac{d\sigma(\vec{k},\vec{k}')}{d\Omega} = |f(\vec{k},\vec{k}')|^2. \tag{14}$$

In the ordinary potential scattering, $\hat{V}$ is a local operator:

$$\langle x'|\hat{V}|x''\rangle = V(x')\,\delta(\vec{x}'-\vec{x}'').$$

Consider expressing the Hamiltonian (5) in the form (6). In view of Eq. (8), this leads to

$$\langle x'|\hat{V}|x''\rangle = \frac{\hbar^2}{2m}\left[(\delta^{ij}-g^{ij})\partial_i'\partial_j' - \frac{\partial_i'(\sqrt{g}g^{ij})}{\sqrt{g}}\partial_j' + 2\lambda R\right]\delta(\vec{x}'-\vec{x}''), \tag{15}$$

where ∂_i' means partial derivation with respect to x'^i. The curved space scattering is seen to correspond to an ultralocal potential.

We conclude this section by noting that the correspondence with the potential scattering on $\mathbb{R}^2$ is justified, for M is assumed to be asymptotically flat and topologically trivial. The latter allows one to use a single coordinate patch in performing the computations.

III. SCATTERING DUE TO A CYLINDRICALLY SYMMETRIC GEOMETRY

Consider a surface $M\subset\mathbb{R}^3$ defined by the equation

$$z = f(r), \tag{16}$$

where (r,θ,z) are cylindrical coordinates on $\mathbb{R}^3$ and $f:[0,\infty)\rightarrow\mathbb{R}$ is a smooth function with vanishing first derivative at the origin, i.e., $\dot{f}(0)=0$. This is the condition which makes M a differentiable manifold. Furthermore, assume that f has a (physically) compact support. Thus M is asymptotically flat.

The implicit geometry of M is described by the (induced) metric (from $\mathbb{R}^3$),

$$(g_{ij}) = \begin{pmatrix} F^2 & 0 \\ 0 & r^2 \end{pmatrix}, \tag{17}$$

where $F^2 := 1+\dot{f}^2$. Given the metric, one can easily compute the terms in Eq. (15). In view of Eq. (13), one then has

$$\langle x'|\hat{V}|k\rangle = \frac{\hbar^2}{4\pi m}\left\{\left(1-\frac{1}{F^2}\right)\frac{\partial^2}{\partial r'^2} + \left[\frac{1}{r'}\left(1-\frac{1}{F^2}\right) - \frac{\dot{F}}{F^3}\right]\frac{\partial}{\partial r'} + \frac{4\lambda\dot{F}}{r'F^3}\right\}e^{i\vec{k}\cdot\vec{x}'},$$
$$= \frac{\hbar^2}{4\pi m}\left\{-\left(1-\frac{1}{F^2}\right)\left(\frac{\vec{k}\cdot\vec{x}'}{r'^2}\right)^2 + i\left[\frac{1}{r'}\left(1-\frac{1}{F^2}\right) - \frac{\dot{F}}{F^3}\right]\left(\frac{\vec{k}\cdot\vec{x}'}{r'}\right) + \frac{4\lambda\dot{F}}{r'F^3}\right\}e^{i\vec{k}\cdot\vec{x}'}.$$

The latter formula, together with Eq. (12), leads to the expression for $f^{(1)}(\vec{k}',\vec{k})$:

$$f^{(1)}(\vec{k}',\vec{k}) = \frac{e^{-3\pi i/4}}{\sqrt{8\pi k}}\int d^2x' e^{i(\vec{k}-\vec{k}')\cdot\vec{x}'}\left\{-\left(1-\frac{1}{F^2}\right)\left(\frac{\vec{k}\cdot\vec{x}'}{r'}\right)^2 + i\left[\frac{1}{r'}\left(1-\frac{1}{F^2}\right) - \frac{\dot{F}}{F^3}\right]\left(\frac{\vec{k}\cdot\vec{x}'}{r'}\right) + \frac{4\lambda\dot{F}}{r'F^3}\right\}. \tag{18}$$

