Generalized center-of-mass coordinate and relative momentum operators studied through unitary transformations

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A method for studying the generalized center-of-mass coordinate and relative momentum operators is provided. This method is based on unitary transformations. The eigenvectors of these operators can be easily obtained. Their properties and relationship with one-body system eigenvectors are also discussed. [S1050-2947(96)08508-3]

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I. INTRODUCTION

In quantum mechanics, we often have to construct the eigenvectors of a complete set of operators. For one-body systems, the most useful basis vectors are often $|q\rangle$ and $|p\rangle$, where $|q\rangle$ is the eigenvector of the coordinate operator Q, and $|p\rangle$ is the eigenvector of the momentum operator P. In Fock space, we have

$$|q\rangle=\pi^{-1/4}e^{-(q^2/2)+\sqrt{2}qa^\dagger-[(a^\dagger)^2/2]}|0\rangle, \tag{1}$$

$$|p\rangle=\pi^{-1/4}e^{-(p^2/2)+i\sqrt{2}pa^\dagger+[(a^\dagger)^2/2]}|0\rangle, \tag{2}$$

$$Q|q\rangle=q|q\rangle,\quad P|p\rangle=p|p\rangle,\quad [Q,P]=i. \tag{3}$$

The creation and annihilation operators have been introduced to express Q and P as

$$Q=\frac{a+a^\dagger}{\sqrt{2}},\quad P=\frac{a-a^\dagger}{i\sqrt{2}}. \tag{4}$$

For two-body systems, the coordinate and momentum operators are Q_j and P_j, $j=1,2$, for which $[Q_j,P_k]=i\delta_{jk}$. For two-body systems we can also introduce creation and annihilation operators to express the coordinate and momentum operators,

$$Q_j=\frac{a_j+a_j^\dagger}{\sqrt{2}},\quad P_j=\frac{a_j-a_j^\dagger}{i\sqrt{2}},\quad j=1 \text{ and } 2, \tag{5}$$

It can be shown that the relative coordinate Q_1-Q_2 commutes with the total momentum P_1+P_2. Thus common eigenvectors exist for the two operators. These eigenvectors appear in Maslov's semiclassical quantization scheme [1], and the famous scheme proposed by Einstein, Podolsky and Rosen (EPR) [2] in their study of quantum measurement. Another commuting set of operators for two-body systems is the center-of-mass coordinate $\mu_1Q_1+\mu_2Q_2$ and the mass-weighted relative momentum $\mu_2P_1-\mu_1P_2$, where

$$\mu_1=\frac{m_1}{m_1+m_2},\quad \mu_2=\frac{m_2}{m_1+m_2},\quad \mu_1+\mu_2=1.$$

Common eigenvectors can be constructed for this pair of operators as well. How do we construct these eigenvectors? What is the relationship between the two-body system eigenvectors and the one-body system eigenvectors? Our goal in this paper is to present a simple method for systematically deriving these eigenvectors by means of unitary transformations. Through our method, the relationship between the two-body system eigenvectors and the one-body system eigenvectors can be found easily. Although properties of these eigenvectors have been discussed before [3,4], to our knowledge this relationship has not been revealed previously.

II. EIGENVECTORS OF Q_1+Q_2 AND P_1-P_2

The common eigenvector of Q_1 and P_2 is $|q_1,p_2\rangle$:

$$Q_1|q_1,p_2\rangle=q_1|q_1,p_2\rangle,\quad P_2|q_1,p_2\rangle=p_2|q_1,p_2\rangle, \tag{6}$$

$|q_1,p_2\rangle$ can be expanded in Fock space:

$$|q_1,p_2\rangle=\frac{1}{\sqrt{\pi}}e^{-(q_1^2/2)+\sqrt{2}q_1a_1^\dagger-[(a_1^\dagger)^2/2]}\times e^{-(p_2^2/2)+i\sqrt{2}p_2a_2^\dagger+[(a_2^\dagger)^2/2]}|00\rangle, \tag{7}$$

Since we can prove that

$$[Q_1+Q_2,P_1-P_2]=0, \tag{8}$$

there exist common eigenvectors for the pair of operators of Q_1+Q_2 and P_1-P_2. We introduce the unitary transformation

$$U=e^{[\pi(a_1a_2^\dagger-a_1^\dagger a_2)/4]}=e^{[i\pi(Q_2P_1-Q_1P_2)/4]}, \tag{9}$$

$$UU^\dagger=U^\dagger U=1, \tag{10}$$

By virtue of the relation

$$e^LFe^{-L}=F+[L,F]+\frac{[L,[L,F]]}{2!}+\cdots, \tag{11}$$

we can derive

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$$UQ_1U^\dagger=\frac{Q_1+Q_2}{\sqrt{2}},\quad UQ_2U^\dagger=\frac{Q_2-Q_1}{\sqrt{2}},$$
$$UP_1U^\dagger=\frac{P_1+P_2}{\sqrt{2}},\quad UP_2U^\dagger=\frac{P_2-P_1}{\sqrt{2}}. \tag{12}$$

