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Relativistic electron moving in a multimode quantized radiation field

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We derive a set of exact solutions of the Dirac equation for a single electron moving in a multimode quantized radiation field. All the modes in this field are propagating in the same direction. Our method entails the application of the Lie-algebra procedure. We discuss the physical meaning and possible applications of these solutions. [S1050-2947(96)09507-8]

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I. INTRODUCTION

Searching for exact and approximate solutions for a single electron interacting with an electromagnetic (em) wave has a long history. In Gordon's and Volkov's [1] pioneering work and other authors' successive work [2–4], the em waves were treated as classical fields. These solutions are of great importance in treatments of multiphoton ionization (MPI) and multiphoton scattering processes [4–6].

Rapid increases of the power of laser equipment may result in more photon absorption by the electron in MPI processes, making the photoelectron more energetic, to the point of possessing relativistic speeds. Some work has been done on the relativistic photoelectron produced in strong-field MPI [7,8]. Multicolor and multimode MPI experiments have been performed by some groups [9,10]. It is necessary to obtain wave functions for a relativistic electron moving in multimode electromagnetic waves. These kinds of wave functions can also be used for, as mentioned by Rosenberg and Zhou [4], the construction of wave packets, fields that are localized in space as well as time and hence capable of providing a more realistic description.

The em waves in the Volkov-type solutions have been treated as quantized fields only in recent decades [11]. Berson and Fedorov and Kazakov found solutions of the Dirac equation in the case of the quantized single-mode linearly polarized em plane wave in the Bargman representation. Bialynicki-Birula found solutions in the phase representation that correspond to solutions in the large-photon-number limits. Bergou and Ehlotzky and Filipowicz found solutions of the Dirac equation in the quantized single-mode em plane wave, which is independent of the field operator representation, but only in the case where the em wave is circularly polarized.

In recent years, in a series of papers, we obtained exact solutions for a relativistic electron interacting with an arbitrarily polarized single-mode photon field [7] and solutions for a nonrelativistic electron interacting with multimode photon fields with multiple propagation directions [12].

Classical Volkov solutions have been used to describe the final state of the electron in MPI according to the Keldysh ansatz in Keldysh-Faisal-Reiss theory [6]. Being in the field is a feature of the electron described by Volkov states, contrary to the fact that all photoelectrons are collected in a situation where there is no radiation field. A close comparison, which shows the discrepancy between the theoretical result based on the Keldysh ansatz and the experimental result obtained by Bucksbaum *et al.* [10] in standing-wave multiphoton ionization including the half Kapitza-Dirac effect [13], has been made by Guo and Drake (GD) [14]. The GD analysis shows that a photoelectron does undergo two processes. The first process is the ionization process of the electron from the atom into the radiation field, while the second process is the escape process from the radiation field to free space. Bucksbaum *et al.*'s angular distribution splitting does not occur in the first process, which is the Keldysh-Faisal-Reiss (KFR) process. It occurs in the second process predicted by the scattering theory of Guo, Åberg, and Crasemann (GAC) [15]. The agreement between our scattering theory and the Bucksbaum *et al.*'s experiment shows the correctness of quantum field wave functions, and also that the Volkov states or Volkov-like states cannot be the final states in MPI, but play a role as intermediate states.

The original Volkov solutions [1] are for a relativistic electron interacting with a multimode electromagnetic wave propagating in the same direction. The parallel solutions in the case of a multimode photon field propagating in the same direction have not been obtained.

In this paper we start from the most general case, the Dirac equation for an electron interacting with a multimode photon field propagating in different directions, then reduce this equation to an algebraic equation. In the case where all photon modes propagate in the same direction, we solve the Dirac equation exactly. A brief discussion addresses the physical meaning and possible applications of the resulting exact solutions.

II. PHOTON MODES PROPAGATING IN DIFFERENT DIRECTIONS

When a relativistic electron moves in a multimode quantized electromagnetic plane wave, its wave function satisfies the Dirac equation

$$\left[i\gamma\partial-e\sum_\alpha \gamma A_\alpha(k_\alpha x)-m_e\right]\Psi(x)=0, \tag{1}$$

where α is the index for the modes of the radiation field and

$$A_\alpha(k_\alpha x)=g_\alpha(\epsilon_\alpha a_\alpha e^{-ik_\alpha x}+\epsilon_\alpha^* a_\alpha^\dagger e^{ik_\alpha x}), \tag{2}$$

with $g_\alpha=(2V_\alpha\omega_\alpha)^{-1/2}$. In this paper, there is no implied summation on repeated photon mode indices.

The polarization four-vectors ϵ_α can be defined by the following relations:

$$\begin{aligned}\epsilon_\alpha^*\epsilon_\alpha&=-1,\\ \epsilon_\alpha\epsilon_\alpha&=-\cos\xi_\alpha e^{i\Theta_\alpha},\\ \epsilon_\alpha^*\epsilon_\alpha^*&=-\cos\xi_\alpha e^{-i\Theta_\alpha},\end{aligned} \tag{3}$$

where $\epsilon_\alpha=(0,\boldsymbol{\epsilon}_\alpha)$. In a coordinate system, we define

$$\begin{aligned}\boldsymbol{\epsilon}_\alpha&=[\boldsymbol{\epsilon}_x\cos(\xi_\alpha/2)+i\boldsymbol{\epsilon}_y\sin(\xi_\alpha/2)]e^{i\Theta_\alpha/2},\\ \boldsymbol{\epsilon}_\alpha^*&=[\boldsymbol{\epsilon}_x\cos(\xi_\alpha/2)-i\boldsymbol{\epsilon}_y\sin(\xi_\alpha/2)]e^{-i\Theta_\alpha/2}.\end{aligned} \tag{4}$$

In this paper we use the metric $(g_{\mu\nu})=\mathrm{diag}(1,-1,-1,-1)$. The scalar productor of two 4-vectors is defined as $ab=g_{\mu\nu}a^\mu b^\nu$ and the scalar product γa is defined as $\gamma_\mu a^\mu$; γ stands for 4×4 Dirac matrices.

A canonical transformation

$$\Psi(x)=\exp\left(i\sum_\alpha k_\alpha x N_\alpha\right)\phi(x), \tag{5}$$

where $N_\alpha\equiv\frac{1}{2}(a_\alpha a_\alpha^\dagger+a_\alpha^\dagger a_\alpha)$ is the number operator for each photon mode, can be applied to Eq. (1) to eliminate the explicit dependence on the coordinates in the Dirac equation. After performing this transformation, Eq. (1) becomes

$$\left(i\gamma\partial-\sum_\alpha \gamma k_\alpha N_\alpha-e\gamma A-m_e\right)\phi(x)=0, \tag{6}$$

where the vector potential is

$$A=\sum_\alpha g_\alpha(\epsilon_\alpha a_\alpha+\epsilon_\alpha^* a_\alpha^\dagger). \tag{7}$$

We set

$$\phi(x)=\phi\ \exp(-ipx); \tag{8}$$

then,

$$\left(\gamma p-\sum_\alpha \gamma k_\alpha N_\alpha-e\gamma A-m_e\right)\phi=0. \tag{9}$$

This is the equation in the most general case where photon modes can have different directions of propagation.

