Phase states and phase operators for the quantum harmonic oscillator

Antoine Royer
Département de Génie Physique, Ecole Polytechnique, Montréal, Québec H3C 3A7, Canada
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How does the classical notion of "phase" apply to a quantum harmonic oscillator $H=\frac{1}{2}(\hat{q}^2+\hat{p}^2)$, $[\hat{q},\hat{p}]=i\hbar$, which cannot have sharp position *and* momentum? A quantum state $\hat{\rho}$ can be assigned a definite *classical* phase only if it is a *large-amplitude localized state*. Our only demand, therefore, on a (Hermitian) phase operator $\hat{\phi}$ is that the phase distribution $P(\varphi)=\mathrm{Tr}\{\delta(\hat{\phi}-\varphi)\hat{\rho}\}$ attribute the correct sharp phase to any such "classical phase" state. This requires that the Weyl symbol $[\hat{\phi}]_w(q,p)$ of $\hat{\phi}$ tend to $\theta \bmod 2\pi$ as $r\to\infty$, where $\theta=\tan^{-1}(p/q)$ and $r=(q^2+p^2)^{1/2}$. There are infinitely many such phase operators. Each is expressible as $\hat{\phi}=[\tan^{-1}(\hat{p}/\hat{q})]_\Omega$, where Ω specifies an *ordering rule* for $\hat{q}$ and $\hat{p}$. The commutator $-i[\hat{H},\hat{\phi}]=1-2\pi[\delta(\tan^{-1}\hat{p}/\hat{q})]_\Omega$ corresponds to the Poisson bracket $\{H,\phi_{\mathrm{cl}}\}_{\mathrm{PB}}=1-2\pi\delta(\theta)$ for the *single-valued* classical phase $\phi_{\mathrm{cl}}=\theta \bmod 2\pi$. *Phase states* $\hat{\Gamma}(\varphi)$ are defined by the condition that their Weyl symbols $[\hat{\Gamma}(\varphi)]_w(r,\theta)\to\delta(\theta-\varphi)$ as $r\to\infty$. If moreover $\int_0^{2\pi}d\varphi\hat{\Gamma}(\varphi)=\hat{1}$, then $\hat{\Gamma}(\varphi)$ is a phase probability operator measure (POM). In particular, $\delta(\hat{\phi}-\varphi)$ is a phase POM. Normalizable *approximate* phase states $\hat{\Gamma}^\varepsilon(\varphi)$ are defined by $[\hat{\Gamma}^\varepsilon(\varphi)]_w(r,\theta)=2\pi\varepsilon^2e^{-\varepsilon r}[\hat{\Gamma}(\varphi)]_w(r,\theta)$, $\varepsilon\ll 1$. Phase states are not, in general, "pure orthogonal" in the sense $\hat{\Gamma}(\varphi)\hat{\Gamma}(\varphi')=\delta(\varphi-\varphi')\hat{\Gamma}(\varphi)$, unless they are of the form $\delta(\hat{\phi}-\varphi)$. However, any phase state $\hat{\Gamma}_1(\varphi)$ is *trace orthogonal* to any phase POM $\hat{\Gamma}_2(\varphi)$, in the sense that $\mathrm{Tr}\{\hat{\Gamma}_2(\varphi')\hat{\Gamma}_1^\varepsilon(\varphi)\}\to\delta(\varphi-\varphi')$ as $\varepsilon\to 0$. This implies that measurement of the POM $\hat{\Gamma}_2(\varphi')$ on the state $\hat{\Gamma}_1^\varepsilon(\varphi)$ yields the outcome φ with probability 1 as $\varepsilon\to 0$. Phase measurements of the first kind are possible *in principle*; they would allow one to prepare (approximate) phase states and monitor their phase evolution in a quantum (phase) nondemolishing manner. Cases of special interest are the Susskind-Glogower and the Cahill-Glauber ordered phase states and operators. The energy-phase (or number-phase) uncertainty relation is $\Delta H\Delta\phi\geq 0$, the lower limit $\Delta H\Delta\phi=0$ being realized by pure number states; however, for states whose Wigner functions are localized away from the origin and from the extremities of the (single-valued) phase window $(0,2\pi)$, the uncertainty relation is effectively $\Delta H\Delta\phi\geq\frac{1}{2}$.