The common eigenvector of Q_1+Q_2 and P_2-P_1 can then be constructed as $U|q_1,p_2\rangle$. We want to derive the explicit form of this eigenvector. For this purpose, we rewrite U as

$$U=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\frac{q_1'-q_2'}{\sqrt{2}},\frac{q_1'+q_2'}{\sqrt{2}}\right\rangle\langle q_1',q_2'|dq_1'dq_2'. \tag{13}$$

It is easy to prove, using the recently developed technique of IWOP [5], that (13) is consistent with (9). Now we can calculate the common eigenvector of Q_1+Q_2 and P_1-P_2:

$$\begin{aligned}U|q_1,p_2\rangle&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\frac{q_1'-q_2'}{\sqrt{2}},\frac{q_1'+q_2'}{\sqrt{2}}\right\rangle\langle q_1',q_2'|q_1,p_2\rangle dq_1'dq_2'\\&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\frac{q_1'-q_2'}{\sqrt{2}},\frac{q_1'+q_2'}{\sqrt{2}}\right\rangle\delta(q_1'-q_1)\,\frac{1}{\sqrt{2\pi}}\,e^{ip_2q_2'}dq_1'dq_2'\\&=\frac{1}{\sqrt{2}\pi}\int_{-\infty}^{+\infty}e^{ip_2q_2'-[(q_1-q_2')^2/4]+(q_1-q_2')a_1^\dagger-[(a_1^\dagger)^2/2]-[(q_1+q_2')^2/4]+(q_1+q_2')a_2^\dagger-[(a_2^\dagger)^2/2]}|00\rangle dq_2'\\&=\frac{1}{\sqrt{\pi}}\,e^{-(p_2^2+q_1^2)/2+(q_1-ip_2)a_1^\dagger+(q_1+ip_2)a_2^\dagger-a_1^\dagger a_2^\dagger}|00\rangle.\end{aligned} \tag{14}$$

Using (12), we obtain

$$(Q_1+Q_2)U|q_1,p_2\rangle=\sqrt{2}q_1U|q_1,p_2\rangle, \tag{15}$$

$$(P_1-P_2)U|q_1,p_2\rangle=-\sqrt{2}p_2U|q_1,p_2\rangle, \tag{16}$$

We define $\eta_1=q_1$, $\eta_2=-p_2$, and $\eta=\eta_1+i\eta_2$. The common eigenvector $U|q_1,p_2\rangle$ can now be rewritten as

$$|\eta\rangle=\frac{1}{\sqrt{\pi}}\,e^{-(|\eta|^2/2)+\eta a_1^\dagger+\eta^* a_2^\dagger-a_1^\dagger a_2^\dagger}|00\rangle, \tag{17}$$

which is consistent with the result of [3].

Similarly, we know that the eigenvector of Q_1-Q_2 and P_1+P_2 is $U|p_1,q_2\rangle$. The eigenvector can be written in concise form as

$$|\zeta\rangle=\frac{1}{\sqrt{\pi}}\,e^{-(|\zeta|^2/2)+\zeta a_1^\dagger-\zeta^* a_2^\dagger+a_1^\dagger a_2^\dagger}|00\rangle. \tag{18}$$

$|\zeta\rangle$ obeys

$$(Q_1-Q_2)|\zeta\rangle=\sqrt{2}\zeta_1|\zeta\rangle, \tag{19}$$

$$(P_1+P_2)|\zeta\rangle=\sqrt{2}\zeta_2|\zeta\rangle, \tag{20}$$

where

$$\zeta_1=-q_2,\quad \zeta_2=p_1,\quad \zeta=\zeta_1+i\zeta_2,$$

We understand that $\{|\eta\rangle\}$ and $\{|\zeta\rangle\}$ are orthonormal and complete as a result of the unitary transformation:

$$\langle\eta'|\eta\rangle=\langle q_1',p_2'|U^\dagger U|q_1,p_2\rangle=\delta(\eta'-\eta), \tag{21a}$$

$$\int|\eta\rangle\langle\eta|d^2\eta=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}U|q_1,p_2\rangle\langle q_1,p_2|U^\dagger dq_1dp_2=1, \tag{21b}$$

$$\langle\zeta'|\zeta\rangle=\langle p_1',q_2'|U^\dagger U|p_1,q_2\rangle=\delta(\zeta'-\zeta), \tag{21c}$$

$$\int|\zeta\rangle\langle\zeta|d^2\zeta=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}U|p_1,q_2\rangle\langle p_1,q_2|U^\dagger dp_1dq_2=1. \tag{21d}$$