III. PHOTON MODES PROPAGATING IN THE SAME DIRECTION

If all the modes propagate in the same direction, Eq. (9) can be solved exactly by extending the method for solving the single-mode equation. We assume that for all k_α there is a k such that

$$\begin{aligned}k_\alpha&=\omega_\alpha k,\\ k_\alpha k&=0,\\ |\mathbf{k}|&=1.\end{aligned} \tag{10}$$

The following relations hold for all the modes:

$$k_\alpha k_\beta=0, \tag{11}$$

$$k_\alpha A=kA=0. \tag{12}$$

We can introduce the projection operators as in the single-mode case,

$$\mathcal{P}=(\gamma p+m_e)\gamma k/2kp,\quad \mathcal{Q}=\gamma k(\gamma p-m_e)/2kp. \tag{13}$$

The following eigenvector equations for $\mathcal{P}$ and $\mathcal{Q}$ can be observed:

$$\begin{aligned}&\mathcal{P}\gamma k=0,\\ &\mathcal{P}\left(\gamma p+m_e+\frac{m_e^2-p^2}{2kp}\gamma k\right)=\gamma p+m_e+\frac{m_e^2-p^2}{2kp}\gamma k,\\ &\gamma k\mathcal{P}=\gamma k,\\ &\left(\gamma p-m_e+\frac{m_e^2-p^2}{2kp}\gamma k\right)\mathcal{P}=0,\\ &\mathcal{Q}\gamma k=\gamma k,\\ &\mathcal{Q}\left(\gamma p+m_e+\frac{m_e^2-p^2}{2kp}\gamma k\right)=0,\\ &\gamma k\mathcal{Q}=0,\\ &\left(\gamma p-m_e+\frac{m_e^2-p^2}{2kp}\gamma k\right)\mathcal{Q}=\gamma p-m_e+\frac{m_e^2-p^2}{2kp}\gamma k.\end{aligned} \tag{14}$$

Now we want to express ϕ in terms of $\mathcal{P}\phi$. Using γk to operate on Eq. (9) from the left, we get

$$(\gamma k\gamma p-e\gamma k\gamma A-m_e\gamma k)\phi=0;$$

thus

$$\phi=(2kp)^{-1}(\gamma p\gamma k+m_e\gamma k+e\gamma k\gamma A)\phi. \tag{15}$$

Since $\gamma k\gamma A=-\gamma A\gamma k$ and $\gamma k\mathcal{Q}=0$, as in the single-mode case, we still get

$$\phi=(1+e\gamma k\gamma A/2kp)\mathcal{P}\phi. \tag{16}$$

Multiplying Eq. (9) from the left by $(\gamma p+m_e)/2kp$, we have

$$\mathcal{P}\sum_\alpha \omega_\alpha N_\alpha\phi+\left[e\frac{(\gamma p+m_e)}{2kp}\gamma A-\frac{p^2-m_e^2}{2kp}\right]\phi=0. \tag{17}$$

The second term in Eq. (17) can be treated by using the right-hand side of Eq. (16) for ϕ

$$\left(\frac{e(\gamma p+m_e)}{2kp}\gamma A-\frac{p^2-m_e^2}{2kp}\right)\phi=\left(\frac{e(\gamma p+m_e)}{2kp}\gamma A-\frac{p^2-m_e^2}{2kp}\right)\left(1+\frac{e\gamma k\gamma A}{2kp}\right)\mathcal{P}\phi$$

$$=\left[\frac{e(\gamma p+m_e)}{2kp}\gamma A+\frac{e^2(\gamma p+m_e)}{2kp}\frac{\gamma A\gamma k\gamma A}{2kp}-\frac{p^2-m_e^2}{2kp}-\frac{e(p^2-m_e^2)\gamma k\gamma A}{(2kp)^2}\right]\mathcal{P}\phi$$

$$=\left[\frac{2epA}{2kp}-\frac{e\gamma A(\gamma p-m_e)}{2kp}-\mathcal{P}\frac{e^2\gamma A\gamma A}{2kp}-\frac{p^2-m_e^2}{2kp}-e\frac{(p^2-m_e^2)\gamma k\gamma A}{(2kp)^2}\right]\mathcal{P}\phi$$

$$=\left[\frac{2epA}{2kp}-\mathcal{P}\frac{e^2\gamma A\gamma A}{2kp}-\frac{p^2-m_e^2}{2kp}-e\gamma A\left(\frac{\gamma p-m_e}{2kp}+\frac{(m_e^2-p^2)\gamma k}{(2kp)^2}\right)\right]\mathcal{P}\phi$$

$$=\left[\frac{2epA}{2kp}-\mathcal{P}\frac{e^2A^2}{2kp}-\mathcal{P}\frac{e^2\gamma_\mu\gamma_\nu[A^\mu,A^\nu]}{4kp}-\frac{p^2-m_e^2}{2kp}\right]\mathcal{P}\phi, \tag{18}$$

where we have used one of the ‘‘eigenvector relations,’’ namely,

$$\left(\gamma p-m_e+\frac{m_e^2-p^2}{2kp}\gamma k\right)\mathcal{P}=0,$$

and the algebraic identity

$$\gamma A\gamma A=\gamma_\mu\gamma_\nu A^\mu A^\nu=\tfrac{1}{2}(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu)A^\mu A^\nu$$
$$+\tfrac{1}{2}\gamma_\nu\gamma_\mu(A^\nu A^\mu-A^\mu A^\nu)$$
$$=A^2+\tfrac{1}{2}\gamma_\nu\gamma_\mu[A^\nu,A^\mu]. \tag{19}$$

The equation for $\mathcal{P}\phi$ is then

$$\left(2kp\sum_\alpha\omega_\alpha N_\alpha+2epA-e^2A^2+e^2g^2\mathcal{P}\Sigma_z-p^2+m_e^2\right)\mathcal{P}\phi$$
$$=0, \tag{20}$$

where Σ_z is a 4×4 diagonal matrix with element $(1,-1,1,-1)$ or (σ_z,σ_z), and g^2 is defined as

$$g^2\equiv\sum_\alpha g_\alpha^2\sin\xi_\alpha. \tag{21}$$

The derivation leading to Σ_z is as follows:

$$\gamma_\nu\gamma_\mu[A^\nu,A^\mu]=\gamma_\nu\gamma_\mu\sum_{\alpha\beta}g_\alpha g_\beta[(\epsilon_\alpha^\nu a_\alpha+\epsilon_\alpha^{\nu*}a_\alpha^\dagger)(\epsilon_\beta^\mu a_\beta+\epsilon_\beta^{\mu*}a_\beta^\dagger)-(\epsilon_\alpha^\mu a_\alpha+\epsilon_\alpha^{\mu*}a_\alpha^\dagger)(\epsilon_\beta^\nu a_\beta+\boldsymbol{\epsilon}_\beta^{\nu*}a_\beta^\dagger)]$$

$$=\gamma_\nu\gamma_\mu\sum_{\alpha\beta}g_\alpha g_\beta[(\epsilon_\alpha^{\nu*}\epsilon_\beta^\mu-\epsilon_\alpha^{\mu*}\epsilon_\beta^\nu)a_\alpha^\dagger a_\beta+(\epsilon_\alpha^\nu\epsilon_\beta^{\mu*}-\epsilon_\alpha^\mu\epsilon_\beta^{\nu*})a_\alpha a_\beta^\dagger]$$