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I. INTRODUCTION

Phase changes in harmonic oscillators underlie interferometry, and the measurement of time. Indeed, phase shifts correspond to time displacements. A phase shift $\Delta\theta$, induced on a quantum state by some device, or just by the passage of time, is a *parameter* in a time evolution operator, whose measurement (e.g., by interferometry) need not involve a *phase operator*. Still, it would be desirable to dispose of a phase operator $\hat{\phi}$, whose measurement (of the first kind) might, hopefully, allow to *prepare* sharply phased states, and *monitor* their evolution in a "quantum nondemolishing" (QND) manner.

Classically, the phase of an oscillator with sharp position q and momentum p is the angle

$$\theta=\theta(q,p)=\tan^{-1}(p/q) \tag{1.1}$$

(in units where the mass and frequency of the oscillator are unity). But what is the phase, or phase distribution, of a quantum oscillator (we use $\hbar=1$),

$$\hat{H}=\tfrac{1}{2}(\hat{q}^2+\hat{p}^2),\quad [\hat{q},\hat{p}]=i, \tag{1.2}$$

which cannot have sharp position *and* momentum? Although the notion of "phase shift" is clear in the quantum context (a parameter in a time evolution operator), that of "phase" itself appears rather fuzzy.

In any case, since the Hamiltonian $\hat{H}$ generates time evolution, hence phase shifts, one would expect a phase operator $\hat{\theta}$ to satisfy

$$[\hat{H},\hat{\theta}]=i\Leftrightarrow e^{-it\hat{H}}\hat{\theta}e^{it\hat{H}}=\hat{\theta}+t. \tag{1.3}$$

Eigenstates of such a $\hat{\theta}$ would remain eigenstates under phase shifts, so that it would be possible to monitor them nondestructively by repeated $\hat{\theta}$ measurements. Unfortunately, the requirement (1.3), initially proposed by Dirac [1], is impossible to satisfy [2–4], leading to the belief, for some time, that a quantum phase operator does not exist [which is true if one defines it by (1.3)].

Various approaches to the "quantum phase" have been proposed, eliciting enormous literature for such a circumscribed subject. Some of these approaches will be reviewed in the first part of this paper. We restrict ourselves to those approaches which are more or less directly relevant to our own. In particular, we do not discuss very interesting treatments by Shapiro and Shepard [5], and by Ban [6], making use of "doubled" Hilbert spaces, or of Liouville space.

In this paper, we acknowledge that "phase" is an essentially *classical* notion, as was emphasized by Bergou and Englert [7]. A classical phase can be assigned unambiguously to a quantum state only if it is a *large amplitude localized state*. By this we mean a state $\hat{\rho}_{q'p'}$ for which the expectations $\langle q|\hat{\rho}_{q'p'}|q\rangle$ and $\langle p|\hat{\rho}_{q'p'}|p\rangle$ are localized within inter-

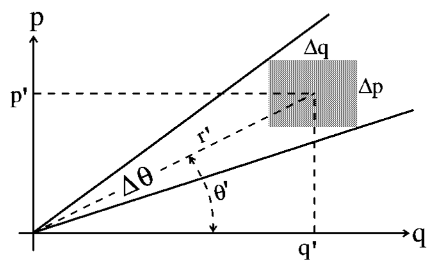

FIG. 1. The shaded box represents a quantum state $\hat{\rho}_{q'p'}$ whose position and momentum transition probabilities, $\langle q|\hat{\rho}_{q'p'}|q\rangle$ and $\langle p|\hat{\rho}_{q'p'}|p\rangle$, are localized about $(q',p')=(r',\theta')$, with spreads (uncertainties) Δq and Δp. The box subtends an angle $\Delta\theta$ at the origin of phase space. If the amplitude r' is large enough that the phase uncertainty $\Delta\theta$ is negligably small, then $\hat{\rho}_{q'p'}$ can be assigned the phase θ'.

