Decoherence and dephasing in a quantum measurement process

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We numerically simulate the quantum measurement process by modeling the measuring apparatus as a one-dimensional Dirac comb that interacts with an incoming object particle. The global effect of the apparatus can be well schematized in terms of the total transmission probability and the *decoherence parameter*, which quantitatively characterizes the loss of quantum-mechanical coherence and the wave-function collapse by measurement. These two quantities alone enable one to judge whether the apparatus works well or not as a detection system. We derive simple theoretical formulas that are in excellent agreement with the numerical results, and can be very useful in order to make a ''design theory'' of a measuring system (detector). We also discuss some important characteristics of the wave-function collapse. [S1050-2947(96)07507-5]

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I. INTRODUCTION

Quantum mechanics is the most fundamental physical theory developed during the last seventy years. Nevertheless, physicists still debate the quantum measurement problem [1,2], which has caused them to ponder over some very fundamental questions. While the original formulation, essentially due to von Neumann, introduced the so-called ''collapse'' of the wave function as a postulate, currently most physicists think that it is necessary to analyze and derive the quantum measurement process by starting from fundamental laws. A quantum measurement must be analyzed as a concrete physical process, by means of quantum mechanics itself, and must not be dealt with by postulating a ''collapse'' *à la* von Neumann [3–5].

The seminal formulation, due to von Neumann, is very unsatisfactory from the point of view of the internal consistency of quantum theory, because the act of ''observation,'' namely the very possibility of obtaining information about the quantum properties of the system under investigation, requires the existence of ''classical'' devices that, by definition, do not follow the quantum mechanical laws. It would be baffling, in our opinion, if such a fundamental theory such as quantum mechanics would need classical (i.e., nonquantum) objects in order to ascertain the very value of quantum observables.

A quantum measurement is a *very complicated* physical process, because it involves the interaction of an object system Q with a macroscopic apparatus A, made up of a huge number of elementary constituents. The interactions taking place between Q and the constituents of A will provoke ''dephasing'' or ''decoherence'' on the former. This loss of quantum-mechanical coherence is the physical process that underlines, in our opinion, the so-called ''collapse'' of the wave function.

In order to analyze the above-mentioned loss of quantum-mechanical coherence, we introduced a ''decoherence parameter'' [4,5], which estimates the ''degree of collapse'' of the wave function. This can be regarded as an attempt at giving a quantitative estimate of the dephasing occurring in a quantum measurement process. A similar philosophy has been followed by several authors at different times. Among others, quantitative measures for decoherence were also proposed by Caldeira and Leggett [6] and Paz, Habib, and Zurek [7]. It should be emphasized that the present paper follows an approach originally put forward in 1980 [3].

Strictly speaking, the decoherence parameter should be estimated by means of a dynamical-statistical analysis of the above-mentioned complicated interactions taking place inside the apparatus. This kind of theoretical work is, of course, very hard to carry out in a satisfactory way. For this reason, in the present paper, we shall study the interaction between an object particle Q and a linear array A of δ-shaped potentials (a Dirac comb), representing the elementary constituents of the macroscopic apparatus. The collision centers (δ potentials) are regarded as heavy and structureless in the present model, so that every elementary collision is elastic and we cannot expect genuine dissipation effects, such as absorption, directly stemming from the elementary interactions. Nevertheless, since reflections in the one-dimensional case correspond to leakage provoked by elastic deflections in a three-dimensional apparatus, we can naturally expect to have some kind of ''dissipation'' for Q, provoked by successive reflections from the constituents of A. In fact, we shall observe a certain kind of irreversibility by taking into account the randomness of the macroscopic system A, schematized by the fluctuations of the parameters that characterize the Dirac comb. This is only a caricature of a measurement process, which can only roughly represent the complicated interactions taking place in a real measurement process. Nonetheless, we shall see that even such a rough example can yield, via the evaluation of the decoherence parameter, useful insights into the phenomenon of decoherence, mainly through numerical simulations. At the same time, we shall see that our analysis can yield a possible and preliminary ''design theory'' in order to ''tailor'' an apparatus to be used as a detector in a quantum measurement process.

II. DEPHASING AND INTERACTION IN THE APPARATUS

In the von Neumann–Wigner approach [1,8] the quantum measurement process is described as

$$|\Phi_I\rangle=|\psi\rangle\otimes|A_I\rangle\rightarrow|\Phi_F\rangle=\sum_k c_k|\psi_k\rangle\otimes|A_k\rangle, \quad (2.1)$$

where $|\Phi\rangle,|\psi\rangle,|A\rangle$ are the wave functions of the total $(Q+A)$ Q and A systems, respectively, the subscripts I and F stand for initial and final, respectively, $|\psi\rangle=\Sigma_k c_k|\psi_k\rangle$, $\{|\psi_k\rangle\}_k$ is a complete set of eigenfunctions of Q, and $|A_k\rangle$ denotes the apparatus state displaying the kth result. This is often called a von Neumann measurement process of the first kind.

Many physicists start their discussion on the measurement problem from the von Neumann process (2.1). However, as a matter of fact, Eq. (2.1) does *not* represent any *collapse* of the wave function. It simply displays a *spectral decomposition* [3–5,8], namely, a physical process in which the states of the apparatus A become entangled with those of the object system Q. No measurement has occurred in (2.1), because the phase correlation is still perfectly kept in $|\Phi_F\rangle$: Indeed, the final density matrix

$$\rho_F\equiv|\Phi_F\rangle\langle\Phi_F|=\sum_k |c_k|^2|\psi_k\rangle\langle\psi_k|\otimes|A_k\rangle\langle A_k| + \sum_i \sum_{j\neq i} c_i c_j^*|\psi_i\rangle\langle\psi_j|\otimes|A_i\rangle\langle A_j| \quad (2.2)$$

contains all its off-diagonal terms, which can give rise to any sort of coherent quantum-mechanical effects, such as interference.

It should also be emphasized that in order to view the above process as a ''measurement,'' in some sense, one has to require the orthogonality condition among the apparatus states:

$$\langle A_j|A_k\rangle=\delta_{jk}, \quad \forall j,k. \quad (2.3)$$

The above requirement is of paramount importance both in the von Neumann–Wigner approach and in the so-called environment theories [9]. It also plays a fundamental role in the ''many-worlds interpretation'' [10,11]. However, it is a postulate that cannot be proved and is subject to many criticisms [4,5,12].

By contrast with the approach outlined above, We analyzed the measurement process by describing the apparatus in terms of density matrices (which is physically more sensible), estimated the decoherence parameter ϵ, and discussed under which conditions the off-diagonal terms of the total density matrix can be shown to vanish: What we need is a (macroscopic) detector that is able to act as a *dephaser*, namely, to erase the phase correlations between different states of A [4,5].

In this paper, we shall concentrate on a very particular aspect of this problem. Strictly speaking, the wave-function collapse should be described in terms of the apparatus states, by showing explicitly the disappearance of the off-diagonal part of the total density matrix, as in Eq. (3.9) of the first paper in [5] or Eq. (38) of [4]. This problem is very involved, in general. For this reason, we shall simply model the phenomenon of decoherence by concentrating only on the fundamental role played by the many fine interactions taking place between Q and the elementary constituents of A. To this end, we shall suppress the apparatus states in the following description. This is obviously a drastic simplification. In order to avoid any misunderstandings, we stress that this simplification aims only at performing a *heuristic* treatment of decoherence.

III. THE DECOHERENCE PARAMETER

A. Definition

Let us briefly summarize some of our previous results [4] and reintroduce the *decoherence parameter*, in terms of which a *quantitative* definition of wave-function collapse can be given. Consider a typical Young-type experiment, in which an incoming Q ''particle'' (namely, a Schrödinger wave function relative to a single detection event) is split in two states ψ_1 and ψ_2, corresponding to the two possible routes in the interferometer. We place the apparatus A along the second path, so that the wave function of Q reads

$$\psi=\psi_1+T\psi_2, \quad (3.1)$$

where T is the transmission coefficient. Following our discussion in the preceding section, the apparatus state has been suppressed, although the action of the apparatus on the Q particle has been properly taken into account via T. For simplicity, we assume that ψ_1 and ψ_2 are very close to a plane wave with wave number k. If we want to take into account wave-packet effects, we have to average our results, as, for instance, Eqs. (3.2) or (3.3), with the k-weight factor $|a(k)|^2$ characterizing the wave packet.

Equation (3.1) holds for every single incoming particle. Moreover, every incoming particle is described by the *same* wave function $\psi=\psi_1+\psi_2$ immediately before interacting with the apparatus. However, after the interaction, the apparatus transmission coefficient T will depend on the particular apparatus state at the *very instant* of the passage of the particle. Since the apparatus undergoes random fluctuations (which reflect the internal motion of its elementary quantum constituents) we label the incoming particle with j ($j=1,\ldots,N_p$, where N_p is the total number of particles in an experimental run), and rewrite the transmission coefficient as T_j $(j=1,\ldots,N_p)$. Notice that the same macroscopic state of the apparatus will correspond to many different microscopic states. Consequently, different incoming particles will be affected differently by the interaction with the apparatus, and will be described by slightly different values of T. Accordingly, the probability of detecting the jth particle after recombination reads

$$P^{(j)}\equiv|\psi^{(j)}|^2=|\psi_1+T_j\psi_2|^2=|\psi_1|^2 + |T_j|^2|\psi_2|^2+2\mathrm{Re}(\psi_1^*T_j\psi_2). \quad (3.2)$$

After many particles have been detected, the average probability will be given by

$$\overline{P} \equiv \frac{1}{N_p} \sum_{j=1}^{N_p} P^{(j)} = |\psi_1|^2 + \overline{|T|^2} |\psi_2|^2 + 2\,\mathrm{Re}(\psi_1^* \overline{T} \psi_2), \tag{3.3}$$

where the overbar denotes an *event average* over all events in one experimental run (which is made up of N_p events). Note that $|\overline{T}|^2 \leqslant \overline{|T|^2}$, and that a necessary and sufficient condition for observing no interference (collapse of the wave function) is $\overline{T}=0$. We define the *decoherence parameter* as

$$\epsilon = 1 - \frac{|\overline{T}|^2}{\overline{|T|^2}}, \quad 0 \leqslant \epsilon \leqslant 1, \tag{3.4}$$

and rewrite Eq. (3.3) as

$$\overline{P} = |\psi_1|^2 + \overline{t}\,|\psi_2|^2 + 2\sqrt{\overline{t}}\sqrt{1-\epsilon}\ \mathrm{Re}(\psi_1^* e^{i\beta} \psi_2), \tag{3.5}$$

where $\overline{T} \equiv |\overline{T}| e^{i\beta}$ and $\overline{t} \equiv \overline{|T|^2}$. Notice that $\overline{t}$ is the *experimentally measured* value of the transmission probability. A nonvanishing ϵ is a consequence of the statistical fluctuations in the apparatus. Interference is lost, and hence the wave-function collapse takes place, in the limit $\epsilon=1$. All effects provoked by A on Q are properly taken into account by $\overline{t}=\overline{|T|^2}$ and ϵ.

One can also consider the visibility of the interference pattern

$$V = \frac{\overline{P}_{\max} - \overline{P}_{\min}}{\overline{P}_{\max} + \overline{P}_{\min}} = V_0 \sqrt{(1-\epsilon)} = \frac{2\sqrt{\overline{t}(1-\epsilon)}}{1+\overline{t}}, \tag{3.6}$$

where $V_0 = 2\sqrt{t}/(1+t)$ is the value in the absence of fluctuations (observe that in such a case $t=\overline{t}$). Once again we see that coherence between the two branch waves is totally lost when $\epsilon=1$, in which case the visibility is zero.

It is worth stressing that this approach incorporates in a natural way the possibility of investigating those situations in which coherence is *partially* lost or, stated differently, the wave function is *partially* collapsed. These intermediate cases correspond to the values $0<\epsilon<1$.

The loss of quantum coherence is the result of a huge number of rather ''dirty'' interactions, acting randomly for a certain amount of time. We are therefore interested in the effect on Q of a *huge number of dirty (random) interactions*. Each interaction will take place between Q and one (or a bunch of) elementary constituent(s) of A. The global effect on the wave function of Q will be, as we shall see, a loss of phase coherence. This is the idea we shall pursue in the present study. The task of the decoherence parameter ϵ will be to judge whether the global interaction is ''clean'' ($\epsilon \simeq 1$) or ''dirty'' ($\epsilon \ll 1$).

B. Ergodic assumption

We can now formulate an *ergodic hypothesis*: The ''event'' average over many particles in one experimental run (denoted hitherto with an overbar) will be assumed equal to the *statistical ensemble average* over all the possible microstates of the apparatus. If we denote the latter with $\langle \cdots \rangle$, our assumption reads

$$\overline{\cdots} = \langle \cdots \rangle. \tag{3.7}$$

The significance of the ergodic assumption in quantum physics is not as clear as it is in classical physics, where the ergodic theorem states that the long-time average of an observable can be replaced with an ensemble average over the phase space. Ergodicity is not easy to prove mathematically and is therefore often postulated on physical grounds. In the present context, Eq. (3.7) must also be regarded as a physical postulate.

In the following, the two above-mentioned averages will be used interchangeably by virtue of Eq. (3.7). In particular,

$$\epsilon = 1 - \frac{|\langle T \rangle|^2}{\langle |T|^2 \rangle}. \tag{3.8}$$

On the other hand, one should keep in mind that, strictly speaking, the above two averages cannot always be completely identified. The decoherence parameter, if defined via the event average, as in (3.4) (e.g., in a numerical simulation), in general cannot vanish. Indeed,

$$|\overline{T}|^2 = \frac{1}{N_p^2} \sum_{j,k=1}^{N_p} T_j T_k^* = \frac{1}{N_p^2} \sum_{j=1}^{N_p} |T_j|^2 + \frac{1}{N_p^2} \sum_{j \neq k} T_j T_k^*, \tag{3.9}$$

so that even for a completely random sequence of T_j, such that $\Sigma_{j\neq k} T_j T_k^* \simeq 0$, one obtains

$$\epsilon = 1 - \frac{|\overline{T}|^2}{\overline{|T|^2}} \simeq 1 - \frac{1}{N_p}. \tag{3.10}$$

Therefore, one needs many events ($N_p \gg 1$) in order to be able to ascertain whether the quantum-mechanical coherence is lost. If the latter requirement is not satisfied, in general one is not able to give an operationally meaningful definition of (loss of) coherence, as shown by Eq. (3.10). Observe that this condition is *independent* of the physical requirement that the macroscopic apparatus be characterized by a huge number of degrees of freedom and be able to act as a dephaser.