$$=\gamma_\nu\gamma_\mu\sum_{\alpha\beta}g_\alpha g_\beta[(\epsilon_\alpha^{\nu*}\epsilon_\beta^\mu-\epsilon_\alpha^{\mu*}\epsilon_\beta^\nu)a_\alpha^\dagger a_\beta+(\epsilon_\beta^\nu\epsilon_\alpha^{\mu*}-\epsilon_\beta^\mu\epsilon_\alpha^{\nu*})a_\beta a_\alpha^\dagger]$$

$$=-\gamma_\nu\gamma_\mu\sum_{\alpha\beta}g_\alpha g_\beta[(\epsilon_\alpha^{\nu*}\epsilon_\beta^\mu-\epsilon_\alpha^{\mu*}\epsilon_\beta^\nu)(a_\beta a_\alpha^\dagger-a_\alpha^\dagger a_\beta)]$$

$$=-\sum_{\alpha\beta}g_\alpha g_\beta[\gamma\epsilon_\alpha^*,\gamma\epsilon_\beta][a_\beta,a_\alpha^\dagger]=-\sum_{\alpha\beta}g_\alpha g_\beta[\gamma\epsilon_\alpha^*,\gamma\epsilon_\beta]\delta_{\beta\alpha}$$

$$=-\sum_\alpha g_\alpha^2[\gamma\epsilon_\alpha^*,\gamma\epsilon_\alpha]=-i[\gamma\epsilon_x,\gamma\epsilon_y]\sum_\alpha g_\alpha^2\sin\xi_\alpha$$

$$=-i[\gamma_x,\gamma_y]\sum_\alpha g_\alpha^2\sin\xi_\alpha=-2\Sigma_z\sum_\alpha g_\alpha^2\sin\xi_\alpha. \tag{22}$$

Here we can see that the electron spin mainly aligns in the direction of propagation of the photon field, except the $\mathcal{Q}\phi$ part, which is the second term in the Eq. (16) and will vanish at the nonrelativistic limit. This is a different physical consequence.

For the spin operator Σ_z, we have following identities:

$$\mathcal{P}\Sigma_z\mathcal{P}=\mathcal{P}\Sigma_z, \tag{23}$$

which can be proven as follows. It is straightforwardly verified that

$$\gamma k\Sigma_z=\Sigma_z\gamma k. \tag{24}$$

With

$$\gamma k\mathcal{P}=\gamma k,$$

we have

$$\begin{aligned}\mathcal{P}\Sigma_z\mathcal{P}&=\frac{(\gamma p+m_e)\gamma k}{2kp}\Sigma_z\mathcal{P}\\&=\frac{(\gamma p+m_e)\Sigma_z\gamma k}{2kp}\mathcal{P}\\&=\frac{(\gamma p+m_e)\Sigma_z\gamma k}{2kp}\\&=\frac{(\gamma p+m_e)\gamma k}{2kp}\Sigma_z=\mathcal{P}\Sigma_z,\end{aligned}$$

which is Eq. (23).

Since Σ_z is a 4×4 diagonal matrix with elements $(1,-1,1,-1)$, its eigenvectors v_s satisfy the eigenvalue equation

$$\Sigma_z v_s=s v_s, \quad s=\pm 1. \tag{25}$$

The eigenvalue equation of $\mathcal{P}\Sigma_z$ follows from Eqs. (23) and (25) directly,

$$\mathcal{P}\Sigma_z(\mathcal{P}v_s)=s(\mathcal{P}v_s), \tag{26}$$

which makes Eq. (20) separable from the bispinor space. We factorize the wave function $\mathcal{P}\phi$ as

$$\mathcal{P}\phi=|\rho\rangle(\mathcal{P}v_s). \tag{27}$$

The $|\rho\rangle$ is a wave function belonging to the photon Hilbert space, satisfying the following equation:

$$\left[2kp\sum_\alpha \omega_\alpha N_\alpha+2epA-e^2A^2+e^2g^2s-p^2+m_e^2\right]|\rho\rangle=0. \tag{28}$$

In our earlier work [12] we developed a Lie-algebra method, by which we obtained analytical wave functions for a single nonrelativistic electron moving in a multimode arbitrarily polarized photon field. By using this method, we obtained analytical wave functions and energy levels for a charged harmonic oscillator in a photon field. The results for the energy levels of an irradiated harmonic oscillator were used in modeling the energy shifts of atomic bound states irradiated by a laser field [16].

Equation (28) involves only quadratic photon operators, which, in the case of a finite number of modes, form a Lie algebra with finite dimensions. The problems to solve just fits our Lie-algebra method.

To show the method, we treat the two-mode case as an example. First of all, we define a Lie algebra **g** by the basis set

$$\{N_1,N_2,I,a_1,a_2,a_1^\dagger,a_2^\dagger,a_1^2,a_2^2,a_1^{\dagger 2},a_2^{\dagger 2},\; a_1a_2,a_1^\dagger a_2^\dagger,a_1a_2^\dagger,a_1^\dagger a_2\}, \tag{29}$$

and

$$\begin{aligned}h=&2\mathcal{B}_1N_1+\mathcal{A}_1a_1^2+\mathcal{A}_1^*a_1^{\dagger 2}+2\mathcal{B}_2N_2+\mathcal{A}_2a_2^2+\mathcal{A}_2^*a_2^{\dagger 2}\\&+2\mathcal{C}a_1a_2+2\mathcal{C}^*a_1^\dagger a_2^\dagger+2\mathcal{D}a_1a_2^\dagger+2\mathcal{D}^*a_1^\dagger a_2+\mathcal{F}I\\&+\mathcal{G}_1a_1+\mathcal{G}_1^*a_1^\dagger+\mathcal{G}_2a_2+\mathcal{G}_2^*a_2^\dagger\end{aligned} \tag{30}$$

is an element of **g**. By defining the coefficients

$$\begin{aligned}\mathcal{B}_2&=kp\omega_1+e^2g_1^2,\\ \mathcal{B}_2&=kp\omega_2+e^2g_2^2,\\ \mathcal{A}_1&=e^2g_1^2\cos\xi_1e^{i\Theta_1},\\ \mathcal{A}_2&=e^2g_2^2\cos\xi_2e^{i\Theta_2},\\ \mathcal{C}&=e^2g_1g_2\cos\frac{\xi_1+\xi_2}{2}e^{i(\Theta_1+\Theta_2)/2},\\ \mathcal{D}&=e^2g_1g_2\cos\frac{\xi_1-\xi_2}{2}e^{i(\Theta_1-\Theta_2)/2},\\ \mathcal{F}&=m_e^2-p^2+e^2g^2s,\\ \mathcal{G}_1&=2eg_1p\epsilon_1,\\ \mathcal{G}_2&=2eg_2p\epsilon_2,\end{aligned} \tag{31}$$

Eq. (28) can be expressed as

$$h|\rho\rangle=0. \tag{32}$$

There is a trivial Cartan subalgebra $\mathbf{h}\subset\mathbf{g}$, with the number operators and the identity operator as the basis set

$$\{N_1,N_2,I\}, \tag{33}$$

such that the other basis elements in **g** are eigenoperators of an element of **h** in commutator relations; e.g.,

$$[2\mathcal{B}_1 N_1 + 2\mathcal{B}_2 N_2, a_1 a_2^\dagger] = (-2\mathcal{B}_1 + 2\mathcal{B}_2) a_1 a_2^\dagger. \tag{34}$$

The eigenstates for an element of the trivial Cartan algebra are very easy to find, but the problem is that h is an element of the trivial Cartan subalgebra only in the noninteracting case. The question becomes: Can we find an alternative representation such that the operator h of an interacting system will be an element of a trivial Cartan subalgebra in the alternative representation? If we can, the problem is solved.