vals Δq and Δp about q' and p', and such that the box $\Delta q\times\Delta p$ subtends a *vanishing* angle at the origin of the (q,p) plane (Fig. 1). Our only demand, therefore, on a (Hermitian) phase operator $\hat{\phi}$ is that the *phase distribution*

$$P(\varphi)=\mathrm{Tr}\{\delta(\hat{\phi}-\varphi)\hat{\rho}\} \tag{1.4}$$

attribute the correct sharp phase to any such "classical phase" state. Infinitely many different Hermitian operators then qualify as "phase operators." Each is expressible as

$$\hat{\phi}=[\tan^{-1}(\hat{p}/\hat{q})]_{\Omega}, \tag{1.5}$$

where Ω specifies an *ordering rule* [8] for the noncommuting operators $\hat{q}$ and $\hat{p}$. Thereby, $\hat{\phi}$ is a "natural" quantization of the classical observable $\theta=\tan^{-1}(p/q)$. The "canonical" ordering is Weyl ordering [8]. The Weyl ordered phase operator was first introduced and studied extensively (both for its fundamental physical significance and mathematical interest) by Smith, Dubin, and Hennings [9]. More general ordering rules were considered independently in [10].

We will find that phase measurements of the first kind are possible, in *principle*: They would allow to prepare sharply phased states, and monitor them in a "quantum (phase) nondemolishing" (QND) manner [11].

Since the paper is rather long, a self-contained summary is given in the two concluding sections XVIII and XIX. The rest of the paper is organized as follows: Sections II–VII review some of the existing literature, and also serve to set down a number of results that we shall need, and announce general features to come out of our own treatment. Sections VIII–XVII constitute the subject proper. Detailed calculations have been relegated to appendixes, to let the discussion flow more smoothly. Appendixes A–I contain details of calculations and complementary information to that given in the text. Appendix J is purely mathematical. Appendix K regroups various known formulas which we require.

II. MULTIPLE- AND SINGLE-VALUED PHASES

We will use polar coordinates in the (q,p) phase plane:

$$\theta=\tan^{-1}(p/q),\quad R=H(q,p)=\tfrac{1}{2}(q^2+p^2), \tag{2.1a}$$

$$r=(q^2+p^2)^{1/2},\quad \rho=|\alpha|=2^{-1/2}|q+ip|=2^{-1/2}r, \tag{2.1b}$$

where we defined three different radial variables: r is the radial distance; $\rho=|\alpha|$ is widely used in the quantum optics literature; R, equal to the classical oscillator Hamiltonian, is the "action variable" canonically conjugate to the angle θ: Indeed, the *canonical* transformation $(q,p)\to(R,\theta)$ preserves the form of Poisson brackets [12]

$$\{f,g\}_{\mathrm{PB}}=\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}=\frac{\partial f}{\partial R}\frac{\partial g}{\partial\theta}-\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial R} \tag{2.2}$$

so that

$$\{R,\theta\}_{\mathrm{PB}}=\{H,\theta\}_{\mathrm{PB}}=\frac{\partial\theta}{\partial\theta}=1. \tag{2.3}$$

The radial measures are $dR=rdr=2\rho d\rho$, and integrals over phase space will be $\int dqdp=\int_0^\infty rdr\int_0^{2\pi}d\theta$, etc.