To perform the integral on the right-hand-side of (18), we choose a coordinate system in which $\Delta\vec{k} := \vec{k}-\vec{k}'$ is along the x' axis. Then switching to polar coordinates (r',θ'), one can evaluate the angular integration. This results in

$$f^{(1)}(\vec{k}',\vec{k}) = \sqrt{\frac{\pi}{2k}}e^{-3\pi i/4}\int_0^\infty dr'\left\{\left[-r'\left(1-\frac{1}{F^2}\right)k_x^2 + \frac{4\lambda\dot{F}}{F^3}\right]J_0(r'|\Delta\vec{k}|) + \left[\frac{k_y^2-k_x^2}{|\Delta\vec{k}|} - k_x\left(1-\frac{1}{F^2}-\frac{r'\dot{F}}{F^3}\right)\right]J_1(r'|\Delta\vec{k}|)\right\}, \tag{19}$$

where J_0 and J_1 are Bessel functions. This expression may be further simplified by noting that $\vec{k}' := k\hat{r}$, i.e., $k = k'$. Denoting the angle between $\vec{k}$ and $\vec{k}'$ by Θ, one has

$$|\Delta\vec{k}| = 2k\sin\left(\frac{\Theta}{2}\right) = 2k_x, \quad k_y^2 - k_x^2 = k^2\cos\Theta.$$

In view of these relations, Eq. (19), and making extensive use of the properties of the Bessel functions [10] and the fact that $F(r=0) = F(r=\infty) = 1$, one finally arrives at the following expression for the scattering amplitude:

$$f^{(1)}(\vec{k}',\vec{k}) = \sqrt{2\pi k}e^{-3\pi i/4}\int_0^\infty dr\left(1 - \frac{1}{F^2}\right) \times\left[-k\sin\left(\frac{\Theta}{2}\right) rJ_0\left(2kr\ \sin\frac{\Theta}{2}\right) + 2\left(\lambda - \frac{1}{8\sin^2(\Theta/2)}\right)J_1\left(2kr\ \sin\frac{\Theta}{2}\right)\right]. \tag{20}$$

Here the term proportional to λ signifies the contribution of the scalar curvature factor, whereas the other terms reflect the effect of the kinetic-energy term.

For the forward scattering $(\Theta = 0)$, this expression simplifies to yield

$$f^{(1)}(\vec{k},\vec{k}) = \sqrt{\frac{\pi}{8}}e^{-i3\pi/4}k^{3/2}\int_0^\infty dr'\left[-r'\left(1 - \frac{1}{F^2}\right)\right]. \tag{21}$$

As seen from Eq. (21), the scalar curvature factor does not contribute to the forward scattering. It does, however, contribute to the nonforward $(\Theta \neq 0)$ scattering. For example, consider the backscattering, where

$$f^{(1)}(\vec{k}' = -\vec{k},\vec{k}) = \sqrt{2\pi k}e^{-3\pi i/4}\int_0^\infty dr\left(1 - \frac{1}{F^2}\right) \times[-krJ_0(2kr) + 2(\lambda - \tfrac{1}{8})J_1(2kr)].$$

To see in more detail how the effects due to the kinetic energy and the scalar curvature terms compare, consider the Gaussian shape for the function f, i.e., let

$$f(r) = \delta\ e^{-\mu r^2/2}, \tag{22}$$

where δ and μ are real parameters. For convenience, let us also introduce the dimensionless parameter $\eta := \mu\delta^2$ and evaluate the integral in Eq. (20) by first expanding the integrand in powers of η. This involves integrals of the form

$$\int_0^\infty r^n e^{-\alpha r^2}J_m(\nu r)dr, \quad (m = 0,1, \quad n \in \mathbb{Z}^+, \quad \alpha \in \mathbb{R}^+), \tag{23}$$

which may be obtained using Ref. [10].

Carrying out the computations to the first nonvanishing order in η, one has

$$f^{(1)}(\vec{k},\vec{k}) = \sqrt{\frac{2\pi}{k}}e^{-i3\pi/4}\left(\frac{k^2}{\mu}\right) \times\left\{\left[\lambda - \frac{1}{4} - \frac{1}{8\sin^2(\Theta/2)} + \frac{k^2\sin^2(\Theta/2)}{4\mu}\right] \times\sin^2\frac{\Theta}{2}\exp\left(-\frac{k^2\sin^2(\Theta/2)}{\mu}\right)\eta + O(\eta^2)\right\}. \tag{24}$$

As seen from Eq. (24), the scalar curvature and the kinetic-energy-term contributions to the scattering amplitude are comparable unless one specializes to forward scattering.