To find the alternative representation, we follow the following numbered steps:

(i) We truncate the Lie-algebra element h expressed by Eq. (30) by keeping only quadratic photon operators

$$h \to \overline{h} = 2\mathcal{B}_1 N_1 + \mathcal{A}_1 a_1^2 + \mathcal{A}_1^* a_1^{\dagger 2} + 2\mathcal{B}_2 N_2 + \mathcal{A}_2 a_2^2 + \mathcal{A}_2^* a_2^{\dagger 2} + 2\mathcal{C} a_1 a_2 + 2\mathcal{C}^* a_1^\dagger a_2^\dagger + 2\mathcal{D} a_1 a_2^\dagger + 2\mathcal{D}^* a_1^\dagger a_2. \tag{35}$$

(ii) We calculate the commutators of $\overline{h}$ with operators a_1, a_2, $a_1^\dagger$, and $a_2^\dagger$ successively, and write the result according to the order

$$\begin{aligned}
[\overline{h}, a_1] &= -2\mathcal{B}_1 a_1 - 2\mathcal{D}^* a_2 - 2\mathcal{A}_1^* a_1^\dagger - 2\mathcal{C}^* a_2^\dagger,\\
[\overline{h}, a_2] &= -2\mathcal{D} a_1 - 2\mathcal{B}_2 a_2 - 2\mathcal{C}^* a_1^\dagger - 2\mathcal{A}_2^* a_2^\dagger,\\
[\overline{h}, a_1^\dagger] &= 2\mathcal{A}_1 a_1 + 2\mathcal{C} a_2 + 2\mathcal{B}_1 a_1^\dagger + 2\mathcal{D} a_2^\dagger,\\
[\overline{h}, a_2^\dagger] &= 2\mathcal{C} a_1 + 2\mathcal{A}_2 a_2 + 2\mathcal{D}^* a_1^\dagger + 2\mathcal{B}_2 a_2^\dagger.
\end{aligned} \tag{36}$$

(iii) We write the transpose of the coefficient matrix of the right-hand side of the above equation:

$$2\begin{pmatrix} -\mathcal{B}_1 & -\mathcal{D} & \mathcal{A}_1 & \mathcal{C} \\ -\mathcal{D}^* & -\mathcal{B}_2 & \mathcal{C} & \mathcal{A}_2 \\ -\mathcal{A}_1^* & -\mathcal{C}^* & \mathcal{B}_1 & \mathcal{D}^* \\ -\mathcal{C}^* & -\mathcal{A}_2^* & \mathcal{D} & \mathcal{B}_2 \end{pmatrix}. \tag{37}$$

(iv) We obtain four eigenvalues and the corresponding four eigenvectors of the above matrix. The four eigenvalues are

$$\begin{aligned}
\rho_{-1}^{(1)} &= -\rho_{+1}^{(1)},\\
\rho_{+1}^{(1)} &= 2\sqrt{\tfrac{1}{2}(\mathcal{B}_1'^2 + \mathcal{B}_2'^2) - |\mathcal{C}'|^2 + |\mathcal{D}'|^2 - \mathcal{Y}},\\
\rho_{-1}^{(2)} &= -\rho_{+1}^{(2)},\\
\rho_{+1}^{(2)} &= 2\sqrt{\tfrac{1}{2}(\mathcal{B}_1'^2 + \mathcal{B}_2'^2) - |\mathcal{C}'|^2 + |\mathcal{D}'|^2 + \mathcal{Y}},
\end{aligned} \tag{38}$$

where

$$\mathcal{Y} = \sqrt{\frac{(\mathcal{B}_2'^2 - \mathcal{B}_1'^2)^2}{4} - (\mathcal{B}_2' - \mathcal{B}_1')^2 |\mathcal{C}'|^2 + (\mathcal{B}_1' + \mathcal{B}_2')^2 |\mathcal{D}'|^2},$$

$$\begin{aligned}
\mathcal{B}_1' &= \mathcal{B}_1 \cosh(2\chi_1) + \mathcal{A}_1 \sinh(2\chi_1) e^{-i\Theta_1}\\
&= [(kp\omega_1 + e^2 g_1^2)^2 - e^4 g_1^4 \cos^2 \xi_1]^{1/2},\\
\mathcal{B}_2' &= \mathcal{B}_2 \cosh(2\chi_2) + \mathcal{A}_2 \sinh(2\chi_2) e^{-i\Theta_2}\\
&= [(kp\omega_2 + e^2 g_2^2)^2 - e^4 g_2^4 \cos^2 \xi_2]^{1/2},\\
\mathcal{C}' &= \mathcal{C}\cosh(\chi_1 + \chi_2) + \mathcal{D}\sinh(\chi_1 + \chi_2) e^{i\Theta_2},\\
\mathcal{D}' &= \mathcal{C}\sinh(\chi_1 + \chi_2) e^{-i\Theta_2} + \mathcal{D}\cosh(\chi_1 + \chi_2),
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
\tanh(2\chi_1) &= -\frac{|\mathcal{A}_1|}{\mathcal{B}_1} = -\frac{e^2 g_1^2 \cos\xi_1}{kp\omega_1 + e^2 g_1^2},\\
\tanh(2\chi_2) &= -\frac{|\mathcal{A}_2|}{\mathcal{B}_2} = -\frac{e^2 g_2^2 \cos\xi_2}{kp\omega_2 + e^2 g_2^2}.
\end{aligned} \tag{40}$$

In most cases the following inequalities hold:

$$\begin{aligned}
kp\omega_1 &\gg e^2 g_1^2,\\
kp\omega_2 &\gg e^2 g_2^2.
\end{aligned} \tag{41}$$

Hence, we can see that χ_1 and χ_2 are small numbers, and $|\mathcal{B}_1|$ and $|\mathcal{B}_2|$ are much larger than $|\mathcal{C}|$ and $|\mathcal{D}|$. These conditions lead to $|\mathcal{B}_1'|$ and $|\mathcal{B}_2'|$ being much larger than $|\mathcal{C}'|$ and $|\mathcal{D}'|$. Thus all four roots are real numbers.