Considering (2.3), we see that the demand $[\hat{H},\hat{\theta}]=i$, set by Dirac [1] for a quantum phase operator $\hat{\theta}$, accords with his cherished correspondence [13]

$$\{.,.\}_{\mathrm{PB}}\leftrightarrow -i[.,.] \tag{2.4}$$

between classical Poisson brackets and quantum commutators. As already mentioned, $[\hat{H},\hat{\theta}]=i$ is impossible to satisfy [2–4]. This is readily seen by considering its matrix elements between number states $|n\rangle$, where $\hat{H}|n\rangle=(n+\tfrac{1}{2})|n\rangle$:

$$\langle n'|[\hat{H},\hat{\theta}]|n\rangle=(n'-n)\langle n'|\hat{\theta}|n\rangle=i\delta_{n'n} \tag{2.5}$$

implying that $\hat{\theta}=\infty\hat{1}$. Moreover, $[\hat{H},\hat{\theta}]=i$ would imply that

$$e^{i\lambda\hat{\theta}}\hat{H}e^{-i\lambda\hat{\theta}}=\hat{H}+\lambda, \tag{2.6}$$

where λ is any real number, so that the kets $e^{-i\lambda\hat{\theta}}|n\rangle$ would be *normalizable* (if $\hat{\theta}$ is Hermitian) eigenkets of $\hat{H}$ with *continuous* eigenvalues $n+\tfrac{1}{2}+\lambda$.

The above is really not surprising: The phase θ used in (2.3) is a *multiple-valued* function of (q,p). Indeed, to get (2.2), it is essential that θ be a *continuous* function of q and p. Therefore, no well-behaved quantum observable can correspond to θ, because a quantum measurement can only have definite (single-valued) outcomes. One should really refer to a *single-valued* classical phase, such as

$$\phi=\phi(q,p)=\theta\ \mathrm{mod}2\pi. \tag{2.7}$$

This satisfies, instead of $\{H,\theta\}_{\mathrm{PB}}=1$,

$$\{H,\phi\}_{\mathrm{PB}}=\frac{\partial\phi}{\partial\theta}=1-2\pi\delta_{2\pi}(\theta), \tag{2.8}$$

where the Dirac comb

$$\delta_{2\pi}(\theta)\equiv\sum_{n=-\infty}^{\infty}\delta(\theta-2\pi n)=\delta(\theta\ \mathrm{mod}2\pi) \tag{2.9}$$

expresses the sudden -2π jumps suffered by ϕ as θ passes multiples of 2π. One might therefore expect a phase operator $\hat{\phi}$ to satisfy, rather than $[\hat{H},\hat{\theta}]=i$, the commutation relation

$$-i[\hat{H},\hat{\phi}]=\hat{1}-2\pi\hat{\Gamma}, \tag{2.10}$$

where $\hat{\Gamma}$, defined by (2.10), i.e., $2\pi\hat{\Gamma}=\hat{1}+i[\hat{H},\hat{\phi}]$, should be some quantum analog of $\delta_{2\pi}(\theta)$. This will indeed turn out to be the case.

III. QUANTUM PHASE FROM NUMBER SHIFT OPERATORS

In this section, we give a brief overview of what has been the main line of thought on the quantum phase. We will need the results

$$(2\pi)^{-1}\int_0^{2\pi} d\varphi\, e^{i(n'-n)\varphi}=\delta_{nn'}, \tag{3.1a}$$

$$(2\pi)^{-1}\int_0^{2\pi} \varphi\, d\varphi\, e^{i(n'-n)\varphi}=\pi\delta_{nn'}-(1-\delta_{nn'})\frac{i}{n'-n}. \tag{3.1b}$$

The relation (2.6), though impossible, points to energy shift operators, i.e., number shift operators, as quantum analogs of $e^{\pm i\theta}$. Since the latter do not suffer the multivaluedness that θ does, one suspects their quantum analogs to be less problematic than $\hat{\theta}$ itself. The familiar number shift operators are the creation and annihilation operators:

$$\hat{a}=2^{-1/2}(\hat{q}+i\hat{p}),\quad \hat{a}^\dagger=2^{-1/2}(\hat{q}-i\hat{p}),\quad \hat{N}=\hat{a}^\dagger\hat{a}=\hat{H}-\tfrac{1}{2}, \tag{3.2a}$$

$$[\hat{a},\hat{a}^\dagger]=1,\quad [\hat{N},\hat{a}]=-\hat{a},\quad [\hat{N},\hat{a}^\dagger]=\hat{a}^\dagger. \tag{3.2b}$$

