Ground state of a weakly interacting Bose gas of atoms in a tight trap

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We study properties of the ground state of a bosonic gas of atoms in a tight trap. The atoms interact via model molecular potentials that may have either positive or negative scattering lengths. We discuss the possibility of Bose-Einstein condensation in such a system. We show that when the size of the trap becomes comparable to the correlation length of the condensate, stable Bose-Einstein condensation might occur independent of the sign of the scattering length.

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I. INTRODUCTION

Recently there has been a great deal of interest in experimental realization of the Bose-Einstein condensate (BEC) [1–5] in various systems of trapped and cooled atoms [6]. These studies have already led to the remarkable observation of Bose-Einstein condensation in the rubidium vapor [7]. Evidence of BEC in a lithium gas with attractive interactions has also been reported [8]. Two of the major questions concerning the physics of such systems are (1) what is the character of atom-atom interactions, and (2) what is their role in the formation of the condensate?

The above two questions can be addressed either for homogeneous gases or for spatially confined systems. In the homogeneous case, a lot of systematic studies, especially concerning question (1), have been carried out. The atom-atom interactions were studied with model potentials or more accurately with Born-Oppenheimer molecular potentials. For spin polarized hydrogen molecular potential curves are known accurately enough that the low-energy s-wave scattering lengths a_{sc} can be computed with confidence. They are positive and represent overall repulsive interactions [9]. For alkali metals the problem is much more complex, since the molecular potential curves which can typically support many bound states are not known precisely. Some of the atoms (e.g., cesium) are believed to have a positive a_{sc}, [10] and some others (e.g., lithium) to have a negative a_{sc} [11].

The knowledge of the scattering length is essential for the description of atom-atom interactions at low energies. In such a situation atom-atom interactions can be modeled within a framework of *shape independent approximation* (SIA) [12], which implies that as the relative momentum k of the scattering particles goes to zero, only the s-wave scattering is relevant and the s-wave phase shift δ_0 behaves as [13]

$$\tan\delta_0/k = -a_{sc} - \frac{1}{2} r_{\rm eff} a_{sc}^2 k^2 + \cdots , \tag{1}$$

with $r_{\rm eff}$ denoting the *effective range* of the potential. Since the range of the atom-atom interactions is believed to be short in comparison to the typical length scale of variations of atomic wave functions, one can in fact neglect the second term on the right-hand side of Eq. (1) and set the scattering amplitude to be constant in this limit. Heuristically, the bare atom-atom potential $V(\vec{R}-\vec{R}')$ is then replaced by an *effective zero-range* potential [14]

$$V(\vec{R}-\vec{R}') \rightarrow V_{\rm eff}(\vec{R}-\vec{R}') = \tilde{B}\,\delta(\vec{R}-\vec{R}'). \tag{2}$$

Note that such an effective potential leads automatically to s-wave scattering only. If the bare potential is weak, and can be treated using the Born approximation, one can set

$$\tilde{B} = \int d^3\vec{R}\, V(\vec{R}). \tag{3}$$

In this case $\tilde{B} = 4\pi\hbar^2 a_B/M$ with M the atomic mass and a_B denotes the scattering amplitude calculated under the Born approximation. More generally, one sets

$$\tilde{B} = 4\pi\hbar^2 a_{sc}/M. \tag{4}$$

The simple heuristic arguments presented here can be derived more rigorously within the framework of many-body theory [15]. To this aim one employs a so-called T-matrix approximation, which consists of a resummation of the *ladder* diagrams in the calculation of the single particle Green's functions [16]. From this point of view the above formulated question (1) can be answered provided the scattering length is known.

One should stress, however, that in the original derivation of the effective potential Galitskii [17] and Beliaev [18] assumed a bare hard core potential. The theory can be generalized to the case of a potential which supports bound states provided (a) it has a positive scattering length, (b) its effective range $r_{\rm eff}$ is not too large. The latter requirement comes from the fact that in order to neglect the momentum dependence of the scattering amplitudes for the momenta of the order $\sqrt{n_0 a_{sc}}$ one needs to have $n_0 a_{sc}^2 r_{\rm eff} \ll 1$. More generally, one expects that for scattering at low but nonzero energy the use of SIA would require the substitution of the zero-range effective potential (2) by a finite-range potential. One of the ways to do it is to represent the effective potential as a series containing derivatives of the Dirac δ function of increasing order,

$$V_{\rm eff}(\vec{R}-\vec{R}')=\tilde{B}\,\delta(\vec{R}-\vec{R}')-\tilde{C}\nabla^2\delta(\vec{R}-\vec{R}')+\cdots. \quad (5)$$