The expressions of the four eigenvectors obtained in our earlier work [12] can be directly used here. For the completeness of this paper, we just rewrite the results as follows:

$$c_1 = t_1 \begin{vmatrix} a_1 & a_2 & a_1^\dagger & a_2^\dagger \\ -2\mathcal{D}^* & -2\mathcal{B}_2 + \rho_{+1}^{(1)} & 2\mathcal{C} & 2\mathcal{A}_2 \\ -2\mathcal{A}_1^* & -2\mathcal{C}^* & 2\mathcal{B}_1 + \rho_{+1}^{(1)} & 2\mathcal{D}^* \\ -2\mathcal{C}^* & -2\mathcal{A}_2^* & 2\mathcal{D} & 2\mathcal{B}_2 + \rho_{+1}^{(1)} \end{vmatrix}, \tag{42}$$

$$c_2 = t_2 \begin{vmatrix} a_1 & a_2 & a_1^\dagger & a_2^\dagger \\ -2\mathcal{B}_1 + \rho_{+1}^{(2)} & -2\mathcal{D} & 2\mathcal{A}_1 & 2\mathcal{C} \\ -2\mathcal{A}_1^* & -2\mathcal{C}^* & 2\mathcal{B}_1 + \rho_{+1}^{(2)} & 2\mathcal{D}^* \\ -2\mathcal{C}^* & -2\mathcal{A}_2^* & 2\mathcal{D} & 2\mathcal{B}_2 + \rho_{+1}^{(2)} \end{vmatrix},$$

$$c_1^\dagger = t_1^* \begin{vmatrix} a_1 & a_2 & a_1^\dagger & a_2^\dagger \\ 2\mathcal{C}^* & 2\mathcal{A}_2^* & -2\mathcal{D} & -2\mathcal{B}_2 + \rho_{+1}^{(1)} \\ 2\mathcal{B}_1 + \rho_{+1}^{(1)} & 2\mathcal{D} & -2\mathcal{A}_1 & -2\mathcal{C} \\ 2\mathcal{D}^* & 2\mathcal{B}_2 + \rho_{+1}^{(1)} & -2\mathcal{C} & -2\mathcal{A}_2 \end{vmatrix},$$

$$c_2^\dagger = t_2^* \begin{vmatrix} a_1 & a_2 & a_1^\dagger & a_2^\dagger \\ 2\mathcal{A}_1^* & 2\mathcal{C}^* & -2\mathcal{B}_1 + \rho_{+1}^{(2)} & -2\mathcal{D}^* \\ 2\mathcal{B}_1 + \rho_{+1}^{(2)} & 2\mathcal{D} & -2\mathcal{A}_1 & -2\mathcal{C} \\ 2\mathcal{D}^* & 2\mathcal{B}_2 + \rho_{+1}^{(2)} & -2\mathcal{C} & -2\mathcal{A}_2 \end{vmatrix},$$

where the normalization constants t_1 and t_2 are chosen such that

$$[c_1,c_1^\dagger]=[c_2,c_2^\dagger]=I,$$
$$(c_1)^\dagger=c_1^\dagger, \tag{43}$$
$$(c_2)^\dagger=c_2^\dagger.$$

The other commutators are zero,

$$[c_1,c_2]=[c_1^\dagger,c_2^\dagger]=[c_1,c_2^\dagger]=[c_2,c_1^\dagger]=0. \tag{44}$$

with c-photon operators, the truncated Lie-algebra element $\overline{h}$ can be expressed in terms of the number operators of c photons,

$$\overline{h}=\rho_{+1}^{(1)}N_{c_1}+\rho_{+1}^{(2)}N_{c_2}, \tag{45}$$

where

$$N_{c_1}=\tfrac{1}{2}(c_1c_1^\dagger+c_1^\dagger c_1),$$
$$N_{c_2}=\tfrac{1}{2}(c_2c_2^\dagger+c_2^\dagger c_2). \tag{46}$$

For convenience, we write the transformation, Eq. (42), as

$$\begin{aligned}
c_1&=\alpha_{11}a_1+\alpha_{12}a_2+\beta_{11}a_1^\dagger+\beta_{12}a_2^\dagger,\\
c_2&=\alpha_{21}a_1+\alpha_{22}a_2+\beta_{21}a_1^\dagger+\beta_{22}a_2^\dagger,\\
c_1^\dagger&=\beta_{11}^*a_1+\beta_{12}^*a_2+\alpha_{11}^*a_1^\dagger+\alpha_{12}^*a_2^\dagger,\\
c_2^\dagger&=\beta_{21}^*a_1+\beta_{22}^*a_2+\alpha_{21}^*a_1^\dagger+\alpha_{22}^*a_2^\dagger.
\end{aligned} \tag{47}$$

The linear operators in Eq. (30) can be written in terms of c-photon operators,

$$\mathcal{G}_1a_1+\mathcal{G}_1^*a_1^\dagger+\mathcal{G}_2a_2+\mathcal{G}_2^*a_2^\dagger=\mathcal{G}_1'c_1+\mathcal{G}_1'^*c_1^\dagger+\mathcal{G}_2'c_2+\mathcal{G}_2'^*c_2^\dagger; \tag{48}$$

thus

$$\begin{pmatrix}\mathcal{G}_1'\\ \mathcal{G}_2'\\ \mathcal{G}_1'^*\\ \mathcal{G}_2'^*\end{pmatrix}=(L^{-1})^T\begin{pmatrix}\mathcal{G}_1\\ \mathcal{G}_2\\ \mathcal{G}_1^*\\ \mathcal{G}_2^*\end{pmatrix}, \tag{49}$$

where

$$L=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12}\\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22}\\ \beta_{11}^* & \beta_{12}^* & \alpha_{11}^* & \alpha_{12}^*\\ \beta_{21}^* & \beta_{22}^* & \alpha_{21}^* & \alpha_{22}^*\end{pmatrix}. \tag{50}$$

The inverse of L can be obtained by evaluating the mixed commutators between c-photon operators and a-photon operators. Thus we have

$$L^{-1}=\begin{pmatrix}\alpha_{11}^* & \alpha_{21}^* & -\beta_{11} & -\beta_{21}\\ \alpha_{12}^* & \alpha_{22}^* & -\beta_{12} & -\beta_{22}\\ -\beta_{11}^* & -\beta_{21}^* & \alpha_{11} & \alpha_{21}\\ -\beta_{12}^* & -\beta_{22}^* & \alpha_{12} & \alpha_{22}\end{pmatrix}. \tag{51}$$

The Lie-algebra element h is then expressed in terms of c-photon operators as

$$h=\rho_{+1}^{(1)}N_{c_1}+\rho_{+1}^{(2)}N_{c_2}+\mathcal{G}_1'c_1+\mathcal{G}_1'^*c_1^\dagger+\mathcal{G}_2'c_2+\mathcal{G}_2'^*c_2^\dagger+\mathcal{F}I. \tag{52}$$

We can introduce now the displacement operator D to eliminate the linear terms of the annihilation and creation operators

$$\begin{gathered}|\rho\rangle\to|\rho'\rangle=D|\rho\rangle,\\ D=\exp(-\delta_1c_1^\dagger+\delta_1^*c_1-\delta_2c_2^\dagger+\delta_2^*c_2).\end{gathered} \tag{53}$$

It is easy to verify that

$$\begin{aligned}
Dc_1D^\dagger&=c_1+\delta_1I,\\
Dc_1^\dagger D^\dagger&=c_1^\dagger+\delta_1^*I,\\
Dc_2D^\dagger&=c_2+\delta_2I,\\
Dc_2^\dagger D^\dagger&=c_2^\dagger+\delta_2^*I.
\end{aligned} \tag{54}$$