The relation $[\hat{N},\hat{a}]=-\hat{a}$ is indeed similar to (2.6) written in the form $[\hat{N},e^{i\hat{\theta}}]=-e^{i\hat{\theta}}$. The eigenstates of $\hat{a}$ are coherent states

$$|qp\rangle^{\mathrm{coh}}=|\alpha\rangle^{\mathrm{coh}}=|\rho e^{i\theta}\rangle^{\mathrm{coh}}=e^{-(1/2)\rho^2}\sum_{n=0}^{\infty}\frac{\rho^n e^{in\theta}}{(n!)^{1/2}}|n\rangle, \tag{3.3a}$$

$$\alpha=2^{-1/2}(q+ip)=\rho e^{i\theta},\quad \hat{a}|\alpha\rangle=\alpha|\alpha\rangle. \tag{3.3b}$$

These are the archetypical localized states, with $\Delta q=\Delta p\approx 1$ (see Fig. 1): They indeed acquire a definite phase as their amplitudes $\rho\to\infty$, but *not* at lesser values of ρ. Let us try, then, to modify $\hat{a}$: Noting that $\hat{a}\hat{N}\hat{a}^\dagger=(N+1)^2$, we see that a closer similarity with (2.6), and also with $e^{\pm i\theta}=r^{-1/2}(q\pm ip)$, will be obtained if, following Susskind and Glogower [3], we define "expθ" operators as

$$\hat{E}=(\hat{N}+1)^{-1/2}\hat{a}=\sum_{n=0}^{\infty}|n\rangle\langle n+1|,$$

$$\hat{E}^\dagger=\hat{a}^\dagger(\hat{N}+1)^{-1/2}=\sum_{n=0}^{\infty}|n+1\rangle\langle n|. \tag{3.4}$$

One then has [compare (3.2)]

$$\hat{E}^\dagger|n\rangle=|n+1\rangle,\quad \hat{E}|n\rangle=(1-\delta_{n0})|n-1\rangle, \tag{3.5a}$$

$$[\hat{N},\hat{E}]=-\hat{E},\quad [\hat{N},\hat{E}^\dagger]=\hat{E}^\dagger, \tag{3.5b}$$

$$\hat{E}\hat{E}^\dagger=\hat{1},\quad \hat{E}^\dagger\hat{E}=\hat{1}-|0\rangle\langle 0|, \tag{3.5c}$$

$$(\hat{E})^m\hat{N}(\hat{E}^\dagger)^m=\hat{N}+m \tag{3.5d}$$

$$\begin{aligned}(\hat{E}^\dagger)^m\hat{N}(\hat{E})^m=\hat{N}-m+m|0\rangle\langle 0|+(m-1)|1\rangle\langle 1|\\ +(m-2)|2\rangle\langle 2|+\cdots+|m-1\rangle\langle m-1|,\end{aligned} \tag{3.5e}$$

where $\hat{E}^m$ indeed appears as a discrete analog of $e^{-i\lambda\hat{\theta}}$ in (2.6). However, since it annihilates the vacuum, $\hat{E}$ is not invertible, hence not unitary: Indeed, it cannot be, for otherwise it could be used to shift $\hat{N}$ down to negative eigenvalues [contrary to (3.5e)].

The Susskind-Glogower number shift operator $\hat{E}$ has the following unormalizable eigenkets:

$$|\varphi\rangle_{\mathrm{SG}}=(2\pi)^{-1/2}\sum_{n=0}^{\infty}e^{in\varphi}|n\rangle,\quad \hat{E}|\varphi\rangle_{\mathrm{SG}}=e^{i\varphi}|\varphi\rangle_{\mathrm{SG}}. \tag{3.6a}$$