This observation has interesting spinoffs: The very concept of quantum-mechanical coherence appears to be statistical. One needs the *accumulation* process of many individual events in order to define ϵ and quantitatively estimate the degree of dephasing. This is true not only when the degree of decoherence is maximum, as in Eq. (3.10), but even for perfectly coherent systems: Consider, for example, a double-slit experiment yielding a perfect interference pattern. Nothing can be said about the coherence properties of the quantum system (the beam of particles entering the interferometer) if one does not accumulate many particles in order to build up the interference pattern [13].

The above considerations might make the reader think that no meaning can be ascribed to *single* detection events. This would not be correct. One can give a sensible definition of ''collapse'' for a single particle via Eq. (3.7). In this way, the coherence properties of a single quantum system can be defined via the ensemble average of the macroscopic system it interacted with. Individual detection events were discussed in Ref. [5].

IV. NUMERICAL SIMULATION

A. Dirac comb

We can now investigate the decoherence process from a numerical point of view. The particle interacts with the apparatus (a macroscopic object made up of a huge number of elementary constituents) according to the laws of quantum mechanics (Schrödinger equation). We will describe every interacting constituent (or every bunch of constituents) with a δ potential, and the whole apparatus with a one-dimensional array of δ potentials (a Dirac comb). This is the well-known Kronig-Penney model [14], which has been investigated from several points of view [15]. Previous numerical simulations of the measurement process by means of a Dirac comb were performed in Refs. [4,16]. In the present paper, however, we shall focus our attention on the role played by some particular combination of the numerical constants characterizing the array of potentials. We shall see that the apparatus acts as a ''dephaser'' when some factors appearing in the expression of the decoherence parameter exceed certain ''critical'' values.

The total barrier will be written as

$$V(x)=\sum_{\ell=1}^{N}\Lambda\,\delta(x-b_\ell), \tag{4.1}$$

where N is the total number of δ potentials, which play the role of ''elementary interactions,'' Λ the strength of the interactions, and b_ℓ their locations.

The transmission and reflection coefficients T and R of the whole barrier are computed in Appendix A. One obtains

$$\begin{pmatrix} T \\ 0 \end{pmatrix}=e^{-ikb_N}Z\prod_{\ell=1}^{N-1}e^{ikd_\ell\tau_3}Z\begin{pmatrix} 1 \\ R \end{pmatrix}, \tag{4.2}$$

where τ_3 is the third Pauli matrix, k the wave number of Q, $d_\ell=b_{\ell+1}-b_\ell$, and

$$Z=\begin{pmatrix} 1-i\Omega & -i\Omega \\ i\Omega & 1+i\Omega \end{pmatrix}, \tag{4.3}$$

where $\Omega=\Lambda/\hbar v$, v being the Q particle speed. We shall set $b_1=0$. [As explained in Appendix A, the factor e^{-ikb_N}, appearing in (4.2), should also be put in front of (58) of Ref. [4] and (8.24) of the first paper in Ref. [5].]

So far, the internal motions of the elementary constituents of the apparatus have not been taken into account. These internal motions will give rise to an intrinsic stochasticity of the parameters describing the constituents themselves. In terms of our Dirac-comb model, this stochasticity will be modeled as follows: since the interactions between the apparatus's constituents and the incoming particles will take place in different parts of the apparatus, the positions b_ℓ and the relative spacings $d_\ell=b_{\ell+1}-b_\ell$ will be subject to statistical fluctuations. Moreover, the total number N of interactions will also vary for different incoming particles.

In a previous numerical simulation [4] we showed that this simple Dirac-comb model is able to reproduce correctly many different physical devices, such as a ''phase shifter'' (that can preserve the quantum coherence) or a ''dephaser'' (that is able to provoke complete dephasing and work as a detector), or an ''absorber'' (that yields a nonvanishing absorption coefficient). We also showed that, in contrast to widespread belief, absorption effects stemming from imaginary potentials are not necessarily significant for the collapse of the wave function, because the loss of coherence (of which ϵ is an estimate) stems only from the noise and the number N of elementary interactions in the apparatus. In the present paper, unlike in [4], we shall introduce fluctuations of N and shall analyze in detail the dependence of ϵ on several numerical constants involved. In the case of large absorptionlike effects due to reflections, we shall see that fluctuations become large enough to yield the wave-function collapse.

B. Adjusting the parameters

Equation (4.2) must be solved for many different incoming particles (''events'') and according to the analysis of Sec. III, the average transmission-reflection coefficient and probability read

$$\langle T\rangle=\overline{T}=\frac{1}{N_p}\sum_{j=1}^{N_p}T_j, \tag{4.4}$$

$$\langle R\rangle=\overline{R}=\frac{1}{N_p}\sum_{j=1}^{N_p}R_j, \tag{4.5}$$

$$\langle t\rangle=\langle|T|^2\rangle=\overline{|T|^2}=\frac{1}{N_p}\sum_{j=1}^{N_p}|T_j|^2, \tag{4.6}$$

$$\langle r\rangle=\langle|R|^2\rangle=\overline{|R|^2}=\frac{1}{N_p}\sum_{j=1}^{N_p}|R_j|^2. \tag{4.7}$$

All quantities will always be computed for $N_p=1000$ and the ensemble average $\langle\ \rangle$ will be taken over a Gaussian distribution of d_ℓ and N.

We must carefully avoid the case of resonance reflection by the latticelike Dirac-comb structure, which may lead to a total reflection probability of order unity. This occurs when $k\langle d\rangle$ (k being the neutron wave number and $\langle d\rangle$ the average spacing between scatterers) is close to an integer multiple of π. Such a situation has been observed and investigated during the initial part of our numerical situation and is uninteresting from the point of view of ''decoherence'' effects. In our simulation, in order to reduce unwanted spurious effects, we shall always set

$$k\langle d\rangle=4.5\pi. \tag{4.8}$$

We shall focus our attention on thermal neutrons interacting with atoms, so that

$$\lambda=\frac{2\pi}{k}=2\ \text{Å},\quad \langle d\rangle=4.5\ \text{Å}. \tag{4.9}$$

The statistical distributions of the spacings and of the number of δ potentials will be taken to be Gaussian, with averages $\langle d\rangle$ and $\langle N\rangle$, respectively, and standard deviations

$$Y \equiv \frac{\delta d}{\langle d \rangle}, \quad \delta d \equiv \sqrt{\langle (d - \langle d \rangle)^2 \rangle}, \tag{4.10}$$

$$\Delta \equiv \frac{\delta N}{\sqrt{\langle N \rangle}}, \quad \delta N \equiv \sqrt{\langle (N - \langle N \rangle)^2 \rangle}. \tag{4.11}$$

In our simulation, the value of these two parameters will be varied between

$$0 \leqslant Y \leqslant .5, \quad 0 \leqslant \Delta \leqslant 1. \tag{4.12}$$

The maximum value of Y does not seem appropriate to describe a rigid lattice. Indeed, as we shall see, there are situations in which coherence is lost for $Y > 10^{-1}$. In such a case, the apparatus cannot be viewed as a solid: A rigid lattice cannot work properly as a ''detector,'' in the sense that it does not provoke decoherence on the incoming particle. This point will be considered and discussed in Sec. IV F, in particular in connection with neutron interferometry. Notice also that many detectors make use of thermodynamically unstable states, such as gases or liquids in critical conditions, whose physical states are characterized by very large statistical fluctuations.

It is not very easy to understand what are the physically most interesting values of the parameter $\Omega = \Lambda/\hbar v$ appearing in Eq. (4.3). Such values must be determined according to the physical problem investigated. One can guess that there are situations in which there is a profound link between Ω and $\langle N \rangle$, which may lead, under some conditions, to a precise relation between Ω and $\langle N \rangle$: This is closely related to van Hove's ''$\lambda^2 T$'' limit and to the occurrence of a dissipativelike behavior in quantum mechanics, and has been

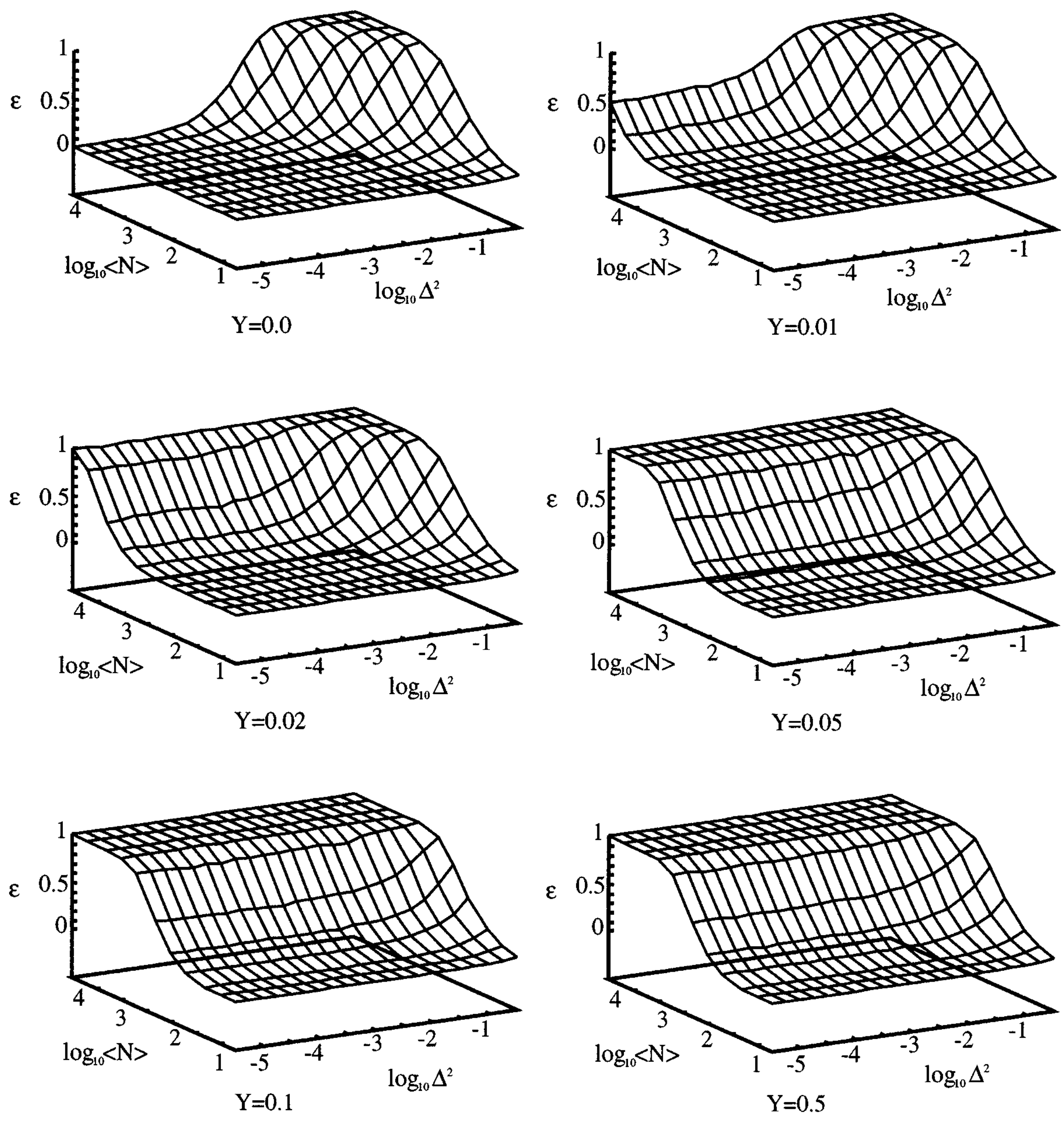

FIG. 1. Dependence of ϵ on $\langle N \rangle$ and Δ, for some values of Y. Observe the sharp transition region from $\epsilon = 0$ (coherence) to $\epsilon = 1$ (decoherence) when $Y = 0$. See text.

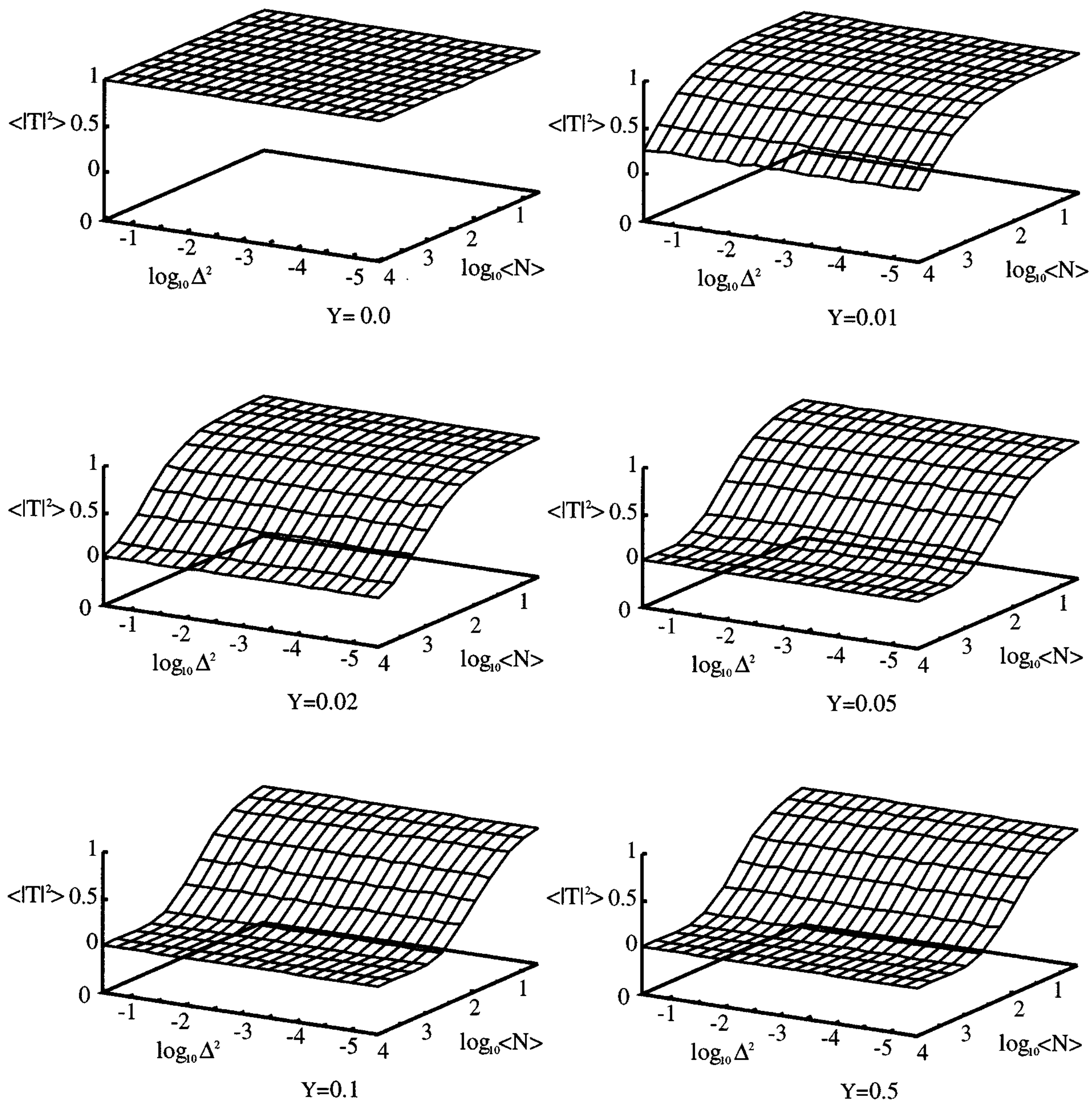

FIG. 2. Transmission probability $\langle |T|^2\rangle$ vs $\langle N\rangle$ and Δ.

discussed elsewhere [17]. We shall briefly reconsider this point in Sec. V.