Because of centrifugal symmetry of the potential $V_{\rm eff}$ there is no first order term in Eq. (5). The nonlocal terms in the above expansion are due to the fact that (i) at nonzero energies scattering amplitude is energy dependent even for the s waves; (ii) higher partial waves contribute. That means that the coefficient $\tilde{C}$ contains contributions of two kinds: those due to the energy dependence of the s-wave amplitude, $\tilde{C}_s=\pi\hbar^2 a_{\rm sc}^2 r_{\rm eff}/M$, and those due to the scattering of higher partial waves. In this paper we examine the conditions under which the expression (5) has to be used instead of (2). As we shall see, such conditions may occur in tight atomic traps.

The problem of the validity of Eq. (2) is also related to the fact of whether one considers a homogeneous or inhomogeneous system. The Galitskii-Beliaev theory was formulated for the case of homogeneous gas. In such a case, and in the presence of repulsive interactions, (i.e., $\tilde{B}>0$), the expression (2) can be used at zero temperature since the condensate wave function is spatially constant. The contribution of the second derivative term in Eq. (5) can then be neglected. On the other hand, it is not clear whether Eq. (2) can be used in tightly confined atomic traps, especially when the trap size becomes comparable to the correlation length of the condensate $l_c=1/\sqrt{n_0 a_{\rm sc}}$. The trap potential introduces then a necessary spatial dependence of the condensate wave function, such that the contribution of the second derivative term in Eq. (5) might become relevant. One may thus ask if and how the presence of the trap affects the validity of Eq. (2). One can also ask whether Eq. (2) can be used for atom-atom potentials that support many bound states since $a_{\rm sc}$ is, in such a case, a rapidly varying function of the details of the potential [10], and can attain any value between $-\infty$ and $+\infty$. In particular it may happen that $a_{\rm sc}$ is close to zero.

Even if the details of the atom-atom potential were known, there would still remain the question (2) as to whether BEC is possible for potentials with negative $a_{\rm sc}$, or for potentials that support bound states. These questions were recently addressed by Stoof [20] for the case of a homogeneous gas. Stoof used Eq. (2) for both positive and negative $\tilde{B}$, and has shown that for $\tilde{B}<0$, BEC of a weakly interacting gas is impossible. Instead, the system undergoes a first order transition to a high-density phase. For $\tilde{B}>0$, BEC takes place at sufficiently low temperature and leads to a build up of the macroscopically coherent atomic field. In the critical case $\tilde{B}=0$, one deals with the ideal Bose gas, for which BEC is still possible, but does not lead to the build up of coherence (condensation occurs into a Fock state, rather than a coherent state). In view of the recent experimental observation of BEC in lithium gas with a negative scattering length [8], it is clear that Stoof's result cannot have an unlimited range of applicability.

The purpose of this paper is to shed more light on the two questions formulated above for the case of tightly confined gases. More precisely, we address here two related questions: (a) Is the standard SIA, Eq. (2), still valid in tightly confined traps? (b) Is BEC in traps possible for potentials that are overall attractive, i.e., have negative $a_{\rm sc}$? As we will show below, for sufficiently small traps the second term in Eq. (5) becomes dominant, and ensures that effective atom-atom interactions are repulsive, allowing for BEC even if $\tilde{B}<0$ [21]. Although our results are to a great extent universal (i.e., independent of the particular form of the atom-atom potential), in order to illustrate them we present a model of N Bose atoms in a harmonic trap interacting via an effective nonlocal potential that can have an arbitrary scattering length, and which reduces to the form (5) in the limit of low-energy scattering. Since the effective potential does not have a zero range, it accounts in principle for scattering of all partial waves. Interestingly, our model is analytically soluble at $T=0$ in the mean field approximation. We show that as $\tilde{B}$ changes sign the system undergoes a phase transition from the weakly interacting gaseous phase (WIG) to a high-density (liquid or solid) phase (HDP). In the WIG phase the standard SIA becomes *exact* in the large N limit.