IV. EFFECT OF A LOCALIZED MAGNETIC FIELD

Consider the system of the preceding section subject to a localized cylindrically symmetric magnetic field $\vec{B}$. The latter may be defined by the vector potential $\vec{A}$ (connection one-form $A = A_r dr + A_\theta d\theta + A_z dz$) with

$$A_r := 0 =: A_z,$$

$$A_\theta := \frac{Br^2}{2}\mathcal{G}(r), \tag{25}$$

where $\mathcal{G}:[0,\infty) \to \mathbb{R}$ is a smooth, compactly supported function and B is a constant parameter with the dimension of magnetic field.

The Hamiltonian operator $\hat{H}_B$ for the constrained (two-dimensional) system subject to such a magnetic field is obtained by replacing $\hat{p}_i$ in Eq. (4) by $\hat{p}_i + (e/c)\hat{A}_i$, where e is the charge of the particle and c is the speed of light. For the particular cylindrically symmetric system of interest, one finds

$$\hat{H}_B = \hat{H} + \frac{e}{2mcr^2}\left(2\hat{A}_\theta\hat{p}_\theta + \frac{e}{c}\hat{A}_\theta^2\right), \tag{26}$$

where $\hat{H}$ is the free Hamiltonian (4).

For convenience, let us denote the sum of the potential terms on the right-hand side of (26) by $\Delta\hat{V}$. Expressing $\Delta\hat{V}$ in the coordinate representation, where $\langle x|\hat{p}_\theta = -i\hbar\partial/\partial\theta\langle x|$, one has

$$\langle x|\Delta\hat{V}|k\rangle = \frac{e}{4\pi mcr^2}\left[2\hbar rA_\theta(k_y\cos\theta - k_x\sin\theta) + \frac{e}{c}A_\theta^2\right]e^{i\vec{k}\cdot\vec{x}}. \tag{27}$$

It is this term that enters the expression for the scattering amplitude, i.e., is added to the terms due to $\hat{H}$. In view of Eqs. (12), (25) and (26), one needs to compute the integral:

$$\int d^2x' e^{-i\vec{k}'\cdot\vec{x}} \langle x'|\Delta\hat{V}|k\rangle = \frac{e\hbar B}{4\pi mc}\int_0^\infty dr\int_0^{2\pi} d\theta [r^2\mathcal{G}(r)(k_y\cos\theta - k_x\sin\theta)e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}] + \frac{e^2B^2}{16\pi mc^2}\int_0^\infty dr\int_0^{2\pi} d\theta[r^3\mathcal{G}^2(r)e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}]. \tag{28}$$

Choosing the x axis in the $\Delta\vec{k}$ direction as in Eqs. (19) and (20), one can perform the angular integral. In terms of the angle Θ between $\vec{k}$ and $\vec{k}'$, one has

$$\int d^2x' e^{-i\vec{k}'\cdot\vec{x}}\langle x'|\Delta\hat{V}|k\rangle = \frac{e^2B^2}{8mc^2}\int_0^\infty dr r^3\mathcal{G}^2(r)J_0\left(2kr\sin\frac{\Theta}{2}\right) + \frac{ie\hbar Bk\cos(\Theta/2)}{2mc}\int_0^\infty dr r^2\mathcal{G}(r)J_1\left(2kr\sin\frac{\Theta}{2}\right). \tag{29}$$

The scattering amplitude is then obtained by adding (29) to the integral in Eq. (12).

Now, consider a Gaussian shape for the function $\mathcal{G}$:

$$\mathcal{G}(r) = e^{-r^2/2\sigma^2}. \tag{30}$$

In this case the integrals appearing in Eq. (29) are again of the form (23) and easily evaluated. In view of Eq. (12), one then has the following expression for the contribution of magnetic field to the scattering amplitude:

$$\Delta f^{(1)}(\vec{k}',\vec{k}) = \sqrt{\frac{\pi}{2k}}e^{3\pi i/4}\left[\frac{e^2\Phi^2}{8\pi^2\hbar^2c^2}\left(1 - k^2\sigma^2\sin^2\frac{\Theta}{2}\right) + \frac{ie\Phi}{\pi\hbar c}(k^2\sigma^2\sin\Theta)\right]e^{-\sigma^2k^2\sin^2(\Theta/2)}, \tag{31}$$

where

$$\Phi := \pi\sigma^2 B$$

is a characteristic magnetic flux[1] and $i = \sqrt{-1}$.