By choosing

$$\delta_1=-\frac{\mathcal{G}_1'^*}{\rho_{+1}^{(1)}},$$
$$\delta_2=-\frac{\mathcal{G}_2'^*}{\rho_{+1}^{(2)}}, \tag{55}$$

the Lie-algebra element h in Eq. (52) becomes

$$h'=\rho_{+1}^{(1)}N_{c_1}+\rho_{+1}^{(2)}N_{c_2}+\mathcal{F}'I, \tag{56}$$

where

$$\mathcal{F}'=m_e^2-p^2+e^2g^2s-\frac{|\mathcal{G}_1'|^2}{\rho_{+1}^{(1)}}-\frac{|\mathcal{G}_2'|^2}{\rho_{+1}^{(2)}}. \tag{57}$$

Thus, Eq. (32) becomes

$$h'|\rho'\rangle=0, \tag{58}$$

which has solutions in the form

$$|\rho'\rangle=|n_1,n_2\rangle_c, \tag{59}$$

if the following energy momentum relation is satisfied:

$$p^2=m_e^2+e^2g^2s+\rho_{+1}^{(1)}(n_1+\tfrac{1}{2})+\rho_{+1}^{(2)}(n_2+\tfrac{1}{2})$$
$$-\frac{|\mathcal{G}_1'|^2}{\rho_{+1}^{(1)}}-\frac{|\mathcal{G}_2'|^2}{\rho_{+1}^{(2)}}. \quad (60)$$

(v) We obtain the vacuum state and number states for the c-photon system. There are two ways to obtain the new vacuum and number states. One may solve the following set of infinite-dimensional linear equations:

$$c_1|0,0\rangle_c=0,$$
$$c_2|0,0\rangle_c=0 \quad (61)$$

to obtain the vacuum state $|0,0\rangle_c$, which has been given in Ref. [12]. One can generate all new number states by using new creation operators $c_1^\dagger$ and $c_2^\dagger$ to act on the new vacuum state. The second method is used to find an exponential transformation in the photon Hilbert space to transform the old vacuum and number states to the new vacuum and number states, as it transforms the a-photon operators to c-photon operators [16]. By applying the second method, we solved the problem of a harmonic oscillator in a radiation field. Now we repeat our method here. We assume that the transformation of Eq. (47) can be written in an exponential form

$$e^f\mathbf{a}e^{-f}=L\mathbf{a}, \quad (62)$$

where f is a quadratic scalar operator satisfying

$$f=\mathbf{a}^TF\mathbf{a}, \quad (63)$$

with $\mathbf{a}$ being a column vector of operators

$$\mathbf{a}=\begin{pmatrix} a_1\\ a_2\\ a_1^\dagger\\ a_2^\dagger\end{pmatrix}, \quad (64)$$

$\mathbf{a}^T$ the transpose of $\mathbf{a}$, and F a 4×4 matrix in the two-mode case. To look for f, we set

$$[f,\mathbf{a}]=B\mathbf{a}, \quad (65)$$

where B is a 4×4 matrix. We can easily identify the following relation:

$$F=-\tfrac{1}{2}SB, \quad (66)$$

where S is the symplectic metric matrix

$$S=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}. \quad (67)$$

By writing power expansions, we find

$$e^f\mathbf{a}e^{-f}=e^B\mathbf{a}. \quad (68)$$

Thus, we have

$$e^B=L. \quad (69)$$

For a given L, the condition for finding B is that L is nonsingular. The L defined by Eq. (47) is indeed nonsingular. In a convenient notation, we write

$$F=-\tfrac{1}{2}S\ \ln L. \quad (70)$$

For an arbitrary number state, we therefore have

$$|n_1,n_2\rangle_c=e^f|n_1,n_2\rangle, \quad (71)$$

where $|n_1,n_2\rangle$ are eigenstates of N_1 and N_2, while n_1 and n_2 are the photon numbers of the two modes.

After taking the above steps numbered (i)–(v), the problem stated is essentially solved. What remains to be done consists of tracing back all the transformations we have made and writing the final solutions of the original Dirac equation. The final solutions in the two-mode case are

$$\Psi(x)=V_e^{-1/2}\exp[i(k_1N_1+k_2N_2-p)x]$$
$$\times(1+e\gamma k\gamma A/2kp)D^\dagger|n_2,n_2\rangle_c\mathcal{P}v_s. \quad (72)$$

By a direct inspection, we see that these solutions can be formally generalized to the case with arbitrary number of photon modes propagating in the same direction k. Thus, the generalized solutions are in the form

$$\Psi(x)=V_e^{-1/2}\exp[i(k_1N_1+k_2N_2+\cdots-p)x]$$
$$\times(1+e\gamma k\gamma A/2kp)D^\dagger|n_1,n_2,\ldots\rangle_c\mathcal{P}v_s. \quad (73)$$

IV. DISCUSSION

A. Stationary states and energy levels

Without losing generality, we limit our discussion in the two-mode case. The obtained solutions are in a Lorentz invariant form where the four coordinates, temporal and spatial coordinates, are treated equally. The solutions look time-dependent, but they are stationary solutions as shown below. The solutions can be factorized as

$$\Psi(x)=\exp(-i\mathcal{E}t+iH_\gamma t)\Psi(\mathbf{r}), \quad (74)$$

where

$$H_\gamma=\omega_1N_1+\omega_2N_2 \quad (75)$$

is the Hamiltonian of the free-photon field. By the transformation of Eq. (74), the original Dirac equation, Eq. (1), can be cast, after multiplying with γ^0, into the time-independent form

$$(H_\gamma+H_e+V)\Psi(\mathbf{r})=\mathcal{E}\Psi(\mathbf{r}), \quad (76)$$

where

$$H_e=\boldsymbol{\alpha}\cdot(-i\nabla)+\beta m_e \quad (77)$$

is the free-electron Hamiltonian, and

$$V=-e\boldsymbol{\alpha}\cdot[\mathbf{A}_1(-\mathbf{k}_1\cdot\mathbf{r})+\mathbf{A}_2(-\mathbf{k}_2\cdot\mathbf{r})] \quad (78)$$

is the electron-photon interaction term, with

$$A_\alpha(-\mathbf{k}_\alpha\cdot\mathbf{r})=\exp(-iH_\gamma t)A_\alpha(k_\alpha x)\exp(iH_\gamma t) \quad (\alpha=1,2) \tag{79}$$

as the photon field operator. The positive-kinetic-energy solutions of Eq. (76) are given by

$$\Psi_{pn_1n_2s}(\mathbf{r})=V_e^{1/2}\exp[i(\mathbf{p}\cdot\mathbf{r}-\mathbf{k}_1\cdot\mathbf{r}N_1-\mathbf{k}_2\cdot\mathbf{r}N_2)]$$
$$\times(1+e\gamma k\gamma A/2kp)D^\dagger|n_1,n_2\rangle_c\mathcal{P}v_s, \tag{80}$$

where V_e is the normalization volume of the electron.

The corresponding energy eigenvalue $\mathcal{E}$, the total energy of the electron-photon system, is the temporal component of the four-vector p, which can be seen by comparing Eqs. (5) and (8) with Eq. (74). In the single-mode case we have proven that $\mathbf{p}$, the spatial component of p, is the total momentum of the system as an eigenvalue of the total momentum operator $-i\nabla+\mathbf{k}N$. It is easy to generalize the result to the multimode case, such that p still stands for the total four-momentum of the electron-photon system.