The main result of this paper is, however, that for large, but finite, N there exists a finite region of parameters such that $|\tilde{B}|\leqslant\tilde{B}_c$ for which the system is in a new gaseous phase. This regime of parameters corresponds to a situation when the bare size of the trap, i.e., the size of the single atom ground state wave function a, becomes smaller than the condensate correlation length l_c [22]. We call this phase a *super weakly interacting gas* (SWIG), since the net effect of the atomic interactions is repulsive, but particularly weak in this phase. The SWIG can exist for both positive and negative values of $\tilde{B}$. The standard SIA [Eq. (2)] is invalid in the SWIG phase, and one has to use Eq. (5) instead. Interestingly, however, the properties of the SWIG phase in the limit of large N are determined by the values of the coefficients $\tilde{B}$ and $\tilde{C}$, and in this sense do not depend on any other details of the effective potential. We estimate that the critical temperature for BEC in SWIG is higher than in WIG, and that this phase can be realized experimentally in a laser cooled Bose gas in microtraps [23].

The paper is organized as follows. In Sec. II we present our model of a weakly interacting Bose gas in a tight trap and discuss its properties at $T=0$ in one dimension (1D) and 3D. Although exact implementation of the ladder approximation in the trap geometry seems to be hardly possible, we simply define the model by specifying a particular phenomenological form of the effective atom-atom potential that would result from the resummation of the ladder diagrams. We make several plausible assumptions about this potential: (i) we assume that it is real (i.e., neglect the lifetime of the quasiparticles); (ii) we assume that it has a short-range repulsive core, and, finally, (iii) that it has a finite-range attractive part. We use then a self-consistent Bogoliubov-Hartree theory to describe the ground state of the model. Particular choice of the form of the effective potential allows us to construct the ground state wave function analytically. In Sec. III we discuss the properties of the ground state in the limit of a large number of atoms N. We discuss there various phases of the system at $T=0$ as a function of the order parameter which turns out to be related to the value of the scattering length, but depends also explicitly on the parameters of the trap. Finally, in Sec. IV we discuss possibilities of physical realizations of the SWIG phase and present our conclusions.

II. THE MODEL OF A WEAKLY INTERACTING BOSE GAS IN THE TRAP

Let us start our discussion with a 1D model. We consider N atoms of mass M in the ground electronic state located in

a harmonic trap of frequency ω_t. The second quantized Hamiltonian for such a system is

$$\mathscr{H}_{\rm eff}=\int dx\Psi^\dagger(x)\left(-\frac{\hbar^2}{2M}\frac{d^2}{dx^2}+\frac{1}{2}M\omega_t^2x^2\right)\Psi(x) + \frac{1}{2}\int dxdx'\Psi^\dagger(x)\Psi^\dagger(x')V_{\rm eff}(x-x')\Psi(x')\Psi(x), \tag{6}$$

where $\Psi(x)$ and $\Psi^\dagger(x)$ are, respectively, the atomic annihilation and creation operator fields that fulfill canonical bosonic commutation relation. The effective Hamiltonian in Eq. (6) is to be understood as resulting from the resummation of the ladder diagrams and contains thus an effective interatomic potential $V_{\rm eff}(x-x')$. The effective potential reproduces in the Born approximation exact scattering amplitudes of the original bare atom-atom potential. Strictly speaking, the annihilation and creation operators $\Psi(x)$ and $\Psi^\dagger(x)$ therefore describe annihilation and creation of *quasiparticles*, rather than the bare ones [15]. In the mean field approximation, or the Bogoliubov-Hartree approximation [24,25], one first replaces operator fields by c-number fields and at $T=0$ seeks the minimum of the functional (6) under the constraint that the number of atoms is fixed, $\int dx\Psi^\dagger(x)\Psi(x)=N$. Equivalently, one considers the minimization of the free energy functional $\mathscr{F}=\mathscr{H}-E\int dx\Psi^\dagger(x)\Psi(x)$, where E is the ground-state energy (or, alternatively, the many-body chemical potential). This leads to the nonlinear Schrödinger equation (NLSE), which for the case of a nonlocal atom-atom potential $V(x-x')$ has an integro-differential form [15]. In the large N limit the kinetic energy term in the above equation can be neglected provided the potential varies sufficiently slowly [26]. Then we obtain the integral equation

$$\left(\frac{1}{2}M\omega_t^2x^2+N\int dx'V_{\rm eff}(x-x')\rho(x')\right)\Psi(x)=E\Psi(x), \tag{7}$$

with $\rho(x)=|\Psi(x)|^2/N$ the normalized density.

Now we specify the effective model potential to be

$$V_{\rm eff}(x)=B\delta(x)-A\exp(-\Gamma|x|). \tag{8}$$

The potential consists of a zero-range repulsive core, and an attractive part of range $\sim 1/\Gamma$. It is real, i.e., we neglect possible effects of the finite lifetime of the quasiparticles. In the low-energy scattering limit it reduces to the form of Eq. (5) with $\tilde{B}=B-2A/\Gamma$, $\tilde{C}=2A/\Gamma^3$.