These constitute a complete but nonorthogonal set of vectors [4]:

$$\int_0^{2\pi}d\varphi|\varphi\rangle_{\mathrm{SG}}\langle\varphi|=\hat{1}, \tag{3.6b}$$

$$\begin{aligned}{}_{\mathrm{SG}}\langle\varphi'|\varphi\rangle_{\mathrm{SG}}&=(2\pi)^{-1}\sum_{n=0}^{\infty}e^{in(\varphi-\varphi')}\\ &=\frac{1}{4\pi}+\tfrac{1}{2}\delta(\varphi'-\varphi)-\frac{i}{4\pi}\cot\frac{\varphi'-\varphi}{2},\end{aligned} \tag{3.6c}$$

where (3.6b) follows from (3.1a), and the principal part is understood in $\cot\frac{1}{2}(\varphi'-\varphi)$. The "phase states" $|\varphi\rangle_{\mathrm{SG}}$ have many compelling properties, and the "phase distribution" defined by

$$P_{\mathrm{SG}}(\varphi)={}_{\mathrm{SG}}\langle\varphi|\hat{\rho}|\varphi\rangle_{\mathrm{SG}} \tag{3.7}$$

is widely used [14]. However, not being orthogonal, $|\varphi\rangle_{\mathrm{SG}}$ cannot be the eigenkets of a Hermitian operator.

This prompted Pegg and Barnett [15] to consider a finite Hilbert space $\{|n\rangle,\ n=1,2,\ldots,s\}$, wherein

$$|\theta_j\rangle_{\rm PB}=(s+1)^{-1/2}\sum_{n=0}^{s} e^{in\theta_j}, \quad \theta_j=\frac{2\pi j}{s+1}, \quad j=0,1,\ldots,s \tag{3.8a}$$

are orthogonal eigenkets of a Hermitian phase operator

$$\hat{\phi}_{\rm PB}=\sum_{n=0}^{s}|\theta_j\rangle\theta_j\langle\theta_j|, \quad \hat{\theta}_{\rm PB}|\theta_j\rangle=\theta_j|\theta_j\rangle. \tag{3.8b}$$

An elegant formalism ensues [15]. In the limit $s\to\infty$,

$$\left(\frac{s+1}{2\pi}\right)^{1/2}|\theta_j\rangle_{\rm PB}\to|\theta_j\rangle_{\rm SG}, \quad \hat{\phi}_{\rm PB}\to\hat{\phi}_{\rm SG}, \tag{3.9a}$$

where [by (3.1b)]

$$\hat{\phi}_{\rm SG}=\int_0^{2\pi}\varphi\, d\varphi|\varphi\rangle_{\rm SG}\langle\varphi|=\pi+\sum_{n'\neq n}|n'\rangle\frac{i}{n-n'}\langle n|. \tag{3.9b}$$

Still, $|\varphi\rangle_{\rm SG}$ are not eigenkets of $\hat{\phi}_{\rm SG}$, even though the latter is a well behaved operator.

The SG operator $\hat{E}$ and the annihilation operator $\hat{a}$ belong to a general family of "modulated" number shift operators

$$\hat{E}_\lambda=\sum_{n=0}^{\infty}|n-1\rangle\lambda_n\langle n|, \quad \lambda_0=0, \tag{3.10a}$$

$$\hat{E}_\lambda|n\rangle=|n-1\rangle\lambda_n, \quad \hat{E}_\lambda^\dagger|n\rangle=|n+1\rangle\lambda_{n+1} \tag{3.10b}$$

introduced by Lerner, Huang, and Walters [16]. The eigenkets of $\hat{E}_\lambda$ are given by

$$|\beta\rangle=\sum_{n=0}^{\infty}c_n|n\rangle, \quad c_n=\frac{\beta c_{n-1}}{\lambda_n}=\frac{\beta^n c_0}{\Pi_{m=1}^{n}\lambda_m},$$
$$\hat{E}_\lambda|\beta\rangle=\beta|\beta\rangle. \tag{3.11}$$