From Eq. (60), we can see that the total four-momentum p can have a following decomposition:

$$p=P+\kappa k, \tag{81}$$

where P is on the electron-mass shell and k is a null vector, i.e.,

$$P^2=m_e^2,$$
$$k^2=0, \tag{82}$$

and the value of κ is

$$\kappa=\frac{1}{2kP}\left[e^2g^2s+\rho_{+1}^{(1)}(n_1+\tfrac{1}{2})+\rho_{+1}^{(2)}(n_2+\tfrac{1}{2})-\frac{|\mathcal{G}_1'|^2}{\rho_{+1}^{(1)}}-\frac{|\mathcal{G}_2'|^2}{\rho_{+1}^{(2)}}\right]. \tag{83}$$

The four-vector relation Eq. (81) can be written as

$$\mathcal{E}=E+\kappa\omega,$$
$$\mathbf{p}=\mathbf{P}+\kappa\mathbf{k}. \tag{84}$$

In our definition, even $\omega=|\mathbf{k}|=1$; for the convenience of our readers, we still spell it out clearly. The interaction energy and momentum are

$$\Delta\mathcal{E}=\kappa\omega-\sum_\alpha(n_\alpha+\tfrac{1}{2})\omega_\alpha,$$
$$\Delta\mathbf{p}=\kappa\mathbf{k}-\sum_\alpha(n_\alpha+\tfrac{1}{2})\mathbf{k}_\alpha, \tag{85}$$

where we can see that the interaction four-momentum is always along the light propagating direction. In the strong radiation field cases, the interaction energy is mainly the ponderomotive energy. Equation (85) shows also that the ponderomotive energy has its counterpart, which should be called the pondermotive momentum.

The stationary feature of these solutions is due to the quantum field treatment. In the Dirac equation, Eq. (1), the electromagnetic field is described by photon operators, and the photons are thus treated as particles in the closed system as well as the electron. In contrast, the classical Volkov solutions are for an electron interacting with a classical electromagnetic (em) wave. The classical em wave is an external and time-dependent field. Thus, the classical Volkov solutions are not stationary solutions, and one cannot obtain an energy level for the interacting system by the classical-field treatment.

B. Large-photon-number limit and nonrelativistic (NR) limit

Solutions in the large-photon-number limit and NR limit have wide applications in MPI theory [7,15] and the Kapitza-Dirac effect theory [14]. To obtain solutions in the large-photon-number limit we rewrite Eq. (80) by inserting an identity operator into the expression of the solutions,

$$\Psi_{pn_1n_2s}(\mathbf{r})=V_e^{-1/2}\exp[i(\mathbf{p}\cdot\mathbf{r}-\mathbf{k}_1\cdot\mathbf{r}N_1-\mathbf{k}_2\cdot\mathbf{r}N_2)](1+e\gamma k\gamma A/2kp)\exp[i(\mathbf{k}_1\cdot\mathbf{r}N_1+\mathbf{k}_2\cdot\mathbf{r}N_2)]$$
$$\times\exp[i(-\mathbf{k}_1\cdot\mathbf{r}N_1-\mathbf{k}_2\cdot\mathbf{r}N_2)]\sum_{l_1l_2}|l_1,l_2\rangle\langle l_1,l_2|D^\dagger|l_1,l_2\rangle_c\mathcal{P}v_s$$
$$=V_e^{-1/2}\sum_{l_1l_2}\exp\{i[\mathbf{p}\cdot\mathbf{r}-\mathbf{k}_1\cdot\mathbf{r}(l_1+\tfrac{1}{2})-\mathbf{k}_2\cdot\mathbf{r}(l_2+\tfrac{1}{2})]\}$$
$$\times[1+e\gamma k\gamma A(\mathbf{r})/2kp]|l_1,l_2\rangle\langle l_1,l_2|D^\dagger|n_1,n_2\rangle_c\mathcal{P}v_s, \tag{86}$$

with

$$A(\mathbf{r})=\mathbf{A}_1(-\mathbf{k}_1\cdot\mathbf{r})+\mathbf{A}_2(-\mathbf{k}_2\cdot\mathbf{r}).$$

In large-photon-number limit, we let n_1 and n_2 be very large numbers, such that

$$g_i\sqrt{n_i}\to\Lambda_i \quad (i=1,2), \tag{87}$$

while g_i without multiplying by $\sqrt{n_i}$ will be considered as 0. In the limit, $\langle l_1,l_2|D^\dagger|n_1,n_2\rangle_c$ depends only on the transferred photon numbers $j_1=l_1-n_1$ and $j_2=l_2-n_2$. By noting

$$\langle l_1,l_2|D^\dagger|n_1,n_2\rangle_c\to\mathcal{J}_{j_1j_2}(\zeta)^*\exp(-ij_1\phi_1-ij_2\phi_2),$$

the solutions are simplified as

$$\begin{aligned}\Psi_{pn_1n_2s}(\mathbf{r})=V_e^{-1/2}\sum_{j_1j_2}&\exp[i(\mathbf{P}+(u_{p1}-j_1)\mathbf{k}_1\\&+(u_{p2}-j_2)\mathbf{k}_2)\cdot\mathbf{r}][1+e\gamma k\gamma A(\mathbf{r})]/2kp\\&\times|n_1+j_1,n_2+j_2\rangle\mathcal{J}_{j_1j_2}(\zeta)^*\\&\times\exp(-ij_1\phi_1-ij_2\phi_2)\mathcal{P}v_s,\end{aligned}\tag{88}$$

where $\mathcal{J}_{j_1j_2}(\zeta)$ are the generalized Bessel functions obtained by the integration method [12],

$$\begin{aligned}\mathcal{J}_{j_1j_2}(\zeta)=\sum_{q_3,q_4,q_5,q_6}&J_{-j_1+2q_3+q_5+q_6}(\zeta_1)e^{-i(2q_3+q_5+q_6)\phi_1}\\&\times J_{-j_2+2q_4+q_5-q_6}(\zeta_2)e^{-i(2q_4+q_5-q_6)\phi_2}\\&\times J_{-q_3}(\zeta_3)e^{iq_3\phi_3}\cdots J_{-q_6}(\zeta_6)e^{iq_6\phi_6},\end{aligned}\tag{89}$$

and

$$\begin{gathered}\zeta_1=\frac{2|e|\Lambda_1}{k_1P}|\mathbf{P}\cdot\boldsymbol{\epsilon}_1|,\\ \phi_1=\tan^{-1}[(P_y/P_x)\tan(\xi_1/2)]+\tfrac{1}{2}\Theta_1,\\ \zeta_2=\frac{2|e|\Lambda_2}{k_2P}|\mathbf{P}\cdot\boldsymbol{\epsilon}_2|,\\ \phi_2=\tan^{-1}[(P_y/P_x)\tan(\xi_2/2)]+\tfrac{1}{2}\Theta_2,\\ \zeta_3=\tfrac{1}{2}u_{p1}\cos\xi_1,\\ \phi_3=\Theta_1,\\ \zeta_4=\tfrac{1}{2}u_{p2}\cos\xi_2,\\ \phi_4=\Theta_2,\\ \zeta_5=2e^2\Lambda_1\Lambda_2\cos[\tfrac{1}{2}(\xi_1+\xi_2)]/(k_1+k_2)P,\\ \phi_5=\tfrac{1}{2}(\Theta_1+\Theta_2),\\ \zeta_6=2e^2\Lambda_1\Lambda_2\cos[\tfrac{1}{2}(\xi_1-\xi_2)]/(k_1-k_2)P,\\ \phi_6=\tfrac{1}{2}(\Theta_1-\Theta_2),\end{gathered}\tag{90}$$