The main reason in choosing the above form (8), apart from its phenomenological plausibility, is that the nonlocal part of the potential is a Green function of

$$(d^2/dx^2-\Gamma^2)\exp(-\Gamma|x-x'|)=-2\Gamma\delta(x-x'). \tag{9}$$

One can thus seek the solution of Eq. (7) in the interval $[-x_0,x_0]$, i.e., such that $\Psi(x)=0$ for $|x|\geqslant x_0$. Within the interval $[-x_0,x_0]$, $\rho(x)$ must fulfill

$$\left(\frac{d^2}{dx^2}-\tilde{\Gamma}^2\right)\rho(x)=\left(\frac{d^2}{dx^2}-\Gamma^2\right)\left(\frac{E}{NB}-\frac{M\omega_t^2x^2}{2NB}\right), \tag{10}$$

obtained by double differentiation of Eq. (7), where $\tilde{\Gamma}^2=\Gamma^2-2A\Gamma/B$. Note that $\tilde{B}=B\tilde{\Gamma}^2/\Gamma^2$, so that either $\tilde{\Gamma}^2$ or $\tilde{B}$ can be regarded as control parameters. The density $\rho(x)$, which is expected to be an even function of x, must therefore have a general form

$$\rho(x)=\alpha-\beta x^2+\eta\cosh(\tilde{\Gamma}x), \tag{11}$$

where the parameters fulfill

$$\beta=M\omega_t^2\Gamma^2/(2NB\tilde{\Gamma}^2), \tag{12}$$

$$2\beta+\tilde{\Gamma}^2\alpha=E\Gamma^2/(NB)+M\omega_t^2/(NB). \tag{13}$$

Additional constraints for α, β, and η are obtained from the requirements that (a) $\rho(x)$ fulfills the integral equation (7); (b) $\rho(x_0)=0$; (c) $\int_{-x_0}^{x_0}\rho(x)dx=1$ which, together with Eqs. (12) and (13), determine x_0. For $\tilde{\Gamma}^2\geqslant 0$, x_0 is determined uniquely. For $\tilde{\Gamma}^2<0$ there are in principle many possible solutions, but only the smallest one assures that $\rho(x)\geqslant 0$.

Before we turn to the discussion of these solutions, let us briefly discuss the 3D case. Exploiting the same idea we consider the effective potential

$$V_{\rm eff}(\vec{R})=B\delta(\vec{R})-A\exp(-\Gamma R)/R. \tag{14}$$

The attractive part of the potential has the form of the Yukawa potential, and is a Green function of

$$(\nabla^2-\Gamma^2)\exp(-\Gamma R)/R=-4\pi\delta(\vec{R}). \tag{15}$$

Once again, in the limit of low-energy scattering this potential reduces to the form of Eq. (5) with $\tilde{B}=B-4\pi A/\Gamma^2$, $\tilde{C}=4\pi A/\Gamma^4$. Note again that in general the potential (14) accounts for scattering of all partial waves. As we shall see below, however, in the limit of large N the properties of the system are determined fully by the asymptotic behavior of $V_{\rm eff}$ at $R\simeq 0$, or more specifically by the values of $\tilde{B}$ and $\tilde{C}$.

The density $\rho(R)$ in 3D (which must be rotationally invariant, regular at $R=0$, and vanish for $R\geqslant R_0$) has to take the form

$$\rho(R)=\alpha-\beta R^2+\eta\sinh(\tilde{\Gamma}R)/R, \tag{16}$$

with $\tilde{\Gamma}^2=\Gamma^2-4\pi A/B=\Gamma^2\tilde{B}/B$. The equations relating α, β, η, E, and R_0 can be found using a method analogous to that used in 1D. Note that it is easy to generalize our method to the case when the potential is a combination of several (attractive or repulsive) Yukawa terms with various ranges.

III. PROPERTIES OF THE GROUND STATE FOR LARGE N

For large (but finite) N the solutions of NLSE can be divided into three classes describing three different phases (states) of the system. These solutions are *universal* (they do not depend on the details of the effective potential, but only on the trap potential). They can be classified by the critical value $\tilde{\Gamma}_c$ of the parameter $\tilde{\Gamma}$ (to be defined later).