The quantity Ω plays the role of a coupling constant, which can be neither too small nor too big in order to ensure the occurrence of some dephasing. In our simulation $\langle N\rangle$ will reach the value 10^4, and it has been found that if $\Omega \ll 10^{-1}$ no decoherence takes place. We therefore set in the following numerical study

$$\Omega = 10^{-1}. \tag{4.13}$$

The values of the transmission probability and decoherence parameter (or, equivalently, the transmission coefficient) depend on all the physical quantities we have considered so far. Since the numerical values of $\langle d\rangle$ and Ω are *fixed* according to Eqs. (4.9) and (4.13), respectively, we can state that, in general, only Δ, Y, and $\langle N\rangle$ determine $\langle |T|^2\rangle$ and ϵ, i.e.,

$$\langle |T|^2\rangle = \langle |T|^2\rangle(\Delta, Y, \langle N\rangle), \tag{4.14}$$

$$\epsilon = \epsilon(\Delta, Y, \langle N\rangle). \tag{4.15}$$

Our task is to clarify how the wave-function collapse takes place ($\epsilon \to 1$) when the numerical values of Δ, Y, and $\langle N\rangle$ change. For the sake of clarity, we shall perform different simulations: In the first one, we shall let $\langle N\rangle$ and Δ vary, while keeping Y fixed to a few different values. In the second one, $\langle N\rangle$ and Y will be varied while keeping Δ fixed to a few different values. In the third and last one, all parameters shall be varied.

C. Varying *N* and Δ

Figure 1 displays the dependence of ϵ on $\langle N\rangle$ and Δ for several values of Y, Fig. 2 displays the behavior of

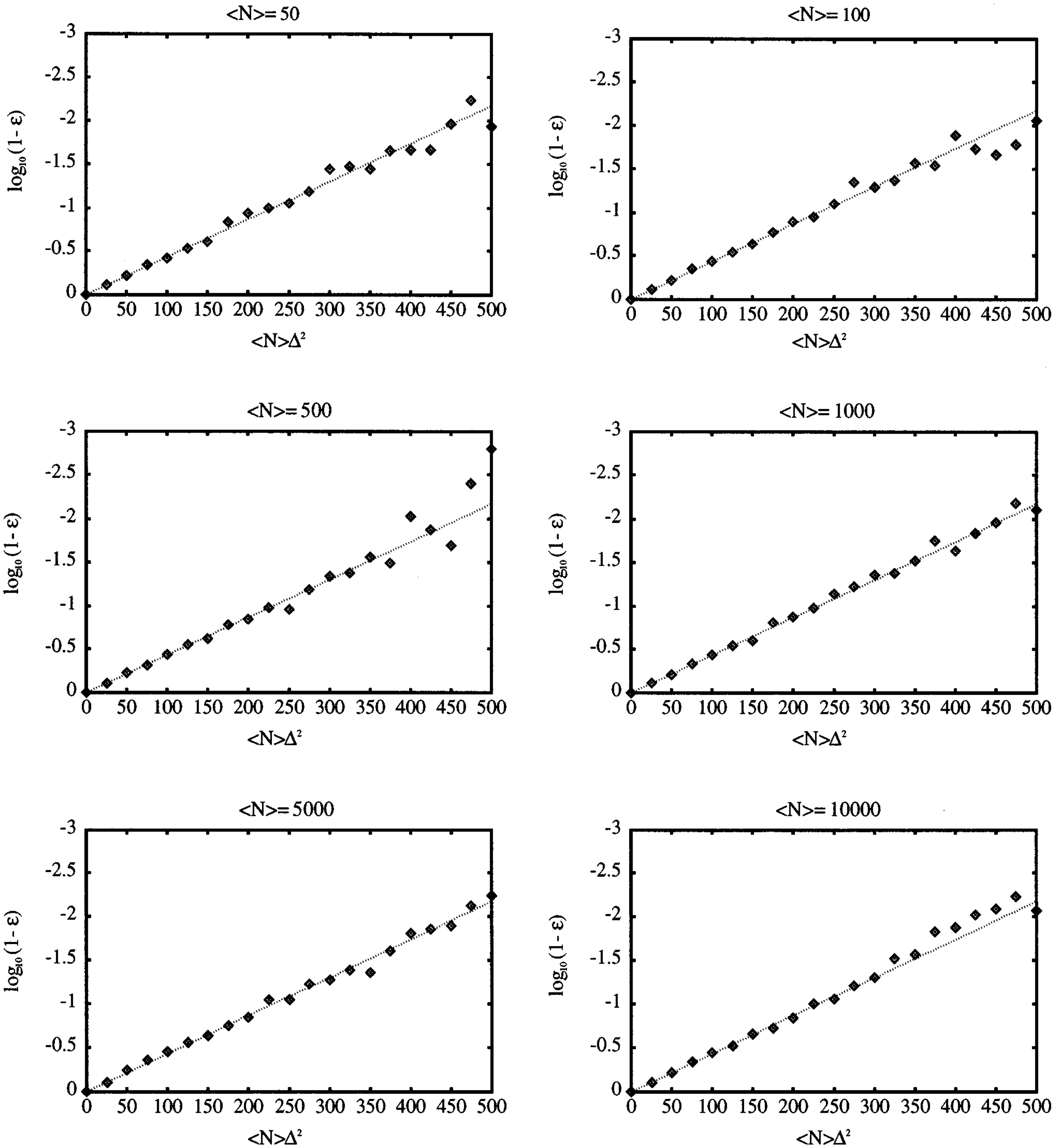

FIG. 3. Dependence of ϵ on $\langle N\rangle\Delta^2$ for several values of $\langle N\rangle$ and for $Y=0$. The dotted line is Eq. (4.19). The diamond points are the numerical results. See text.

$\langle t\rangle=\langle|T|^2\rangle$ as a function of the same parameters, and Fig. 3 the behavior of ϵ versus $\langle N\rangle\Delta^2$ for $Y=0$.

Observe that if $Y=0$ the transition region from coherent behavior ($\epsilon=0$) to totally decoherent behavior ($\epsilon=1$) is very sharp and occurs along the lines $\langle N\rangle\Delta^2=$ const. This behavior can be explained by noting that, for small Ω, the transmission coefficient $\mathcal{T}$ for a single δ [see Eq. (A2)] reads

$$\mathcal{T}=\frac{1}{1+i\Omega}=\frac{1}{\sqrt{1+\Omega^2}}e^{-i\tan^{-1}\Omega}\simeq e^{-(1/2)\Omega^2}e^{-i\Omega}, \tag{4.16}$$

from which we easily derive (see Appendix B)

$$T\simeq e^{-iN\Omega}[1-\tfrac{1}{4}\Omega^2\{1-(-e^{-2i\Omega})^N\}]. \tag{4.17}$$

By taking the average over N with a Gaussian distribution and neglecting small quantities, we obtain

$$\langle|T|^2\rangle\simeq 1, \quad |\langle T\rangle|^2\simeq e^{-\langle N\rangle\Delta^2\Omega^2}, \tag{4.18}$$

so that

$$\epsilon\simeq 1-e^{-\langle N\rangle\Delta^2\Omega^2}. \tag{4.19}$$

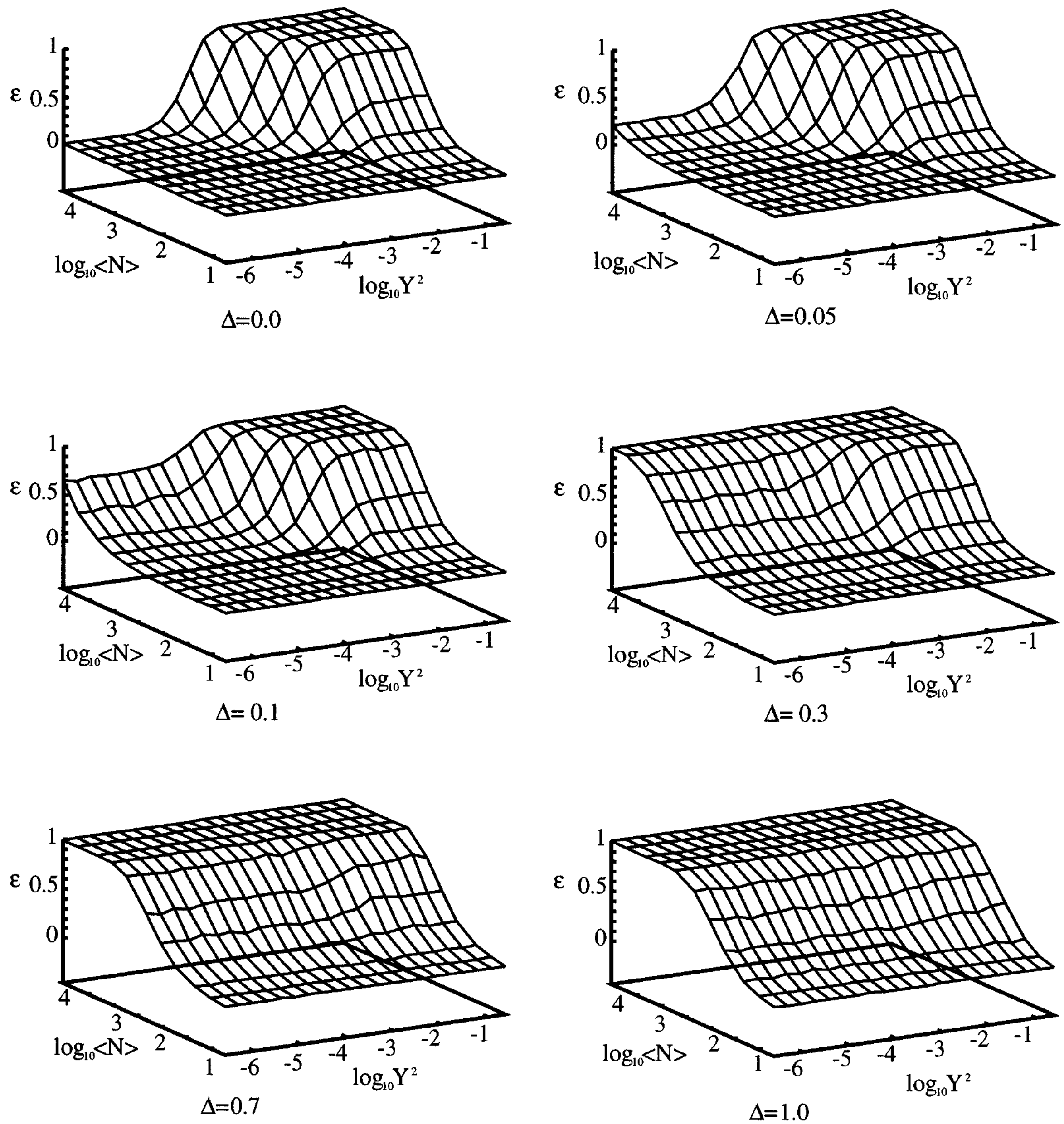

FIG. 4. Dependence of ϵ on $\langle N\rangle$ and Y. Observe the ''saturation'' effect when $Y>10^{-1}$. See text.

In Fig. 3, the ''diamond'' points are the numerical results and the dotted lines are our approximate formula (4.19). The agreement is excellent.

The above reasoning explains the sharp transition along the lines $\langle N\rangle\Delta^2=$ const when $Y=0$. This situation is physically interesting because it corresponds to the case of a neutron beam interacting with a macroscopic solid object, like a crystal: The spacing between different atoms of the macroscopic object is (almost) constant, but different neutrons impinge on different parts of the crystal, at slightly different angles, interacting therefore with a different number of elementary scatterers.

When the condition $Y=0$ (fixed spacings between adjacent potentials) does not hold anymore, additional randomization effects appear and the above approximations break down. This is shown in the other graphs of Figs. 1 and 2.

D. Varying *N* and *Y*

Figure 4 shows the dependence of ϵ on $\langle N\rangle$ and Y for several values of Δ. The behavior of $\langle t\rangle=\langle|T|^2\rangle$ as a function of the same parameters is shown in Fig. 5. For large values of Y, this situation corresponds to a neutron beam interacting with an absorber made up of a liquid or a dilute solution or a gas [18]. (A solid lattice would ''melt'' for such large values of Y.)