$\{|\eta\rangle\}$ and $\{|\zeta\rangle\}$ constitute two different representations in two-mode Fock space. The scalar product of $|\eta\rangle$ and $|\zeta\rangle$ is

$$\begin{aligned}\langle\zeta|\eta\rangle&=\langle p_1,q_2|U^\dagger U|q_1,p_2\rangle=\langle p_1|q_1\rangle\langle q_2|p_2\rangle\\&=\frac{1}{2\pi}\,e^{i(\eta_2\zeta_1-\eta_1\zeta_2)}.\end{aligned} \tag{22}$$

III. EIGENVECTORS OF $\mu_1Q_1+\mu_2Q_2$ and $\mu_2P_1-\mu_1P_2$

For a two-body system with two distinct masses m_1 and m_2, the center-of-mass and the mass-weighted relative momentum are very useful in solving dynamical problems. They can be defined as

$$Q_{\text{c.m.}}=\mu_1Q_1+\mu_2Q_2, \tag{23}$$

$$P_\gamma=\mu_2P_1-\mu_1P_2, \tag{24}$$

where

$$\mu_1=\frac{m_1}{m_1+m_2},\quad \mu_2=\frac{m_2}{m_1+m_2},\quad \mu_1+\mu_2=1.$$

We can prove that $Q_{\mathrm{c.m.}}$ commutes with P_r:

$$[Q_{\mathrm{c.m.}},P_r]=0. \tag{25}$$

Thus it is possible to construct common eigenvectors for the pair of operators $Q_{\mathrm{c.m.}}$ and P_r. In the following derivation, we will treat μ_1 and μ_2 as two arbitrary positive real numbers without the restriction $\mu_1+\mu_2=1$, which was imposed in [4].

Consider the unitary transformation

$$U_1=e^{r[a_1^2-(a_1^\dagger)^2]/2}, \tag{26}$$

$$U_2=e^{\sigma[a_2^2-(a_2^\dagger)^2]/2}, \tag{27}$$

where $\mu_1=e^r$ and $\mu_2=e^\sigma$. It is obvious that U_1 commutes with U_2: $[U_1,U_2]=0$. Using (5) and (11), we can show that

$$(U_1U_2)(Q_1+Q_2)(U_1U_2)^\dagger=\mu_1Q_1+\mu_2Q_2, \tag{28}$$

$$(U_1U_2)(P_1-P_2)(U_1U_2)^\dagger=\frac{\mu_2P_1-\mu_1P_2}{\mu_1\mu_2}. \tag{29}$$

From the result of (15) and (16), we know that the common eigenvector of $Q_{\mathrm{c.m.}}$ and P_r is $U_1U_2U|q_1,p_2\rangle$. By virtue of the relation [5]

$$U_1=e^{r[a_1^2-(a_1^\dagger)^a]/2}=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{\mu_1}}\left|\frac{q_1}{\mu_1}\right\rangle\langle q_1|dq_1, \tag{30}$$

$$U_2=e^{\sigma[a_2^2-(a_2^\dagger)^a]/2}=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{\mu_2}}\left|\frac{q_2}{\mu_2}\right\rangle\langle q_2|dq_2, \tag{31}$$

the common eigenvector can be calculated directly:

$$\begin{aligned}
&U_1U_2U|q_1,p_2\rangle\\
&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{\mu_1\mu_2}}\left|\frac{q_1''}{\mu_1},\frac{q_2''}{\mu_2}\right\rangle\left\langle q_1'',q_2''\right|\left.\frac{q_1-q_2'}{\sqrt{2}},\frac{q_1+q_2'}{\sqrt{2}}\right\rangle\frac{1}{\sqrt{2\pi}}e^{ip_2q_2'}dq_1''dq_2''dq_2'\\
&=\left(\frac{1}{2\pi\mu_1\mu_2}\right)^{1/2}\int_{-\infty}^{+\infty}e^{ip_2q_2'}\left|\frac{q_1-q_2'}{\sqrt{2}\mu_1},\frac{q_1+q_2'}{\sqrt{2}\mu_2}\right\rangle dq_2',\\
&=\frac{1}{\pi\sqrt{2\mu_1\mu_2}}\int_{-\infty}^{+\infty}e^{ip_2q_2'-[(q_1-q_2')^2/(4\mu_1^2)]+[(q_1-q_2')a_1^\dagger/\mu_1]-[(a_1^\dagger)^2/2]-[(q_1+q_2')^2/(4\mu_2^2)]+[(q_1+q_2')a_2^\dagger/\mu_2]-[(a_2^\dagger)^2/2]}|100\rangle dq_2'\\
&=\left(\frac{2\mu_1\mu_2}{\pi(\mu_1^2+\mu_2^2)}\right)^{1/2}\exp\{[(\mu_2^2-\mu_1^2)/2(\mu_1^2+\mu_2^2)][(a_1^\dagger)^2-(a_2^\dagger)^2]+[2\mu_1(q_1-ip_2\mu_2^2)/(\mu_1^2+\mu_2^2)]a_1^\dagger\\
&\quad+[2\mu_2(q_1+ip_2\mu_1^2)/(\mu_1^2+\mu_2^2)]a_2^\dagger-[2\mu_1\mu_2/(\mu_1^2+\mu_2^2)]a_1^\dagger a_2^\dagger\\
&\quad-[(q_1^2+\mu_1^2\mu_2^2p_2^2)/(\mu_1^2+\mu_2^2)]+[ip_2q_1(\mu_2^2-\mu_1^2)/(\mu_1^2+\mu_2^2)]\}|00\rangle.
\end{aligned} \tag{32}$$

Using (15), (16), (28), and (29), we obtain the following result:

$$(\mu_1Q_1+\mu_2Q_2)U_1U_2U|q_1,p_2\rangle=\sqrt{2}q_1U_1U_2U|q_1,p_2\rangle, \tag{33}$$

$$(\mu_2P_1-\mu_1P_2)U_1U_2U|q_1,p_2\rangle=-\sqrt{2}\mu_1\mu_2p_2U_1U_2U|q_1,p_2\rangle. \tag{34}$$

We introduce the complex number $\xi=\xi_1+i\xi_2$, with

$$\xi_1=\left(\frac{2}{\mu_1^2+\mu_2^2}\right)^{1/2}q_1,\quad \xi_2=-\left(\frac{2}{\mu_1^2+\mu_2^2}\right)^{1/2}\mu_1\mu_2p_2. \tag{35}$$

The common eigenvector $U_1U_2U|q_1,p_2\rangle$ can now be rewritten as

$$U_1U_2U|q_1,p_2\rangle=\left(\frac{2\mu_1\mu_2}{\pi(\mu_1^2+\mu_2^2)}\right)^{1/2}e^{[i(\mu_1^2-\mu_2^2)/2\mu_1\mu_2]\xi_1\xi_2}|\xi\rangle, \tag{36}$$

where

$$\begin{aligned}
|\xi\rangle&=e^{-(|\xi|^2/2)+(1/\sqrt{\lambda})[(\mu_1+\mu_2)\xi+(\mu_1-\mu_2)\xi^*]a_1^\dagger+\sqrt{\lambda}[(\mu_1+\mu_2)\xi^*-(\mu_1-\mu_2)\xi]a_2^\dagger+(1/\lambda)\{(\mu_2^2-\mu_1^2)[(a_1^\dagger)^2-(a_2^\dagger)^2]-4\mu_1\mu_2a_1^\dagger a_2^\dagger\}}|00\rangle,\\
\lambda&=2(\mu_1^2+\mu_2^2).
\end{aligned} \tag{37}$$

Next we calculate the overlap $\langle\xi'|\xi\rangle$. According to (35) and (36), we see that

$$\langle\xi'|\xi\rangle=\frac{\pi(\mu_1^2+\mu_2^2)}{2\mu_1\mu_2}\,e^{[i(\mu_1^2-\mu_2^2)(\xi_1'\xi_2'-\xi_1\xi_2)/2\mu_1\mu_2]}\times\langle q_1',p_2'|U^\dagger U_2^\dagger U_1^\dagger U_1U_2U|q_1,p_2\rangle=\delta(\xi'-\xi). \tag{38}$$

In terms of (35) and (36), we know that $|\xi\rangle$ is complete:

$$\int\frac{1}{\pi}|\xi\rangle\langle\xi|d^2\xi=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}U_1U_2U|q_1,p_2\rangle\langle q_1,p_2|U^\dagger U_2^\dagger U_2^\dagger dq_1dp_2=1. \tag{39}$$

$\{|\xi\rangle\}$ constitutes a complete and orthonormal representation in two-mode Fock space. We see that there is a phase factor in our result which was not contained in [4], and that the condition $\mu_1+\mu_2=1$ is not required in our derivation.

In summary, I have derived common eigenvectors of the generalized center-of-mass coordinate, and relative momentum operators using unitary transformations. Their relationship with one-body system eigenvectors can be easily found through this method.

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