For primarily repulsive atom-atom potentials with

$\tilde{\Gamma}^2 > \tilde{\Gamma}_c^2$, atoms at $T=0$ condense into the weakly interacting gaseous phase. In the WIG phase the standard SIA is exact. In 1D the solution becomes $\rho(x)=3(x_0^2-x^2)/(4x_0^3)$, with $x_0=(3NB\tilde{\Gamma}^2/2M\omega_t^2\Gamma^2)^{1/3}$, and energy $E=M\omega_t^2 x_0^2/2$. In 3D the solution is $\rho(R)=15(R_0^2-R^2)/(8\pi R_0^5)$, with $R_0=(15NB\tilde{\Gamma}^2/4\pi M\omega_t^2\Gamma^2)^{1/5}$, and $E=M\omega_t^2 R_0^2/2$. In 3D using semiclassical theory [28] we can estimate the critical temperature for the condensation from the condition that the density in the center of the trap should fulfill $N\rho(0)\Lambda_T^3=2.612$, where $\Lambda_T=h/\sqrt{2\pi M k_B T}$ is the thermal de Broglie wavelength. Of course, this estimate makes sense since we expect that the density in the center of the trap at the critical point does not differ very much from the density at $T=0$. In the WIG phase, one can easily include corrections to our solutions coming from the kinetic energy term by solving the NLSE iteratively [26,27].

For $-\tilde{\Gamma}_c^2 \leqslant \tilde{\Gamma}^2 \leqslant \tilde{\Gamma}_c^2$ the gas is in a critical state, which we term a super weakly interacting gas. Standard SIA cannot be used in this case and one has instead to use Eq. (5). In 1D the solution becomes $\rho(x)=15(x_0^2-x^2)^2/(16x_0^5)$, with $x_0=(45NB/2M\omega_t^2\Gamma^2)^{1/5}$ and energy $E=M\omega_t^2 x_0^2/6$. In 3D the solution is

$$\rho(R)=105(R_0^2-R^2)^2/(32\pi R_0^7), \tag{17}$$

with

$$R_0=(525NB/4\pi M\omega_t^2\Gamma^2)^{1/7} \tag{18}$$

and $E=3M\omega_t^2 R_0^2/10$. Note that SWIG is a low-density gaseous phase in which the volume increases very slowly with the number of particles, as if the repulsion of particles were very weak. Paradoxically, the net effect of atom-atom interactions is repulsive even with a negative $\tilde{B}$. That is, of course, the consequence of the fact that in this phase the second (repulsive) term in Eq. (5) dominates over the first one. The density of atoms in the SWIG is larger than in the WIG, so that the semiclassical estimate gives higher critical temperature. This makes SWIG very interesting from the experimental point of view.

Obviously, the fact that the density in the SWIG phase is larger than that in the WIG phase suggests the possibility that neglecting the kinetic energy term in Eq. (7) may be unreasonable in the SWIG phase. The kinetic energy correction comes essentially from the vicinity of the boundary, at $x=x_0$ in 1D, and $R=R_0$ in 3D. Fortunately, the density in the SWIG phase is continuous with the first derivative, whereas the wave function is continuous at those points in the large N limit. This allows us to prove rigorously that the contribution of the kinetic energy in the SWIG phase is asymptotically negligible (see Appendix A).

Finally, for a primarily attractive potential with $\tilde{\Gamma}^2 < -\tilde{\Gamma}_c^2$ the gas condenses into a high-density phase. In this case the use of the Hamiltonian (6) has to be questioned since it contains only binary interactions. We also note that neglecting the kinetic energy term in Eq. (7) becomes inappropriate in this phase. Nevertheless, denoting $\tilde{\gamma}^2=-\tilde{\Gamma}^2$, the solution in 1D is $\rho(x)\propto[1-\cos(\tilde{\gamma}x)/\cos(\tilde{\gamma}x_0)]$, with x_0 being the smallest positive root of $\tan(\tilde{\gamma}x_0)=-\tilde{\gamma}/\Gamma$. For $\tilde{\gamma} \ll \Gamma$, x_0 becomes close to $\pi/\tilde{\gamma}$, and is N independent. The