It is interesting to notice that there is a ''saturation'' effect in this case: If $Y>0.1$, the values of ϵ and $\langle t\rangle$ do not depend significantly on $\langle N\rangle$. The easiest way to understand this effect is probably to take the ensemble average of both sides of Eq. (4.2): Since the fluctuations of d_ℓ are independent for different ℓ, $\langle T\rangle$ contains factors of the type

$$\langle e^{ikd_\ell}\rangle \simeq e^{ik\langle d\rangle}e^{-k^2Y^2\langle d\rangle^2/2}, \tag{4.20}$$

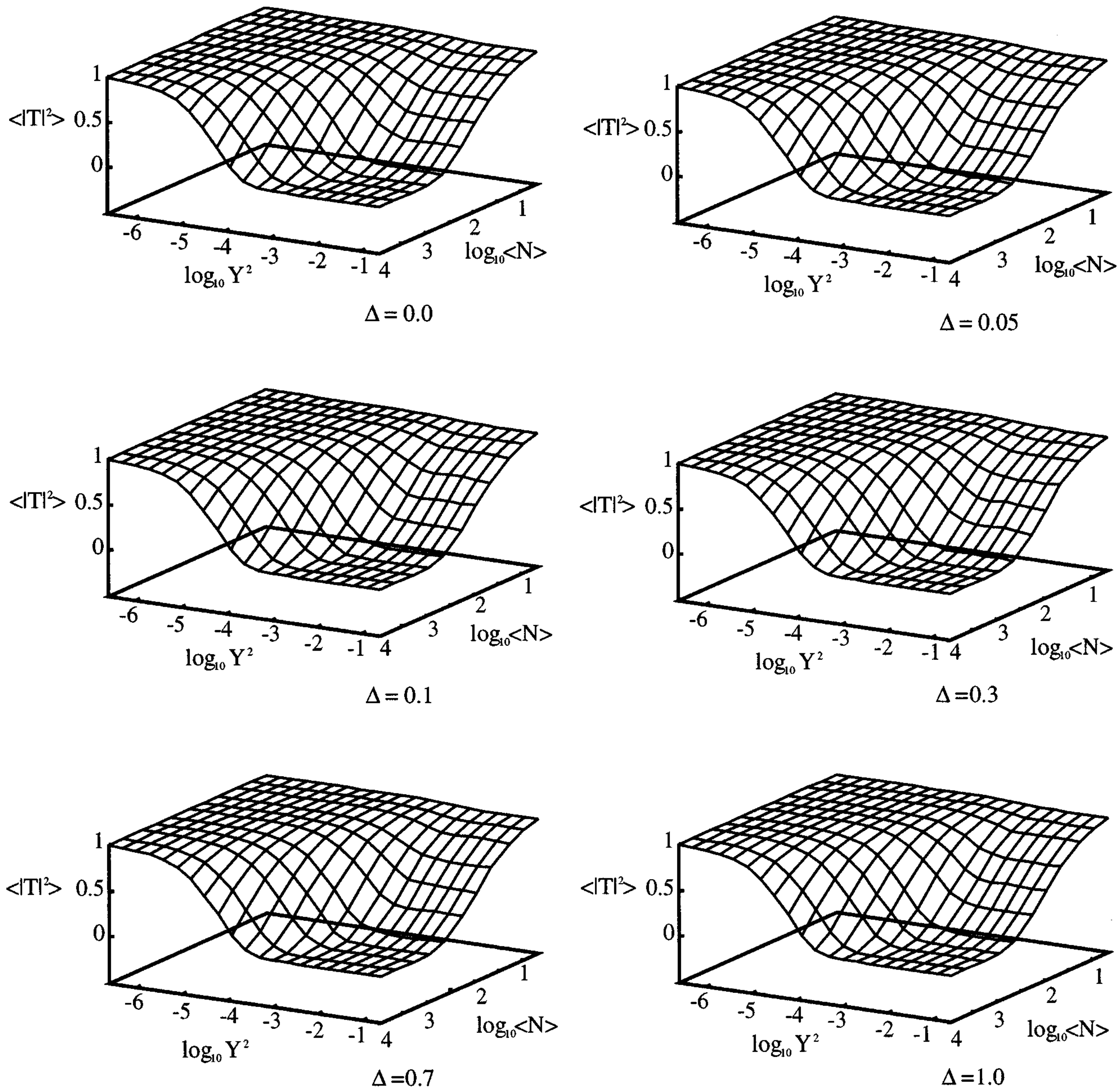

FIG. 5. $\langle|T|^2\rangle$ vs $\langle N\rangle$ and Y.

which vanish (washing away the dependence on Y) when $Y \gg 2/k\langle d\rangle \simeq 10^{-1}$ [see Eq. (4.8)]. A similar situation occurs for $\langle|T|^2\rangle$. Observe that this phenomenon takes place independently of the value of Δ.

A more detailed explanation of this effect is given in Appendix B: One can see that, when $\Delta = 0$, by expanding T in powers of the reflection coefficient $\mathcal{R}$ (see Appendix A) of a single δ potential and writing

$$T = \mathcal{T}^N\{1+\alpha(\mathcal{R})\}, \tag{4.21}$$

one gets at second order in $\mathcal{R}$

$$|\langle T\rangle|^2 \overset{Y\to\text{large}}{\longrightarrow} \exp(-N|\mathcal{R}|^2), \tag{4.22}$$

$$\langle|T|^2\rangle \overset{Y\to\text{large}}{\longrightarrow} \exp(-N|\mathcal{R}|^2)\exp(|\mathcal{R}|^4\mathcal{S}_N^{(4)}), \tag{4.23}$$

$$\epsilon \overset{Y\to\text{large}}{\longrightarrow} 1-\exp(-|\mathcal{R}|^4\mathcal{S}_N^{(4)}). \tag{4.24}$$

where

$$\mathcal{S}_N^{(4)} \overset{Y\to\text{large}}{\longrightarrow} \frac{|\mathcal{T}|^{4N}-N|\mathcal{T}|^4+N-1}{(|\mathcal{T}|^4-1)^2}. \tag{4.25}$$

This result is essentially due to a complete randomization of the phases acquired by multiple reflections, which leads to

$$\langle\alpha\rangle \overset{Y\to\text{large}}{\longrightarrow} 0. \tag{4.26}$$

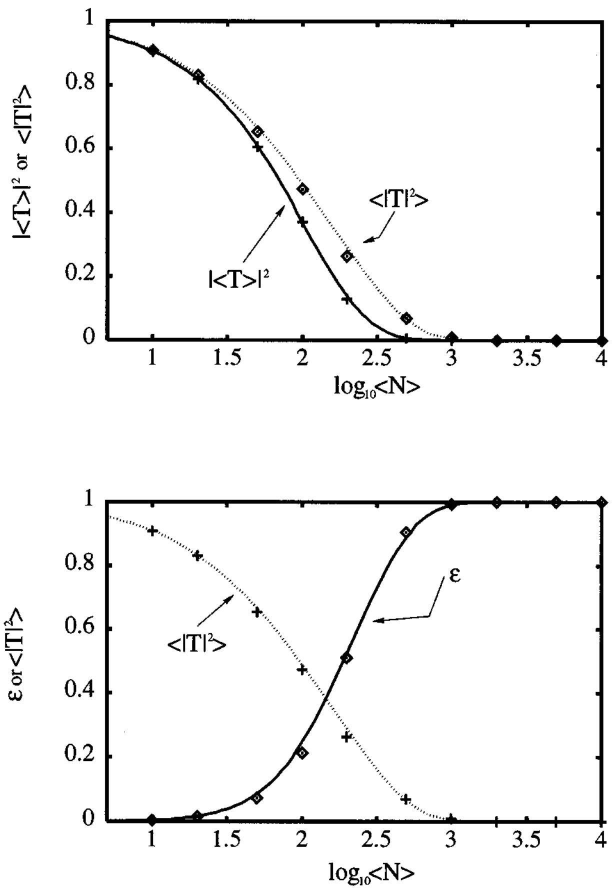

FIG. 6. "Saturation effect": Comparison of the perturbative formulas (4.22)–(4.24) with the numerical results. The dotted and solid curves are given by the perturbative formulas (4.22), (4.23), and (4.24), and the diamond points are the numerical results. $(\Delta=0.)$

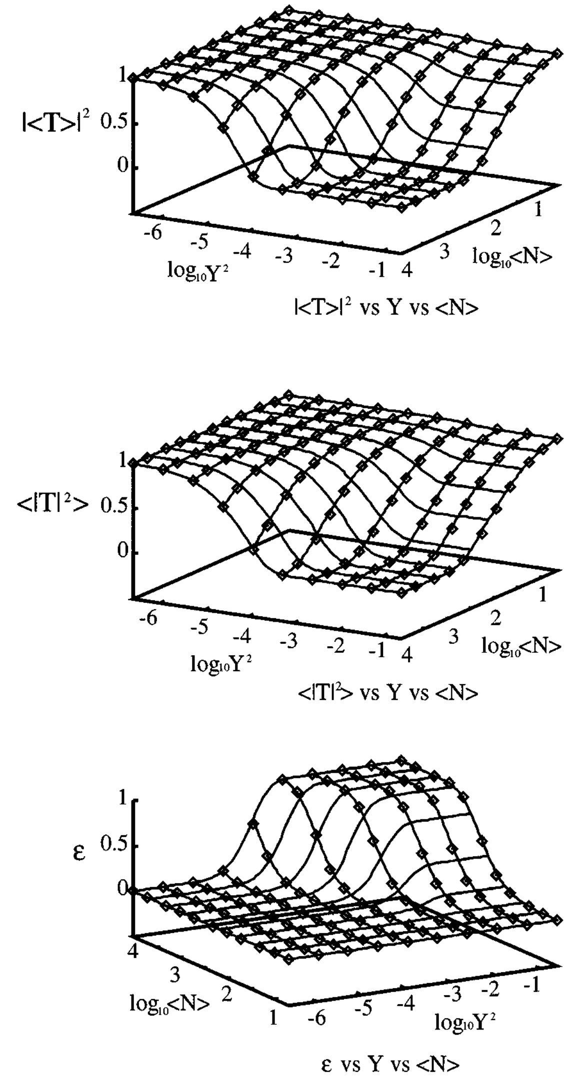

FIG. 7. General case: Comparison of the perturbative formulas (4.28)–(4.30) with the numerical results. The solid curves are given by the perturbative formulas (4.28)–(4.30) and the diamond points are the numerical results. $(\Delta=0.)$

The agreement of the perturbative formulas (4.22)–(4.24) with the numerical results is excellent, as can be seen in Fig. 6.

On the other hand, when there is no "saturation" effect and $\Delta=0$, one finds it convenient to define the number $N_{\rm eff}$ of potentials the Q particle *effectively* interacts with

$$N_{\rm eff}\equiv N+2{\rm Re}\mathcal{S}_N^{(2)}$$

$$\equiv N+2{\rm Re}\left\{\frac{s(s^N-Ns+N-1)}{\mathcal{T}^2(s-1)^2}\right\}, \tag{4.27}$$

where $s\equiv\mathcal{T}^2\langle e^{2ikd}\rangle=-\mathcal{T}^2e^{-2(k\langle d\rangle)^2Y^2}$. By making use of this newly defined parameter, one obtains

$$|\langle T\rangle|^2\simeq\exp(-N_{\rm eff}|\mathcal{R}|^2), \tag{4.28}$$

$$\langle|T|^2\rangle\simeq\exp(-N_{\rm eff}|\mathcal{R}|^2)\exp(|\mathcal{R}|^4\mathcal{S}_{N_{\rm eff}}^{(4)}), \tag{4.29}$$

$$\epsilon\simeq1-\exp(-|\mathcal{R}|^4\mathcal{S}_{N_{\rm eff}}^{(4)}). \tag{4.30}$$

These formulas are identical with those obtained for the *saturated* region [Eqs. (4.22)–(4.24)], when N is replaced by $N_{\rm eff}$. Also in this case, the agreement of the perturbative formulas (4.28)–(4.30) with the numerical results is excellent, as can be seen in Fig. 7.

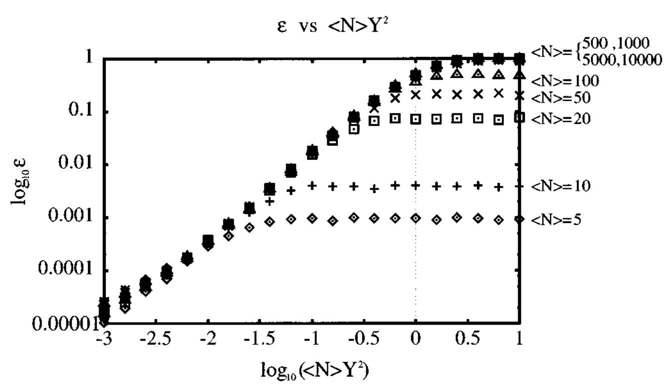

FIG. 8. Dependence of ϵ on $\langle N\rangle Y^2$ for several values of $\langle N\rangle$. $(\Delta=0.)$