and

$$u_{pi}=\frac{e^2\Lambda_i^2}{k_iP}\quad (i=1,2). \tag{91}$$

In the NR limit the solutions expressed by Eq. (88) can be further simplified:

$$\begin{aligned}\Psi_{pn_1n_2}(\mathbf{r})=V_e^{-1/2}\sum_{j_1j_2}&\exp\{i[\mathbf{P}+(u_{p1}-j_1)\mathbf{k}_1\\&+(u_{p2}-j_2)\mathbf{k}_2]\cdot\mathbf{r}\}|n_1+j_1,n_2+j_2\rangle\mathcal{J}_{j_1j_2}(\zeta)^*\\&\times\exp(-ij_1\phi_1-ij_2\phi_2).\end{aligned}\tag{92}$$

In the arguments of the generalized Bessel functions, all k_iP $(i=1,2)$ need to be replaced by $\omega_i m_e$. This result does agree with our earlier result [12].

C. Evaluation of electron self-energy and mass renormalization

Most applications of the nonperturbative quantum electrodynamics (NPQED) developed in recent years are in the strong radiation field regime. Here we present an attempt to use the NPQED approach to treat some typical QED effects in vacuum. The method may initiate an alternative way to evaluate free and bound electron self-energies, electron mass renormalization, and bound-electron Lamb shifts. Here we only show how this method and our results work for the electron mass-renormalization problem. The renormalized mass can be found from the energy shift due to spontaneous photon emission, which should be a part of the energy level of the wave function shown by Eq. (73), in principle. Since Eq. (73) is only for the case where all modes propagate in one direction, the mass-renormalization effect contained in the energy level of the solution Eq. (73) is only due to the modes spontaneously emitted in that direction. To work out a complete mass-renormalization effect, one has to deal with all possible spontaneously emitted modes propagating in all different directions. The following approach is not rigorous, but instructive.

The energy level for an electron interacting with a single-mode photon field is [17]

$$\mathcal{E}=\frac{\mathbf{P}^2}{2m_e}+\frac{C(n+\frac{1}{2})}{m_e}-\frac{e^2g^2(\mathbf{P}\cdot\boldsymbol{\epsilon}_c)(\mathbf{P}\cdot\boldsymbol{\epsilon}_c^*)}{m_eC}, \tag{93}$$

where

$$\begin{gathered}C=[(m_e\omega+e^2g^2)^2-e^4g^4\cos^2\xi]^{1/2},\\ g=(2V_\gamma\omega)^{-1/2}.\end{gathered}\tag{94}$$

The first term in Eq. (93) is the electron kinetic energy. The second term, which is proportional to the background photon number of the field with the inclusion of the zero-point energy of the field, is important in the strong radiation field case. The third term, which originates from the $\mathbf{P}\cdot\mathbf{A}$ term in the Hamiltonian, is due to spontaneous emission and reabsorption. Without considering cross interactions due to different modes in the weak-field case, the energy expression of Eq. (93) can be generalized to the multimode case and reads as

$$\mathcal{E}=\frac{\mathbf{P}^2}{2m_e}+\sum\frac{C(n+\frac{1}{2})}{m_e}-\sum\frac{e^2g^2(\mathbf{P}\cdot\boldsymbol{\epsilon}_c)(\mathbf{P}\cdot\boldsymbol{\epsilon}_c^*)}{m_eC}, \tag{95}$$

where the summation is made over all possible modes propagating in all different directions.

If there is no external radiation field, the background photon number n is zero. By ignoring the zero-point energy, the energy shift due to all possible spontaneous emission and reabsorption is

$$\begin{aligned}\Delta E_{\mathbf{k}}&=-\sum\frac{e^2g^2|\mathbf{P}\cdot\boldsymbol{\epsilon}_c|^2}{m_e(m_e\omega+e^2g^2)}\\&=-\sum\frac{e^2}{m_e^22V_\gamma}\frac{|\mathbf{P}\cdot\boldsymbol{\epsilon}_c|^2}{\omega^2+\omega_0^2}\ (\omega_0=\sqrt{e^2/2V_\gamma m_e}).\end{aligned} \tag{96}$$

The self-energy of the electron can be obtained from the above equation by performing the summation as an integral over the $\mathbf{k}$ space, summing over the two independent polarizations. The result is

$$\Delta E=-\frac{1}{6\pi^2}\frac{e^2}{m_e}|\mathbf{P}|^2\frac{1}{m_e}\left(\omega_{\max}-\omega_0\tan^{-1}\frac{\omega_{\max}}{\omega_0}\right). \tag{97}$$

By taking a cutoff value $\omega_{\max}=m_e$ and letting $V_\gamma\to\infty$, we simplify the above expression as

$$\Delta E=-\frac{1}{6\pi^2}\frac{e^2}{m_e}|\mathbf{P}|^2, \tag{98}$$

which leads to a correct renormalized mass [18]

$$m_e^*=\left[1+\left(\frac{1}{137}\right)\frac{4}{3\pi}\right]m_e, \tag{99}$$

where m_e is the bare mass of the electron.

D. Comments on the quantum-field approach

There are two main reasons for using the quantum-field approach in strong radiation field cases. The first stems from the fact that there might be some quantum-field effects in strong radiation field cases. If there is one, it can only be predicted by a quantum-field treatment; that is something we are still working on. The second reason is as follows. In quantum-field treatments, since the interacting system of atoms and photons has a well-defined total energy and momentum at every step, the quantum-field results provide a precise prescription for formulating the relevant energy-momentum conservation laws as the electron ionizes, acquires ponderomotive energy and momentum, and then ultimately leaves the radiation field. In time-dependent semiclassical theories, those quantities are not always so well defined. The first reason may not convince some readers to use the quantum-field approach in strong radiation field cases, but the second reason is sufficient enough to establish the importance of the quantum-field approach. The modeling of Bucksbaum *et al.*'s experiment on the half Kapitza-Dirac effect by Guo and Drake shows the success of the quantum-field approach to strong-field multiphoton processes.

In the most general case, the Dirac equation for an electron interacting with a multimode photon field propagating in different directions has not been solved. For wider applications and further development on the theory, searching for exact and approximate quantum-field solutions in the most general case is of great importance.

In the approach showed in this paper, many-electron effects, such as vacuum polarization, production of electron-positron pairs, and some other electron-photon effects, are not included. The external field treated in this approach is a plane wave, not a solution of the Maxwell equations with the involved electron as the radiative source of the field. Thus, the reaction of the electron on the field is not included in the treatment. The obtained wave functions with a sign change on the mass can describe a positron, which can be seen from Eq. (1). Using the obtained single-body electron and positron wave functions as basis sets to express the electron wave operator in performing second quantization, as in the traditional QED, many relativistic QED effects might be treated properly by this nonperturbative method. These should be good topics for future research.

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