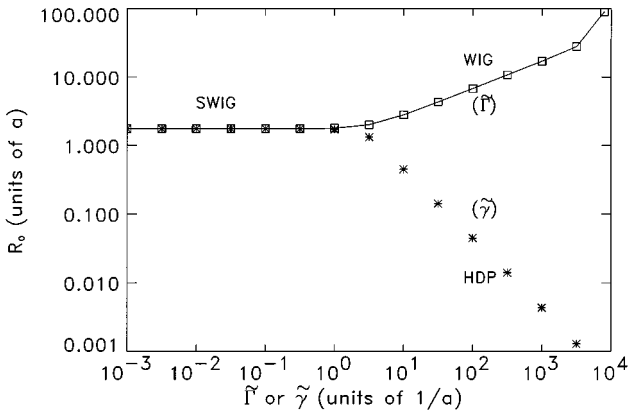

FIG. 1. Dependence of the condensate size R_0 on the control parameter $\tilde{\Gamma}$ for $N=10^4$ with $a=2$ μm. We have used $\Gamma^{-1}=0.25$ nm, $A=16\hbar V_{\rm attr}(R_{\rm attr})/\Gamma$ with $\hbar V_{\rm attr}(R_{\rm attr})=10^{14}$ Hz as discussed in the text. The line connected with open squares is the result for $\tilde{\Gamma}^2>0$, while asterisks represent the points for $\tilde{\Gamma}^2<0$.

energy is $E=-NB\tilde{\gamma}^2/\Gamma^2[2x_0-2\tan(\tilde{\gamma}x_0)/\tilde{\gamma}]$ and is now proportional to N. Similarly, in 3D the density is $\rho(R)\propto[1-R_0\sin(\tilde{\gamma}R)/R\sin(\tilde{\gamma}R_0)]$, with R_0 determined from

$$\tan(\tilde{\gamma}R_0)=-\frac{\tilde{\gamma}R_0\Gamma^2}{(\Gamma^2+\tilde{\gamma}^2)(1+\Gamma R_0)+\Gamma^3 R_0}. \tag{19}$$

The critical value of $\tilde{\Gamma}$ is obtained from the condition $\tilde{\Gamma}_c x_0 \simeq 1$ in 1D and $\tilde{\Gamma}_c R_0 \simeq 1$ in 3D. We obtain (from the expressions for x_0 and R_0 in the WIG)

$$\tilde{\Gamma}_c/\Gamma=(2M\omega_t^2/3NB\Gamma^3)^{1/5}, \tag{20}$$

$$\tilde{\Gamma}_c/\Gamma=(4\pi M\omega_t^2/15NB\Gamma^5)^{1/7}. \tag{21}$$

Physically, the above condition corresponds to the fact that the spatial variations of the condensate density in the trap become so significant that both terms in Eq. (5) are comparable in the center of the trap,

$$|\tilde{B}\rho(0)| \sim |\tilde{C}\rho''(0)|. \tag{22}$$

As we see from Eqs. (20) and (21) the critical region shrinks as N grows, but relatively slowly. The most important feature of the above formulas is that the size of the critical region (i.e., the region of SWIG phase) depends on the trap parameters. Our results are illustrated in Figs. 1 and 2.

IV. PHYSICAL REALIZATION OF THE SWIG PHASE

The main question is whether the SWIG phase can be realized experimentally. One possibility is to control the parameters of the atom-atom potential using external magnetic or electric fields [29]. Such control could, in principle, lead to $a_{\rm sc} \simeq 0$, i.e., $\tilde{B} \simeq 0$. Another possibility is to decrease the size of the atomic trap. The following estimate is for the 3D model, with $B-4\pi A/\Gamma^2=4\pi\hbar^2 a_{\rm sc}/M$, and $a_{\rm sc} \simeq$ 2 nm, M being the mass of a cesium atom. We have defined a characteristic length scale of the attractive part of the potential as $R_{\rm attr}=\int dR R V_{\rm attr}(R)/\int dR V_{\rm attr}(R)=2/\Gamma$, and set it $\simeq 0.5$ nm.

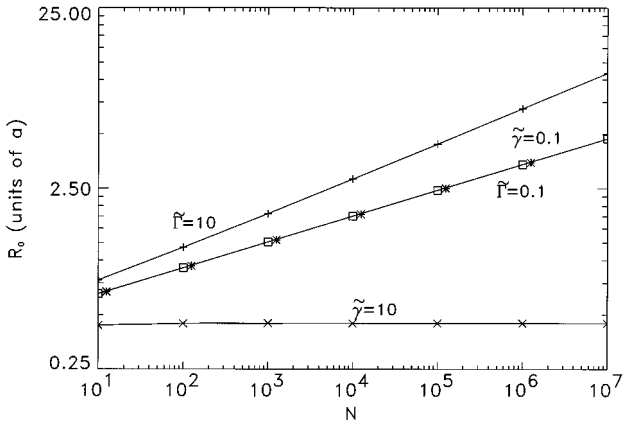

FIG. 2. Dependences of the condensate size R_0 on the number of atoms N with other parameters the same as in Fig. 1. The line connected with plus signs is the result for $\tilde{\Gamma}=10$ (WIG), while the line connected with crosses is that for $\tilde{\gamma}=10$ (HDP). The line connected with open squares is in the SWIG region with $\tilde{\Gamma}=0.1$ which overlays the line connected with asterisks for $\tilde{\gamma}=0.1$.