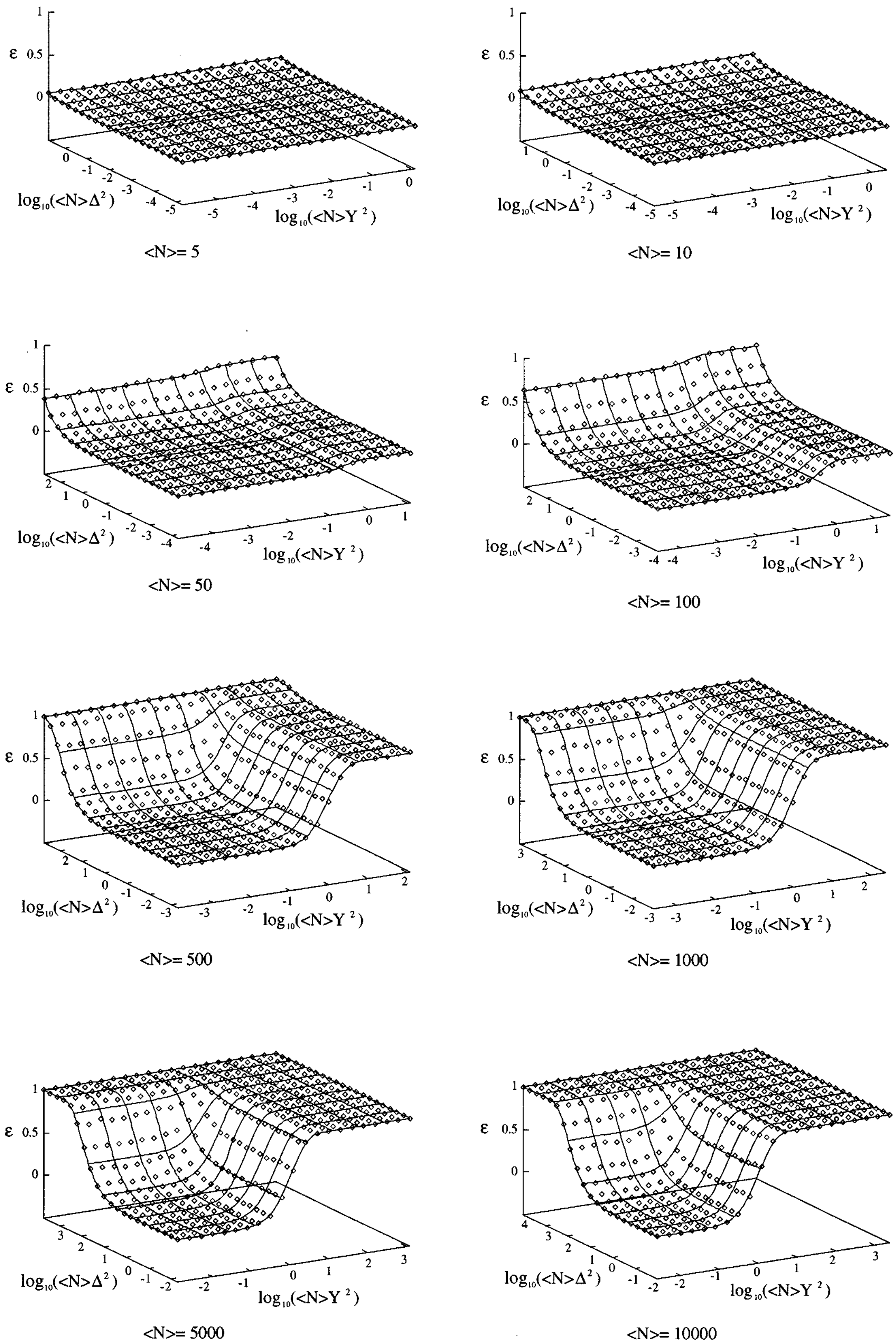

FIG. 9. Decoherence parameter ϵ and transmission probability $\langle |T|^2 \rangle$ vs $\langle N \rangle \Delta^2$ and $\langle N \rangle Y^2$, for several values of $\langle N \rangle$. The solid curves are given by (4.36) and (4.32) and the diamond points are the numerical results.

The decoherence parameter versus NY^2 when $\Delta = 0$ is shown in Fig. 8. It is interesting to notice that, for large N, ϵ becomes a function of the single variable NY^2. As in the case discussed in Sec. IV C, when $\Delta \neq 0$ additional randomization effects appear and the above approximations break down. The resultant effect is shown in Figs. 4 and 5.

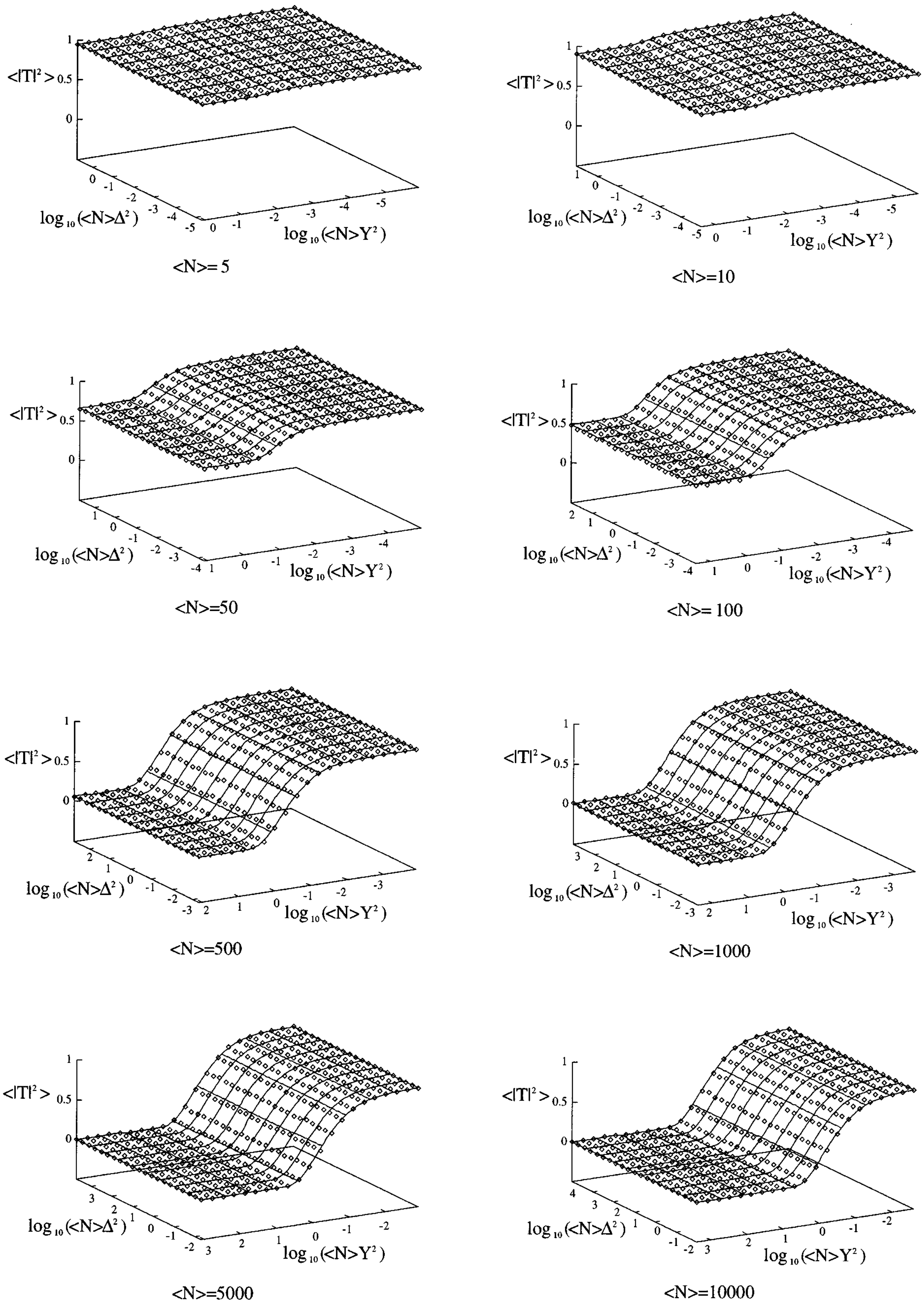

FIG. 9. (*Continued*).

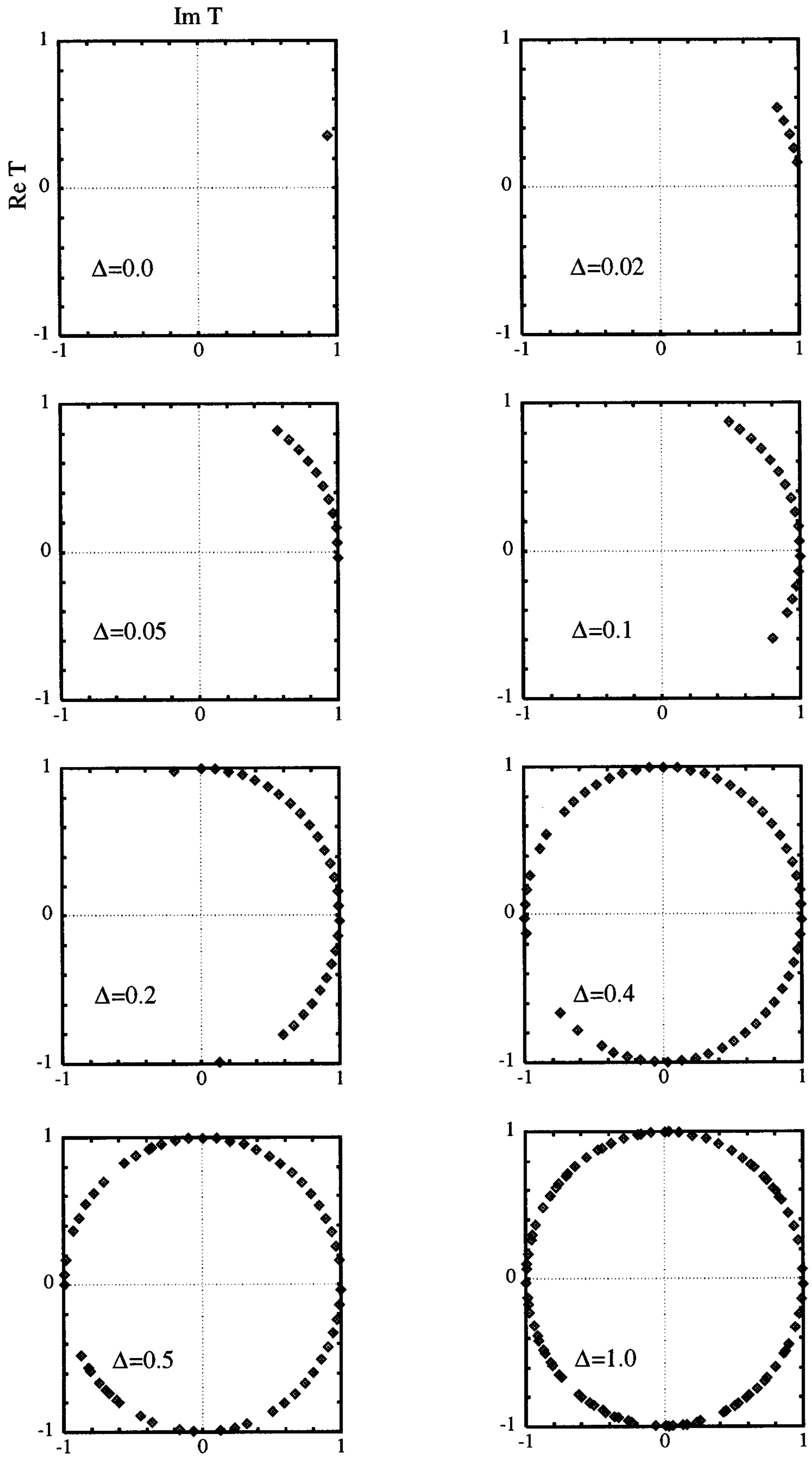

FIG. 10. Phase diagrams for T. The transmission coefficients T_j of $N_p=150$ particles are displayed for $Y=0$ and $\langle N\rangle=2000$. From left to right and top to bottom: $\Delta=$ 0, 0.02, 0.05, 0.1, 0.2, 0.4, 0.5, and 1.0.

E. Varying N, Δ, and Y

We have seen that the decoherence parameter essentially depends on some particular combination of the parameter considered. Namely,

$$\epsilon=\epsilon(\langle N\rangle\Delta^2,\langle N\rangle Y^2). \qquad (4.31)$$

In the present section we shall study the behavior of ϵ as a function of these quantities. This is the most interesting situ-

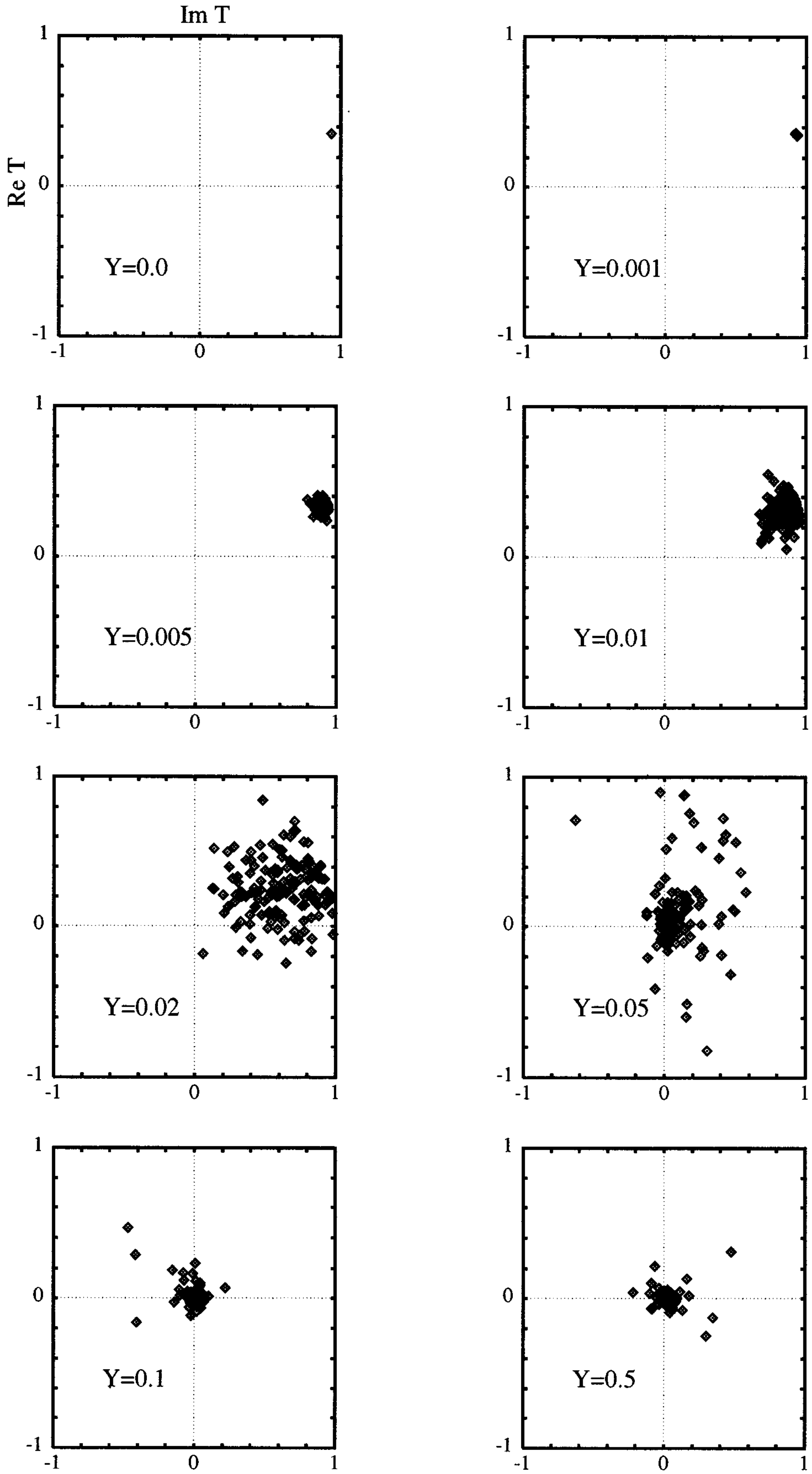

FIG. 11. Phase diagrams for T. The transmission coefficients T_j of $N_p = 150$ particles are displayed for $\Delta = 0$ and $\langle N \rangle = 2000$. From left to right and top to bottom: $Y =$ 0, 0.001, 0.005, 0.01, 0.02, 0.05, 0.1, and 0.5.

ation, from the physical point of view. It corresponds to the case of a neutron interacting with a gas or liquid absorber, in which both the positions and the total number of scatterers change from event to event.