Considering the Gaussian shape for both f and $\mathcal{G}$, one obtains the total scattering amplitude by adding the contributions of the geometry and magnetic field:

$$f_B^{(1)}(\vec{k}',\vec{k}) = f^{(1)}(\vec{k}',\vec{k}) + \Delta f^{(1)}(\vec{k}',\vec{k}). \tag{32}$$

Here $f^{(1)}(\vec{k}',\vec{k})$ is given by Eq. (24).

Equation (32) may be used to give an order-of-magnitude estimate of the size of the contribution of the scalar curvature factor. This may be achieved by comparing the magnitudes of the two terms on the right-hand side of this equation. First, note that for small magnetic fluxes[2] and for the case of nonforward scattering which is of interest here, the term proportional to Φ^2 may be neglected. Next, for simplicity, choose $\mu = 1/\sigma^2$ so that the exponentially decaying factors in (24) and (31) are the same. This reduces the comparison of the two effects to that of the following terms:

$$\text{(I)} \quad 2\left(\lambda - \frac{1}{4} - \frac{1}{8\sin^2(\Theta/2)} + \frac{k^2\sin^2(\Theta/2)}{4\mu}\right)\eta\sin^2\frac{\Theta}{2} + O(\eta^2), \tag{33}$$

$$\text{(II)} \quad \frac{e\Phi\sin\Theta}{\pi\hbar c}. \tag{34}$$

Here use is made of the choice $\mu = 1/\sigma^2$. To simplify further, consider the case of electrons with $\eta \approx 10^{-1}$, $k \ll \sqrt{\mu}$, and $\Theta = \pi/2$. Then a comparable magnetic effect has the following characteristic flux:

$$\Phi \approx \left(\frac{\pi\hbar c\eta}{e}\right)\left(\lambda - \frac{1}{2}\right) \approx \left(\lambda - \frac{1}{2}\right)\times 10^{-8} \ \ (\text{G cm}^2).$$

Thus an experimental setup capable of detection of the effect of the scalar curvature factor in scattering of free particles moving on a curved Gaussian shape surface (22) must have a sensitivity to detect scattering of electrons due to a localized magnetic field (25), (30) of characteristic flux $\approx 10^{-9}$ G cm^2 in the flat $\mathbb{R}^2$ space.

V. DISCUSSION AND CONCLUSION

The physical consequences of the existence of a scalar curvature factor in the scattering of free particles moving on a nonflat surface have been analyzed. Although there have recently been some attempts to study nonflat two-dimensional quantum systems, particularly in the context of the quantum Hall effect [11], the curved spaces considered in the literature are either spaces of constant scalar curvature [12,13] or spaces with exotic topologies and geometries [11]. On the other hand, in all these attempts, the possibility of the existence of a scalar curvature factor in the Schrödinger equation has been ignored.

The particular system investigated in this article is physically more interesting, since there are indeed two-dimensional nonflat systems with $\mathbb{R}^2$ topology in nature. An example of this is the two-dimensional electron system

[1]Note that the total magnetic flux is zero as the topology of the space is $\mathbb{R}^2$ and there are no singularities in the fields.

[2]Consideration of small magnetic fluxes is reasonable because here one tries to find a magnetic effect comparable to the effect of geometry. The latter is an $\hbar^2$ order effect.

formed on the surface of liquid helium [14,15]. Particularly remarkable is the Gaussian shape (22) of the surface of ^{4}He in a dimple electron crystal in the vicinity of the dimples [14]. To arrive at an experimental resolution of the factor-ordering ambiguity, i.e., experimental determination of the value of λ, a more thorough investigation of the available (effectively) two-dimensional systems is needed.

In the comparison of the effect of the curvature with the scattering effects of a cylindrically symmetric magnetic flux in a flat two-dimensional system, such a small magnetic flux corresponds to energy densities of the order of $\sim 10^{-17}$ erg/cm^3 whose effect would likely be swamped by thermal effects in a ^{4}He system. Here the analogy is used to give a very rough order-of-magnitude estimate for the maximum precision required for such an experiment. In practice, one might not need such precision.[3] In fact, one may look at the collective effects such as those of a locally cylindrically symmetric curved surface, i.e., a surface curved at an array of points forming the vertices of a lattice. This is precisely the case in a dimple electron system. The main purpose of this comparison is to demonstrate that the corresponding effect is not several orders of magnitude beyond the experimentally accessible values. This is usually the case where the problems with the quantization on curved spaces are concerned.

[3]For example, in the above analysis the parameter η is taken to be small only to simplify the calculations.

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