We also set $\hbar V_{\rm attr}(R_{\rm attr})$ equal to the characteristic energy of interatomic binding, $\simeq\hbar\times10^{14}$ Hz. With these numbers it is easy to show that $\tilde{\Gamma}/\Gamma$ is indeed small, $\simeq0.008$. For a large trap with trap frequency 10 Hz, and a ground-state size $a\simeq2$ μm, we are well above (WIG) or below (HDP) the critical region, depending on the sign of the scattering length. The situation is very much different, however, for a microtrap of frequency 0.1 MHz, and $a\simeq0.02$ μm. In this case we estimate that the gas at zero temperature will be in the SWIG phase provided the number of atoms is not too large, say $N\leqslant500$. We note that the estimates depend very strongly on the values of A and Γ_c used.

Of course, for such a value of N, the critical temperature for BEC is so low that it is hard to believe that it can be reached with conventional methods. Recently, however, it has been proposed that a gas in a microtrap can be *dynamically* condensed due to the interaction with an external agent—a laser that provides a mechanism of sideband cooling [23]. In such an open system the role of the control parameter (an analog of temperature) is played by the detuning of the laser from the motional sideband resonances. The adjustment of detuning allows the achievement of effective ''temperatures'' that allow for BEC, or even generalized BEC, in which the atoms condense in some of the excited states of the trap even with $N\simeq$ 50–100. Our present analysis strongly suggests that for such a system, if atoms condense, they will necessarily condense into the SWIG phase.

Even with $N<500$, however, one may question the validity of our mean field theory. We expect the mean field approach to be valid when the condensate correlation length l_c is much smaller than the condensate size R_0. For $a_{\rm sc}=2$ nm, $a=20$ nm, and the microtrap frequency $\omega_t=0.1$ MHz, direct application of Eqs. (17) and (18) indicates that $R_0\simeq100$ nm, $n_0=N\rho(0)\simeq5\times10^{17}$ cm^{-3}, and $l_c\simeq30$ nm. As we see, the size of the single atom ground state a is here comparable to, and in fact smaller than, l_c, whereas the actual size of the condensate R_0 is about three times larger than l_c. Although strictly speaking the condition $l_c\ll R_0$ is not fulfilled, nevertheless we think that for $l_c\simeq R_0/3$ the mean field theory should give a correct rough description of the low-temperature physics of the atomic gas in a tight trap. We have to admit, however, that corrections to the mean field theory results might in this case be substantial.

Summarizing, we have presented a phenomenological model of a Bose gas of trapped atoms, which interact via effective potentials that can support bound states and have a scattering length of arbitrary sign. We have found analytic expressions describing the ground state of such a system in the mean field approximation. In the regime of very small scattering length, or very tight atomic traps, the ground state exhibits properties different from those previously discussed [26]. We term the corresponding gaseous phase a super weakly interacting gaseous phase, since the net effect of atom-atom interactions is particularly weak in this phase. The SWIG phase exists even in the case of negative scattering lengths. We have discussed in detail scaling properties of the SWIG phase, and possibilities of its experimental realization.

The main qualitative result of our paper is that atoms with attractive interactions can behave in tight traps as if they repel each other. The question is how general this result is. Is it valid for small, but not very small traps, where the mean field is still applicable, or can one extend it to the limit when $\hbar\omega_t$ is much greater than the energy of atom-atom interactions (i.e., the Lamb-Dicke limit)? To shed some light on this question we discuss the problem of two atoms in a harmonic trap in Appendix B, and show that energy level shifts due to atom-atom interactions of the sort discussed in the paper change the sign as the trap tightens. In this sense our qualitative result can be extended to the mentioned limit, provided corrections to mean field are appropriately treated.