In order to explain these results, it is convenient to observe that, in our numerical simulation, the transmission probability turns out to be largely independent of Δ. We can therefore assume that

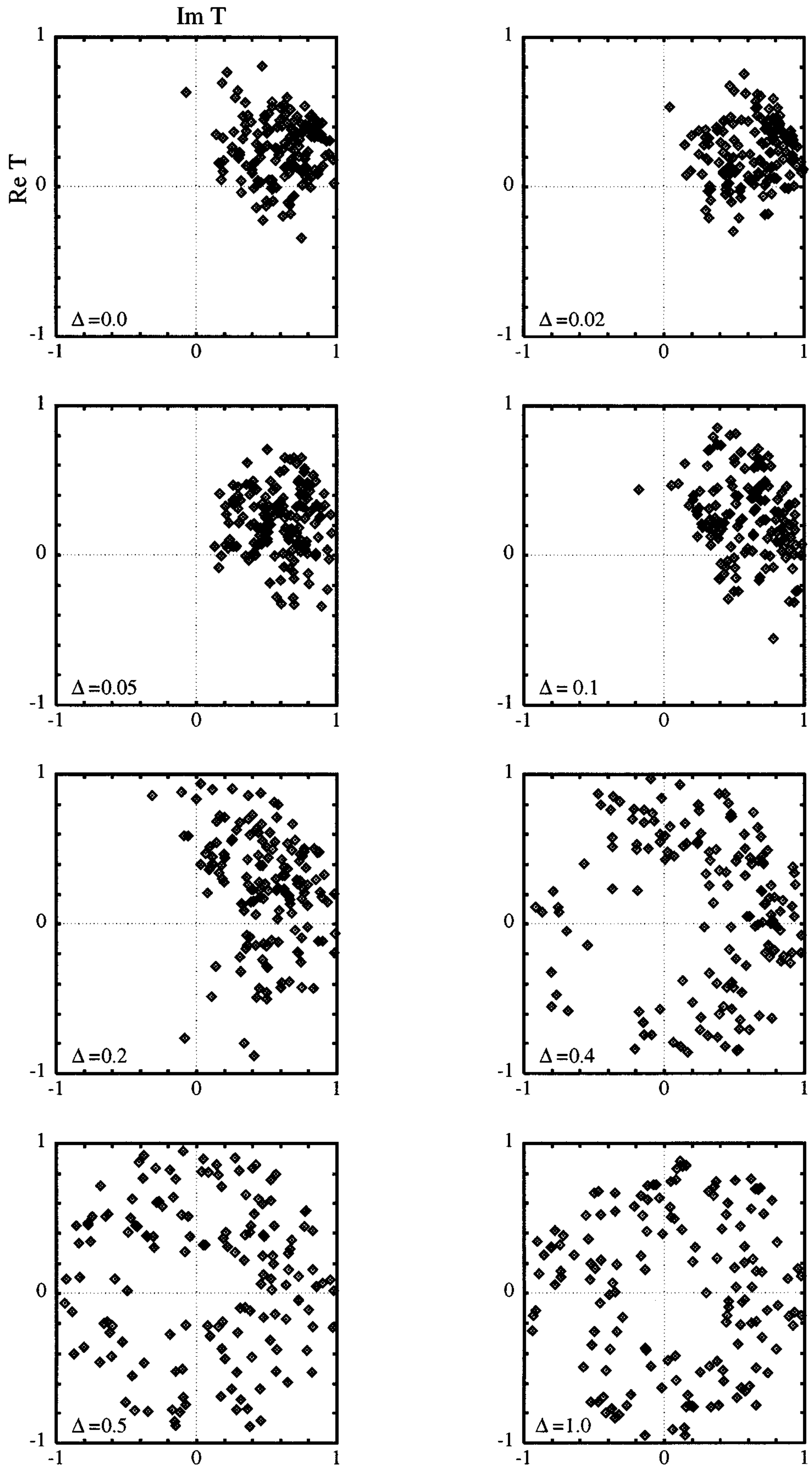

FIG. 12. Phase diagrams for T. As in Fig. 10, but $Y=0.02$.

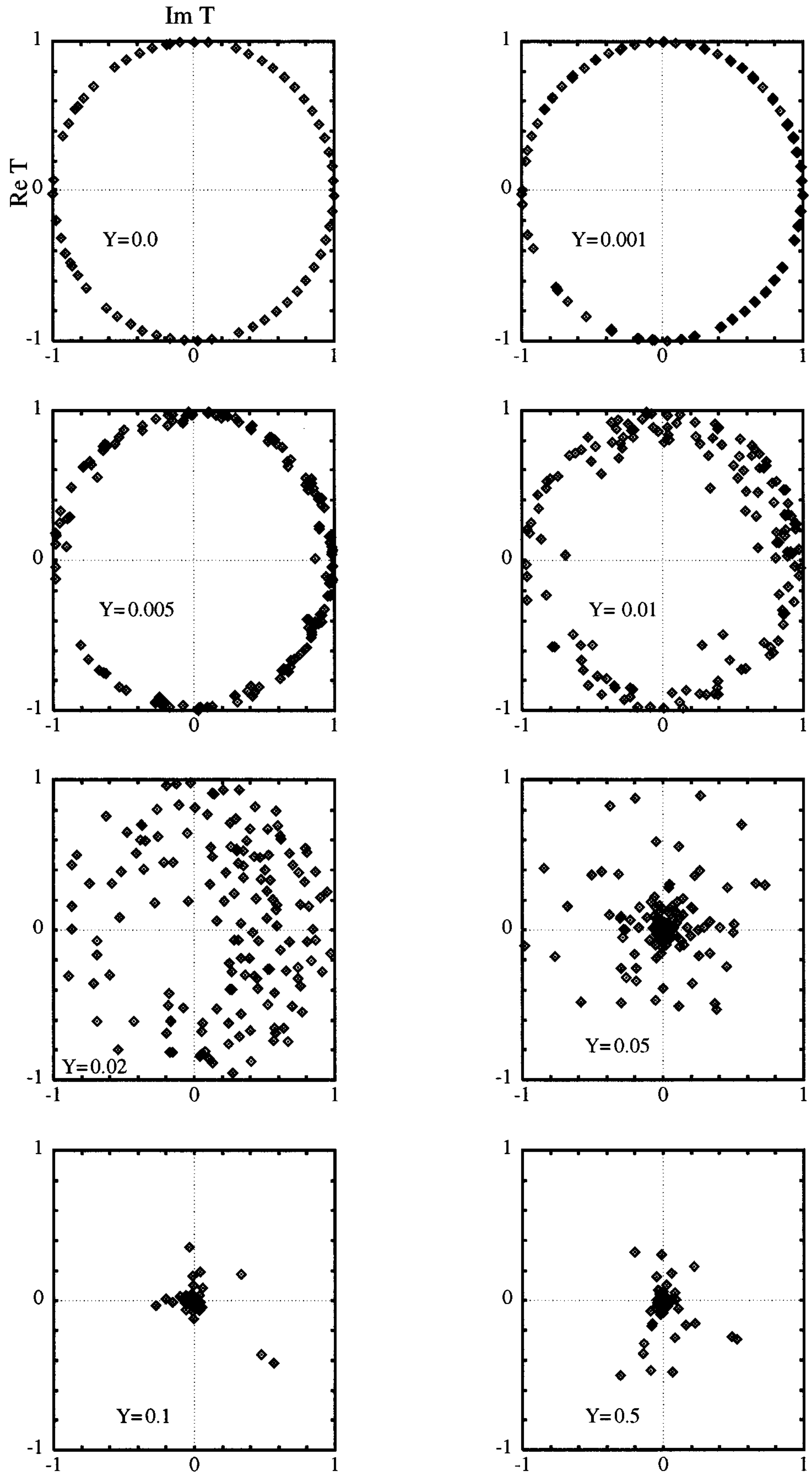

FIG. 13. Phase diagrams for T. As in Fig. 11, but $\Delta=0.5$.

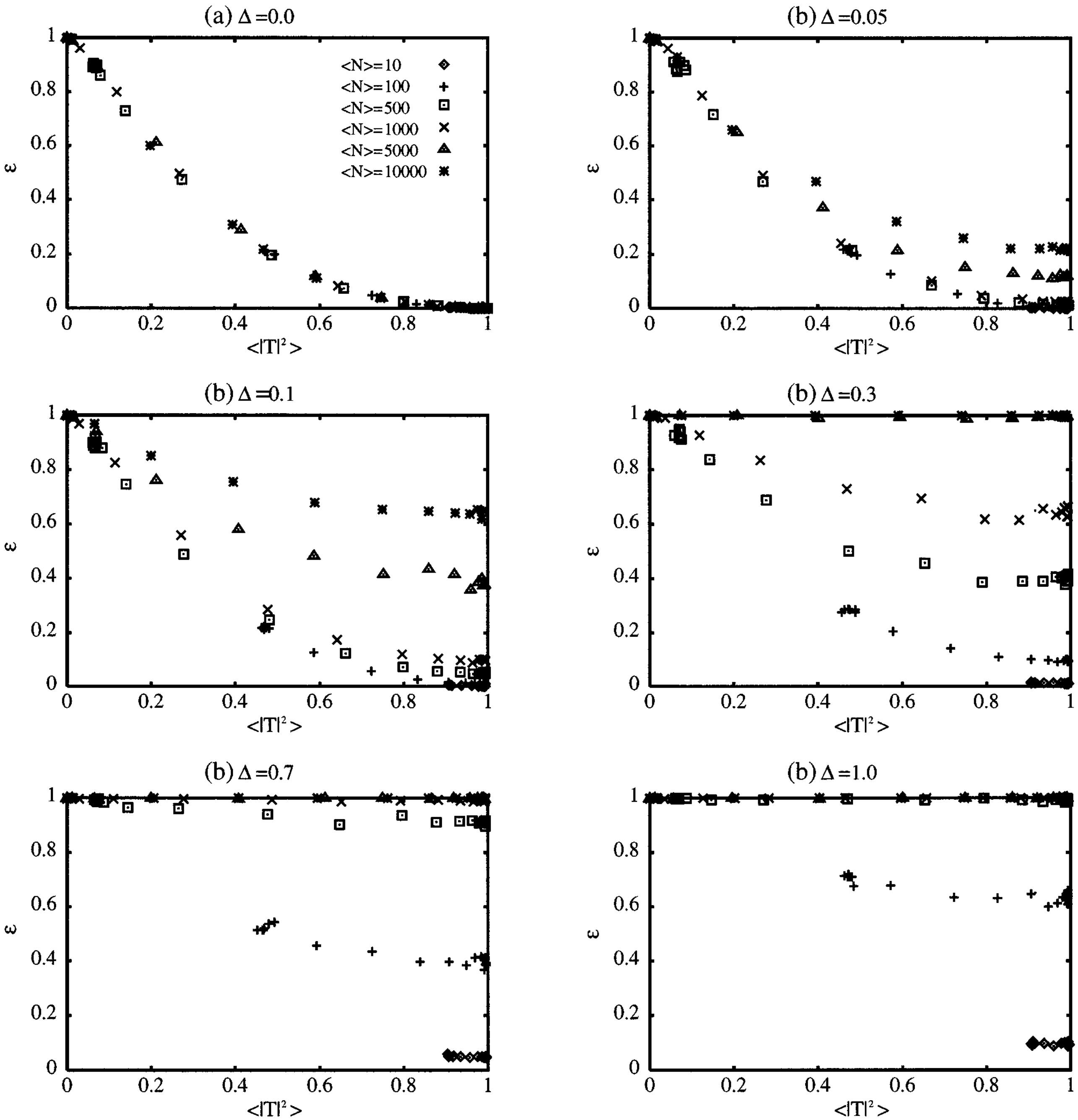

FIG. 14. ''Summary'' of different numerical simulations. The points refer to different values of Δ, Y, and $\langle N\rangle$. In (a), $\Delta=0$, $10\leqslant\langle N\rangle\leqslant 10^4$, and $0\leqslant Y\leqslant 0.5$: All points lie on the same curve. In (b), $0<\Delta\leqslant 1$, $10\leqslant\langle N\rangle\leqslant 10^4$, and $0\leqslant Y\leqslant 0.5$. In all cases, $\epsilon\to 1$ when $\langle|T|^2\rangle\to 0$. *Dephasing and decoherence cannot be avoided at very low transmission probability.*

$$\langle|T|^2\rangle_{N,d}\simeq\langle|T|^2\rangle_d \simeq\exp(-N_{\rm eff}|\mathcal{R}|^2)\exp(|\mathcal{R}|^4\mathcal{S}^{(4)}_{N_{\rm eff}}). \quad (4.32)$$

where we explicitly wrote which variables the averages are taken over. Moreover, the analysis of Appendix B shows that

$$\langle T\rangle_d \simeq \mathcal{T}^N\{1-|\mathcal{R}|^2\mathcal{S}^{(2)}_N\}\simeq e^{-iN\Omega}|\langle T\rangle_d|. \quad (4.33)$$

We therefore assume that

$$\langle T\rangle_{N,d}=\langle e^{-iN\Omega}\rangle_N|\langle T\rangle_d|. \quad (4.34)$$

In this way we obtain

$$|\langle T\rangle_{N,d}|^2\simeq|\langle e^{-iN\Omega}\rangle_N|^2|\langle T\rangle_d|^2 \simeq\exp(-\langle N\rangle\Omega^2\Delta^2)\exp(-N_{\rm eff}\,|\mathcal{R}|^2), \quad (4.35)$$

so that the decoherence parameter turns out to be

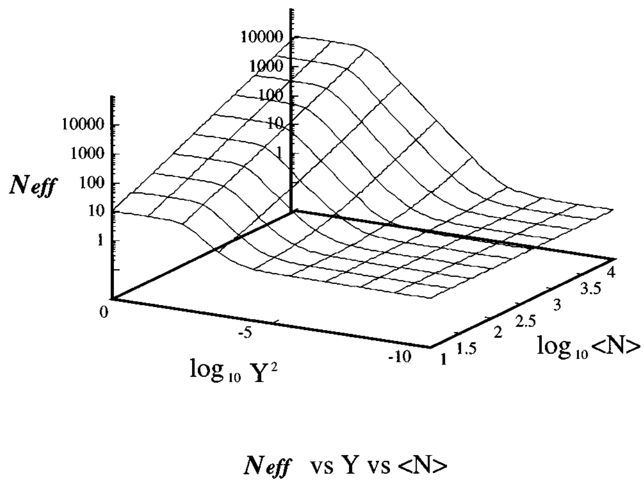

FIG. 15. $N_{\rm eff}$ vs Y and $\langle N\rangle$.

$$\epsilon \simeq 1-\exp(-\langle N\rangle \Omega^2\Delta^2)\exp(-|\mathcal{R}|^4\mathcal{S}^{(4)}_{N_{\rm eff}}). \quad (4.36)$$

In spite of the crudeness of the above approximations, the agreement of these formulas with the numerical results is excellent, as can be seen from Fig. 9. Equations (4.32), (4.35), and (4.36) are our main results and cover all the particular cases hiterto considered.

F. Comments

In order to clarify the mechanism underlying the loss of quantum-mechanical coherence, we have drawn, in Figs. 10–13, the phase diagrams of the transmission probability T. Every figure displays the transmission coefficients T_j of $N_p=150$ particles.

In Fig. 10, $Y=0$, $\langle N\rangle=2000$, and Δ is varied between 0.0 and 1.0. Observe that although the phases spread, the value of the transmission probability is constant and independent of Δ. In the particular case displayed, $\langle |T|^2\rangle$ is close to 1 and the phases are completely randomized for $\Delta=1$ (''collapse'' of the wave function).

In Fig. 11, $\Delta=0$, $\langle N\rangle=2000$, and Y is varied between 0.0 and 0.5. In this case the effect is different: The transmission probability $\langle |T|^2\rangle$ is strongly dependent of Y. The quantum coherence is completely lost for $Y=0.5$ (''collapse'' of the wave function).

In Figs. 12 and 13, both sources of fluctuation are switched on: In the first case, $\langle N\rangle=2000$, $Y=0.02$ (fixed), and Δ is varied as in Fig. 10. The phase coherence is lost already for $\Delta=0.5$ (''collapse'' of the wave function). In the second case, $\langle N\rangle=2000$, $\Delta=0.5$ (fixed), and Y is varied as in Fig. 11. There is complete dephasing already for $Y=0.05$ (''collapse'' of the wave function).

It is interesting to observe that the parameters Δ and Y provoke decoherence in different ways: The value of the transmission probability is practically Δ independent; on the other hand, it is strongly dependent on Y. This is important for two independent reasons. First, this analysis sets the preliminary basis for a ''design theory'' of a quantum-mechanical detector. The combined action of $\langle N\rangle$, Δ, and Y, eventually leading to a complete loss of quantum-mechanical coherence, can be suitably tailored and exploited in order to yield the desired type of dephasing (e.g., by transmitting the Q particle with high or low probability). Second, this analysis can be important for the study of neutron interferometry experiments at very low transmission probability [18,19], because it suggests that different physical devices, such as a silicon crystal, a gaseous or a liquid absorber [18], can modify the coherence properties of an incoming neutron beam in many different ways. For example, one cannot expect a strong dependence on Y (d fluctuations) for a crystal. On the other hand, an interesting dependence of ϵ on Y can certainly be expected when a gaseous (say Ne) neutron absorber or a water solution of some highly absorbing material (say Gd) is inserted in an interferometer [18]. Incidentally, notice that a dependence on Δ is to be expected both for a crystal or a gaseous device, because geometrical factors play here an important role here.

An exhaustive discussion of these effects, beyond the qualitative estimates given here and in Ref. [19], requires a different study and is not easy to perform, because it involves a careful estimate of the most important characteristics of the experimental setup. Such a program is at present under consideration.

However, it is possible to draw some general conclusions even from the results of our simulation: A collection of many different numerical simulations is shown in Fig. 14. In *all the cases investigated*, one observes that when $\langle |T|^2\rangle$ is close to 0, in general $\epsilon\simeq 1$. In other words, it appears that in general, *decoherence effects cannot be avoided at low transmission probability*.

Notice that this phenomenon exists for all values of the parameters Δ, Y, and $\langle N\rangle$. Even though the present numerical simulation neglects many important characteristics of a real experiment, it is difficult to believe that the behavior displayed in Fig. 14 be just a coincidence. It is worth stressing that the statistical fluctuations of a real macroscopic apparatus can *never* be neglected, even in principle, because of finite-temperature effects and of the impossibility of isolating completely the apparatus from its environment.

V. CONCLUDING REMARKS

We have discussed some important characteristics of the quantum measurement process by making use of a schematic representation of a macroscopic device: In our analysis, the complicated dynamical behavior of the elementary constituents of the apparatus (our ''detector'') was represented by means of an array of potentials undergoing (Gaussian) random fluctuations. The ''collapse'' of the wave function is a consequence of the dephasing effects that provoke a loss of quantum-mechanical coherence.

Our main conclusions are Eqs. (4.32), (4.35), and (4.36), which give a realistic estimate of the transmission probability, transmission coefficient, and decoherence parameter, respectively. The values of these parameters alone suffice to outline the essential features of the interaction between the Q particle and the apparatus.

The dephasing effects can always be quantitatively analyzed by means of the decoherence parameter ϵ, which plays the effective role of an ''order parameter'' for the wave-function collapse. This is only an analogy, and must be considered with great care. Indeed, we are not supplying any evidence that the quantum measurement process can be viewed as a phase transition, and are unable, at the present stage, to corroborate this idea with clear-cut arguments. Nevertheless, Figs. 1, 4, and 9 are rather suggestive, in that they lead one to (naively) view the loss of quantum-mechanical coherence as a phase transition of some sort. The idea which the ‘‘collapse’’ of the wave function can be viewed as a sort of phase transition was first proposed, as far as we know, by Ne’eman [20], although in a different context.

The whole measurement process has been analyzed within the quantum-mechanical framework. In this sense, we need neither modify quantum mechanics [21], nor ‘‘complete’’ it by introducing additional parameter (hidden variables) [22]. At the same time, we need not invoke metaphysical concepts [23] in order to explain the measurement process. We believe that quantum mechanics alone suffices to describe *all* the important features of the loss of coherence, which eventually lead to the ‘‘collapse’’ of the wave function. To this end, one needs an effectual operational principle that ‘‘works,’’ yielding the desired decoherence. Such an operational principle is introduced in Eq. (3.7). It enables us both to discuss single events and to give an *operational definition* of quantum coherence, valid both for single events and collections of experimental data. As emphasized in Sec. IIIB, we simply regard (3.7) as a postulate, at the present stage, and hope to substantiate it by more fundamental arguments in the future. It goes without saying that the prescription (3.7) does not imply any fundamental modification of quantum mechanics.

It is worth stressing the analogies and differencies that we think exist between the philosophy underlying the present work and those approaches that make use of the technique of partial tracing. A detector must be an open system [5]: This is the *only way* to circumvent all the well-known no-go theorems [8,24]. The above-mentioned openness of the macroscopic detector makes it similar to a sort of ‘‘environment’’: In this sense, our numerical simulation leads to results that are consistent with the so-called ‘‘environment approach.’’ However, we believe that there is a noteworty difference, because we make no use of projection operators *à la* von Neumann and of partial tracing over the environment states. Decoherence is obtained by applying the ensemble average and the ergodic hypothesis (3.7). Of course, the computation of such an ensemble average (if properly performed over the ‘‘many Hilbert spaces,’’ see [3,5]) leaves us only with the dynamical variables of the Q particle. Needless to say, application of partial tracing has the same effect. Nevertheless, we feel that the underlying philosophy and the physical interpretation are different. Partial tracing is only a convenient ‘‘working rule’’ (see the criticisms put forward against the environment approach in Ref. [25]). By contrast, the ensemble average (3.7) has its own independent logical status. Admittedly, it is a postulate that should be carefully considered and, possibly, justified. Work is now in progress in order to clarify its role in a more delicate context, such as (quantum and classical) chaos. (Notice that the ensemble average endeavors to describe the effects that stem from the macroscopicity of the apparatus.) These issues will be discussed elsewhere.

We would like to conclude this discussion with a perspective on the measurement problem and the general issue of decoherence. We believe that decoherence should be understood as a dissipative phenomenon of some sort [17]. A similar idea was independently put forward by Leggett [26] a few years ago. However, we are not thinking of ‘‘dissipation’’ in the sense of energy loss: Obviously, in the present model (a Dirac comb made up of real potentials) one can expect no energy loss. Rather, we are thinking of an irreversible phenomenon stemming from the complicated and disordered motion of the many elementary constituents of the macroscopic detector, eventually leading to an irretrievable loss of coherence. This problem is at present under investigation.

ACKNOWLEDGMENTS

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APPENDIX A: THE DIRAC COMB

Let the potential $V(x)=\Lambda\,\delta(x)$ and the incident wave amplitude be normalized at 1 for $x=-\infty$. Then

$$\psi=\begin{cases} e^{ikx}+\mathcal{R}e^{-ikx}, & x<0 \\ \mathcal{T}e^{ikx}, & x>0, \end{cases} \tag{A1}$$

where the reflection and transmission coefficients, $\mathcal{R}$ and $\mathcal{T}$ respectively, read

$$\mathcal{R}=-i\Omega(1+i\Omega)^{-1}, \qquad \mathcal{T}=(1+i\Omega)^{-1}, \tag{A2}$$

with $\Omega\equiv\Lambda/\hbar v$ (v is the particle speed). The reflection and transmission probabilities read

$$|\mathcal{R}|^2=\Omega^2(1+\Omega^2)^{-2}, \qquad |\mathcal{T}|^2=(1+\Omega^2)^{-2}. \tag{A3}$$

Analogously, if the wave impinges on the potential from the right and is normalized at 1 for $x=+\infty$, one gets, by explicit calculation or by applying space reflection invariance, $\mathcal{R}=\mathcal{R}'$ and $\mathcal{T}=\mathcal{T}'$.

The total potential barrier is

$$V(x)=\sum_{\ell=1}^{N}\Lambda\,\delta(x-b_\ell), \tag{A4}$$

where N is the total number of δ potentials, Λ their strengths, and b_ℓ their positions. We write the wave function as

$$\psi=\begin{cases} A_1e^{ik(x-b_1)}+B_1e^{-ik(x-b_1)}, & x<b_1 \\ \vdots & \\ A_{\ell+1}e^{ik(x-b_{\ell+1})}+B_{\ell+1}e^{-ik(x-b_{\ell+1})}=C_\ell e^{ik(x-b_\ell)}+D_\ell e^{-ik(x-b_\ell)}, & b_\ell<x<b_{\ell+1} \\ \vdots & \\ C_Ne^{ik(x-b_N)}+D_Ne^{-ik(x-b_N)}, & b_N<x \end{cases} \tag{A5}$$

where we used the coefficients $A_\ell,B_\ell(C_\ell,D_\ell)$ on the very left (right) of the ℓth potential. We get

$$B_\ell=A_\ell\mathcal{R}+D_\ell\mathcal{T}' \qquad C_\ell=A_\ell\mathcal{T}+D_\ell\mathcal{R}'. \tag{A6}$$

By solving for C_ℓ,D_ℓ, we obtain

$$\begin{pmatrix} C_\ell \\ D_\ell \end{pmatrix}=\frac{1}{\mathcal{T}'}\begin{pmatrix} \mathcal{T}\mathcal{T}'-\mathcal{R}\mathcal{R}' & \mathcal{R}' \\ -\mathcal{R} & 1 \end{pmatrix}\begin{pmatrix} A_\ell \\ B_\ell \end{pmatrix}=Z\begin{pmatrix} A_\ell \\ B_\ell \end{pmatrix}. \tag{A7}$$

In our case $\mathcal{R}=\mathcal{R}'$, $\mathcal{T}=\mathcal{T}'$ and

$$Z=\frac{1}{\mathcal{T}}\begin{pmatrix} \mathcal{T}^2-\mathcal{R}^2 & \mathcal{R} \\ -\mathcal{R} & 1 \end{pmatrix}=\begin{pmatrix} 1-i\Omega & -i\Omega \\ i\Omega & 1+i\Omega \end{pmatrix}, \tag{A8}$$

where we made use of Eq. (A2). Finally, we define the *whole* barrier's transmission (T) and reflection (R) coefficients, by setting

$$A_1=1, \quad B_1=R, \quad D_N=0, \quad C_N=Te^{ik(b_N-b_1)} \tag{A9}$$

In Eq. (A9) we eliminated an incorrect kinematical factor appearing in the definition of T, as given in Ref. [4]. Equation (4.2) is easily obtained from Eqs. (A7)–(A9).

APPENDIX B: PERTURBATIVE EXPANSION FOR T

In this appendix we shall concentrate on the d fluctuations of our one-dimensional Dirac comb. The symbol $\langle\ \rangle$ must therefore be understood as an average over the d variables *only*. The reader should also keep in mind that the barrier's transmission coefficient T depends, among others, on N. It is worth stressing that all the theoretical formulas to be derived in this Appendix hold even in the case of a linear array of arbitrary shaped potentials, provided they are separated by force-free regions.

By Eq. (4.2), the exact solution for T reads

$$\begin{pmatrix} T \\ 0 \end{pmatrix}=e^{-ik(d_1+\cdots+d_{N-1})}Z\prod_{\ell=1}^{N-1}e^{ikd_\ell\tau_3}Z\begin{pmatrix} 1 \\ R \end{pmatrix}, \tag{B1}$$

where d_ℓ is the distance between adjacent potentials and we set the first potential at $x=0$.