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APPENDIX A: KINETIC ENERGY CORRECTION IN THE SWIG PHASE

We remind the reader that the first step of the Bogoliubov-Hartree (BH) method consists in the minimization of the energy functional (6) in which the operators Ψ and $\Psi^\dagger$ are regarded as c numbers. The minimum can always be looked for using a variational method, and the density (17) together with wave function $\Psi_{\rm trial}(R)=\sqrt{N\rho(R)}$ is a good trial function. We divide the energy functional into the kinetic and potential parts. The real minimal energy fulfills

$$E_{\min}=\min_{\Psi,\Psi^\dagger}\mathscr{H}[\Psi,\Psi^\dagger]\leqslant\mathscr{H}_{\rm kin}[\Psi_{\rm trial},\Psi^\dagger_{\rm trial}]+\mathscr{H}_{\rm pot}[\Psi_{\rm trial},\Psi^\dagger_{\rm trial}]. \tag{A1}$$

At the same time, due to positiveness of the kinetic energy

$$E_{\min}\geqslant\min_{\Psi,\Psi^\dagger}\mathscr{H}_{\rm pot}[\Psi,\Psi^\dagger]. \tag{A2}$$

Elementary calculations yield that for the SWIG trial function

$$\mathscr{H}_{\rm kin}[\Psi_{\rm trial},\Psi^\dagger_{\rm trial}]=N\frac{\hbar^2}{2M}\frac{21}{2R_0^2}\propto N^{5/7}\quad\text{for}\quad N\to\infty. \tag{A3}$$

On the other hand, as $N\to\infty$, by the definition of $\Psi_{\rm trial}(R)$,

$$\min_{\Psi,\Psi^\dagger}\mathscr{H}_{\rm pot}[\Psi,\Psi^\dagger]=\mathscr{H}_{\rm pot}[\Psi_{\rm trial},\Psi^\dagger_{\rm trial}]=NE, \tag{A4}$$

with $E=3M\omega_t^2R_0^2/10\propto N^{2/7}$. We thus obtain

$$NE\leqslant E_{\min}\leqslant NE+O(N^{5/7}), \tag{A5}$$

which implies that asymptotically the kinetic energy term may be neglected, and $E_{\min}\simeq NE$.

APPENDIX B: TWO ATOMS IN A TIGHT TRAP

In order to illustrate the main qualitative result of this paper, we consider in this appendix two atoms interacting via potential (8) in a harmonic trap in 1D. The Schrödinger equation separates in this case in the center of mass and relative distance variables. Denoting $s=(x_1+x_2)/\sqrt{2}$, $r=(x_1-x_2)/\sqrt{2}$, the Hamiltonian becomes $H=H_s+H_r$, where

$$H_s=-\frac{\hbar^2}{2M}\frac{d^2}{ds^2}+\frac{M\omega_t^2s^2}{2}, \tag{B1}$$

$$H_r=-\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+\frac{M\omega_t^2r^2}{2}+V_{\rm eff}(\sqrt{2}r). \tag{B2}$$

For bosonic atoms (and without spin) only even wave functions of r are allowed. Also, only H_r has eigenvalues that depend on atom-atom interactions. In principle, one can easily follow the dependence of the eigenvalues of H_r as functions of ω_t. In the limit when $\hbar\omega_t$ becomes larger than typical interaction energy, one expects that perturbation theory with respect to $V_{\rm eff}$ should become valid.

The zeroth order eigenstates are the even harmonic oscillator eigenfunctions with eigenenergies $E_n=\hbar\omega_t n$. The energy level shifts are given by $\Delta E_n=\langle n|V_{\rm eff}|n\rangle$. Let us consider, in particular, the ground state shift,

$$\Delta E_0=\frac{B}{\sqrt{2\pi}a}-Ae^{\Gamma^2a^2/2}{\rm erfc}(\Gamma a/\sqrt{2}), \tag{B3}$$

with erfc( ) denoting the error function, and $a=\sqrt{\hbar/2M\omega_t}$. Evidently, for $a\to0$, the energy shift becomes positive (as if the atom-atom interactions were purely repulsive), but remains small in comparison to the zero point energy $\hbar\omega\propto1/a^2$. On the contrary, when $a\to\infty$, the shift is either positive or negative depending on the sign of the scattering length, or in other words depending on the sign of $\tilde{B}$, $\Delta E_0\to(B-2A/\Gamma)/(\sqrt{2\pi}a)\propto\tilde{B}$. When does the transition from the region of negative shift to the region of positive shift take place? Obviously, this happens exactly when the condition for the two-atom analog of the SWIG phase occurs. To this aim we expand the error function up to the second order term, and obtain

$$\Delta E_0\propto B-\frac{2A}{\Gamma}\left(1-\frac{1}{\Gamma^2a^2}\right)\simeq0, \tag{B4}$$

and therefore $\tilde{\Gamma}_c^2\simeq-2A/(B\Gamma a^2)$.

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