If $\mathcal{R}$ is small, the transmission coefficient is expanded as

$$\begin{aligned} T&=\mathcal{T}^N\left\{1+\mathcal{R}^2\sum_j \mathcal{T}\cdots\mathcal{T}e^{iD_j}+\mathcal{R}^4\sum_j \mathcal{T}\cdots\mathcal{T}e^{iD_j'}+O(\mathcal{R}^6)\right\} \\ &\equiv\mathcal{T}^N(1+\alpha_2+\alpha_4+\cdots) \\ &\equiv\mathcal{T}^N(1+\alpha), \end{aligned} \tag{B2}$$

where each term corresponds to a fixed number of reflections inside the Dirac comb and the number of $\mathcal{T}$'s and the phase factors $D_j,D_j',\ldots$ in each summation depend on where the reflections take place (see the following formula).

At the second order in $\mathcal{R}$, the solution for T reads

$$\begin{aligned} T=\mathcal{T}^N\{1+\mathcal{R}^2\{&(e^{2ikd_1}+\cdots+e^{2ikd_{N-1}}) \\ &+\mathcal{T}^2(e^{2ik(d_1+d_2)}+\cdots+e^{2ik(d_{N-2}+d_{N-1})}) \\ &+\cdots+\mathcal{T}^{2(N-3)}(e^{2ik(d_1+\cdots+d_{N-2})}+e^{2ik(d_2+\cdots+d_{N-1})}) \\ &+\mathcal{T}^{2(N-2)}e^{2ik(d_1+\cdots+d_{N-1})}\}\}+O(\mathcal{R}^4), \end{aligned} \tag{B3}$$

which yields, since $\mathcal{R}^2\simeq-|\mathcal{R}|^2$,

$$\begin{aligned} \alpha_2=-|\mathcal{R}|^2\{&(e^{2ikd_1}+\cdots+e^{2ikd_{N-1}}) \\ &+\mathcal{T}^2(e^{2ik(d_1+d_2)}+\cdots+e^{2ik(d_{N-2}+d_{N-1})}) \\ &+\cdots+\mathcal{T}^{2(N-3)}(e^{2ik(d_1+\cdots+d_{N-2})}+e^{2ik(d_2+\cdots+d_{N-1})}) \\ &+\mathcal{T}^{2(N-2)}e^{2ik(d_1+\cdots+d_{N-1})}\}. \end{aligned} \tag{B4}$$

The average transmission coefficient and probability, as given by Eqs. (4.4) and (4.6), read

$$\langle T\rangle=\mathcal{T}^N(1+\langle\alpha\rangle)=\mathcal{T}^N(1+\langle\alpha_2\rangle+\cdots), \tag{B5}$$

$$\begin{aligned} \langle t\rangle=\langle|T|^2\rangle&=|\mathcal{T}|^{2N}(1+2\mathrm{Re}\langle\alpha\rangle+\langle|\alpha|^2\rangle) \\ &=|\mathcal{T}|^{2N}(1+2\mathrm{Re}\langle\alpha_2\rangle+\langle|\alpha_2|^2\rangle+\cdots). \end{aligned} \tag{B6}$$

Notice that ϵ is proportional to the square of the standard deviation $(\delta\alpha)^2$:

$$\begin{aligned} \epsilon&=1-\frac{|\langle T\rangle|^2}{\langle|T|^2\rangle}=\frac{(\delta\alpha)^2}{\langle|1+\alpha|^2\rangle}, \\ (\delta\alpha)^2&=\langle|\alpha|^2\rangle-|\langle\alpha\rangle|^2=\langle|\alpha-\langle\alpha\rangle|^2\rangle. \end{aligned} \tag{B7}$$

1. Calculation of $|\langle T\rangle|^2$: Definition of $N_{\rm eff}$

We explicitly calculate the average value of the transmission coefficient when the spacings d_{ℓ} undergo random fluctuations. We write

$$\langle T\rangle \simeq \mathcal{T}^N\{1+\langle\alpha_2\rangle\} \equiv \mathcal{T}^N\{1-|\mathcal{R}|^2\mathcal{S}_N^{(2)}\}. \tag{B8}$$

where

$$\begin{aligned}\mathcal{S}_N^{(2)} &\equiv -\langle\alpha_2\rangle|\mathcal{R}|^{-2} = \langle(e^{2ikd_1}+\cdots+e^{2ikd_{N-1}})+\mathcal{T}^2(e^{2ik(d_1+d_2)}+\cdots+e^{2ik(d_{N-2}+d_{N-1})})+\cdots+\mathcal{T}^{2(N-3)}\\ &\quad\times(e^{2ik(d_1+\cdots+d_{N-2})}+e^{2ik(d_2+\cdots+d_{N-1})})+\mathcal{T}^{2(N-2)}e^{2ik(d_1+\cdots+d_{N-1})}\rangle\\ &=(N-1)\langle e^{2ikd}\rangle+(N-2)\mathcal{T}^2\langle e^{2ikd}\rangle^2+\cdots\\ &\quad+2\mathcal{T}^{2(N-3)}\langle e^{2ikd}\rangle^{N-2}+\mathcal{T}^{2(N-2)}\langle e^{2ikd}\rangle^{N-1}\\ &=(N-1)s+(N-2)s^2+\cdots+2s^{N-2}+1s^{N-1}\\ &=\frac{s(s^N-Ns+N-1)}{\mathcal{T}^2(s-1)^2},\end{aligned} \tag{B9}$$

where all fluctuations are assumed independent, all spacings identically distributed, and $s\equiv\mathcal{T}^2\langle e^{2ikd}\rangle$.

Moreover, since $|\mathcal{T}|^2=1-|\mathcal{R}|^2$, we obtain

$$\begin{aligned}|\langle T\rangle|^2 &\simeq |\mathcal{T}|^{2N}\{1+2\mathrm{Re}\langle\alpha_2\rangle\}\\ &=\{1-N|\mathcal{R}|^2+O(\mathcal{R}^4)\}\{1-2|\mathcal{R}|^2\mathrm{Re}\mathcal{S}_N^{(2)}+O(\mathcal{R}^4)\}\\ &=1-N|\mathcal{R}|^2-2|\mathcal{R}|^2\mathrm{Re}\mathcal{S}_N^{(2)}+O(\mathcal{R}^4)\\ &=1-|\mathcal{R}|^2\{N+2\mathrm{Re}\mathcal{S}_N^{(2)}\}+O(\mathcal{R}^4)\\ &\equiv 1-|\mathcal{R}|^2N_{\rm eff}+O(\mathcal{R}^4)\\ &\simeq \exp(-N_{\rm eff}\,|\mathcal{R}|^2),\end{aligned} \tag{B10}$$

where we roughly took into account the effects of higher-order reflections, in the last approximate equality. (Notice that this approximation corresponds to the assumption that $\langle *^4\rangle\simeq\langle *^2\rangle\langle *^2\rangle$. This formula holds as a Gaussian rule when all higher-order moments are discarded.) This is Eq. (4.28).

The parameter

$$N_{\rm eff} \equiv N+2\mathrm{Re}\mathcal{S}_N^{(2)} \tag{B11}$$

plays the role of the number of δ potentials the Q particle *effectively* interacts with. Indeed, suppose that the Dirac comb is made up of $N_{\rm eff}$ potentials acting incoherently: We obtain

$$|\langle T\rangle|^2=|\mathcal{T}|^{2N_{\rm eff}}\simeq\langle|T|^2\rangle . \tag{B12}$$

The last approximate equality is fully justified if $N\lesssim 100$, but, surprisingly, yields very accurate results even if $N\sim 10\,000$.

A more rigorous definition of $N_{\rm eff}$ can be given by starting from the perturbative expansion of the square of the average transmission coefficient

$$|\langle T\rangle|^2=1+c_2|\mathcal{R}|^2+c_4|\mathcal{R}|^4+\mathcal{O}(|\mathcal{R}|^6)+\cdots. \tag{B13}$$

In this expansion,

$$N_{\rm eff} \equiv -c_2 . \tag{B14}$$

As a first application of the above formulas, let $Y=0$. Then, by Eq. (4.8), $s=-\mathcal{T}^2$ and

$$\mathcal{S}_N^{(2)}\simeq\frac{1}{4}\{-(-e^{-2i\Omega})^N-2N+1\} \tag{B15}$$

so that

$$\begin{aligned}T &\simeq \mathcal{T}^N\{1-|\mathcal{R}|^2\mathcal{S}_N^{(2)}\}\\ &=(e^{-(1/2)\Omega^2}e^{-i\Omega})^N(1-\tfrac14|\mathcal{R}|^2\{-(-e^{-2i\Omega})^N-2N+1\})\\ &=e^{-iN\Omega}[1-\tfrac14\Omega^2\{1-(-e^{-2i\Omega})^N\}],\end{aligned} \tag{B16}$$

which yields Eq. (4.17).

As a second application, consider the large d-fluctuation case. Notice that, by Eq. (4.8) and the Gaussian property of the fluctuation,

$$s=\mathcal{T}^2\langle e^{2ikd}\rangle=\mathcal{T}^2e^{2ik\langle d\rangle}e^{-2(k\langle d\rangle)^2Y^2}=-\mathcal{T}^2e^{-2(k\langle d\rangle)^2Y^2}. \tag{B17}$$

By Eq. (4.8), if $Y>0.1$, we get $|s|^N\sim 0 \quad (N>1)$. From Eqs. (B9) and (B17), we obtain therefore

$$\mathcal{S}_N^{(2)}\ll N \quad (Y\to \text{large}) , \tag{B18}$$

$$|\langle T\rangle|^2 \overset{Y\to\text{large}}{\longrightarrow} \exp(-N|\mathcal{R}|^2), \tag{B19}$$

which yields Eq. (4.22) and explains the ''saturation'' effect mentioned in Sec. IV D.

It is interesting to observe that in this case

$$N=N_{\rm eff}. \tag{B20}$$

This is in agreement with our physical intuition: when Y is large, the phases are completely randomized, the number of ''effective'' interactions is N, and $|\langle T\rangle|^2\simeq|\mathcal{T}|^{2N}\simeq\exp(-N|\mathcal{R}|^2)$.

We give here a short list of properties of $N_{\rm eff}$. The proofs are left to the reader. (i) If $N=1$, then also $N_{\rm eff}=1$; (ii) if $Y=0$ and $k\langle d\rangle=(n+1/2)\pi$, then $N_{\rm eff}=\sin^2(N\Omega)$ for even N and $N_{\rm eff}=\cos^2(N\Omega)$ for odd N; (iii) if $Y=0$ and $k\langle d\rangle=n\pi$, then $N_{\rm eff}=N^2$ (''resonance'' effects); (iv) if $Y>0.1$ then $N_{\rm eff}=N$.

The behavior of $N_{\rm eff}$ as a function of Y and N is shown in Fig. 15.

2. Calculation of $\langle|T|^2\rangle$

Let us compute the average transmission probability. We consider first the ''saturated'' region. In this case, from Eqs. (B6) and (B18), the average transmission probability becomes

$$\langle|T|^2\rangle \overset{Y\to\text{large}}{\to} |\mathcal{T}|^{2N}\{1+\langle|\alpha_2|\rangle^2\}. \tag{B21}$$

We write

$$\langle|\alpha_2|\rangle^2\equiv|\mathcal{R}|^4\mathcal{S}_N^{(4)}, \tag{B22}$$

where the term $\mathcal{S}_N^{(4)}$ stems from multiple reflections and is explicitly given by

$$\begin{aligned}\mathcal{S}_N^{(4)} \overset{Y\to\text{large}}{\to} & \langle\,|(e^{2ikd_1}+\cdots+e^{2ikd_{N\text{-}1}})\\ &+\mathcal{T}^2(e^{2ik(d_1+d_2)}+\cdots+e^{2ik(d_{N-2}+d_{N-1})})+\cdots\\ &+\mathcal{T}^{2(N-3)}(e^{2ik(d_1+\cdots+d_{N-2})}\\ &+e^{2ik(d_2+\cdots+d_{N-1})})+\mathcal{T}^{2(N-2)}e^{2ik(d_1+\cdots+d_{N-1})}|^2\rangle\\ \simeq & 1(N-1)+|\mathcal{T}|^4(N-2)+\cdots+|\mathcal{T}|^{4(n-2)}1\\ \simeq & \frac{|\mathcal{T}|^{4N}-N|\mathcal{T}|^4+N-1}{(|\mathcal{T}|^4-1)^2},\end{aligned} \tag{B23}$$

In conclusion, by Eq. (B19)

$$\begin{aligned}\langle|T|^2\rangle \overset{Y\to\text{large}}{\to} & |\mathcal{T}|^{2N}(1+|\mathcal{R}|^4S_N^{(4)})\\ &\simeq|\mathcal{T}|^{2N}\exp(|\mathcal{R}|^4S_N^{(4)})\simeq\exp\\ &\quad(-N|\mathcal{R}|^2)\exp(|\mathcal{R}|^4\mathcal{S}_N^{(4)})\\ &=|\langle T\rangle|^2\exp(|\mathcal{R}|^4\mathcal{S}_N^{(4)}).\end{aligned} \tag{B24}$$

where we partially took into account the effects of higher-order reflections, such as in Eq. (B10). This is Eq. (4.23).

Let us now turn to the general case in which there is no saturation ($Y\lesssim0.1$). Such a situation is difficult to tackle in full generality. We shall therefore follow a heuristic approach, and observe that comparison of Eqs. (B10) and (B19) enables us to evince that one can formally obtain the expression for $|\langle T\rangle|^2$ in the general case by substituting

$$N\to N+2\,\mathrm{Re}\mathcal{S}_N^{(2)}\equiv N_{\rm eff} \tag{B25}$$

into the expression for the ''saturated'' case. Roughly speaking, this assumption can be justified as follows: In our numerical simulation, both the transmission probability and decoherence parameter depend with very good approximation on NY^2. Moreover, in the ''saturated'' region, the average transmission coefficient and its square depend only on N. Therefore a change of the value of N in the nonsaturated case can be thought of as equivalent to a change of the value of Y^2. In this way one obtains, from Eq. (B10),

$$\begin{aligned}\langle|T|^2\rangle&\simeq|\langle T\rangle|^2\exp(|\mathcal{R}|^4\mathcal{S}_{N_{\rm eff}}^{(4)})\\ &=\exp(-|\mathcal{R}|^2N_{\rm eff})\exp(|\mathcal{R}|^4\mathcal{S}_{N_{\rm eff}}^{(4)}),\end{aligned} \tag{B26}$$

which is Eq. (4.29).

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