Susskind-Glogower corresponds to $\lambda_n=1$ and $\beta=e^{i\varphi}$. The annihilation operator $\hat{a}$ corresponds to $\lambda_n=n^{1/2}$. Not all states (3.11) have a definite phase [e.g., the coherent states (3.3) at small ρ]: In fact, *any* pure state $|\Psi\rangle=\Sigma_n c_n|n\rangle$ *is an eigenket* $|\beta\rangle$ *of some operator* $\hat{E}_\lambda$, given by (3.10a) with $\lambda_n=\beta c_{n-1}/c_n$. Lerner *et al.* [16] argued that for (3.11) to qualify as "phase states," they should have $\beta=e^{i\varphi}$, and $\lambda_n\to1$ as $n\to\infty$, so that $c_n\to(\text{const})e^{in\varphi}$, that is, they should tend to the Susskind-Glogower $|\varphi\rangle_{\rm SG}$ as $n\to\infty$.

Leonhardt, Vaccaro, Bohmer, and Paul [17] discuss several approaches of the above kind in a unified way, using as underlying notions (i) covariance under phase shifts $e^{i\theta\hat{H}}$, and (ii) invariance under the Susskind-Glogower number shift $\hat{E}^\dagger$. Note, however, that although it has many compelling properties, the number shift $\hat{E}^\dagger$ is not unique, as indicated above.

IV. CLASSICAL PHASE DISTRIBUTIONS

Another line of attack on the quantum phase emphasizes *phase distributions* rather than *operators* for phase or functions of phase. We first discuss, in this section, *classical* phase distributions.

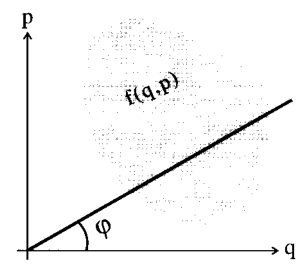

FIG. 2. The shaded region represents a classical state $f(q,p)$. The phase distribution is the overlap of $f(q,p)$ with the infinitely narrow wedge $\delta_\varphi(q,p)$, represented by the radius at angle φ.

A central role is played here by the angular δ function

$$\delta(\theta-\varphi)=\delta_\varphi(q,p)=\frac{q}{\cos\varphi}\delta(q\sin\varphi-p\cos\varphi)\Theta\left(\frac{q}{\cos\varphi}\right), \tag{4.1}$$

where

$$\Theta(x)=\begin{cases}0 & (x<0)\\ 1 & (x>0).\end{cases} \tag{4.2}$$

The step function in (4.1) assures that $q/\cos\varphi>0$, i.e., that $\delta_\varphi(q,p)$ is localized over only half of the straight line $q\sin\varphi=p\cos\varphi$. By contrast,

$$\delta(q\sin\varphi-p\cos\varphi)=\frac{\delta(\theta-\varphi)+\delta(\theta-\varphi-\pi)}{r} \tag{4.3}$$

is localized along the complete straight line $q\sin\varphi=p\cos\varphi$. The extra radial weight $r=q/\cos\varphi$ in (4.1), as against (4.3), reflects the fact that an angular increment $\Delta\theta$ defines a "wedge" in phase space (see Fig. 1). Its manifestations will be a recurrent theme in this paper.

A classical statistical state is a *phase space* probability distribution $f_{\rm cl}(q,p)=f_{\rm cl}(r,\theta)$. Its *phase* probability distribution $P_{\rm cl}(\varphi)$ is the expectation of $\delta(\theta-\varphi)$, that is, in the picturesque language of Schleich and Wheeler [18], the "overlap" of $f_{\rm cl}(q,p)$ with $\delta_\varphi(q,p)$ (Fig. 2):

$$P_{\rm cl}(\varphi)=\int dq\, dp\, \delta_\varphi(q,p)f_{\rm cl}(q,p) \tag{4.4a}$$

$$=\int_0^\infty r\, dr\, f_{\rm cl}(r,\varphi), \tag{4.4b}$$

the radial integral of $f_{\rm cl}(r,\varphi)$. The unormalizable "states" $\delta_\varphi(q,p)$ thus act as *phase probability measures*. By analogy with the quantum formalism, they may be called "classical phase eigenstates."

The phase distribution (4.4) may be compared with the "orientation" distribution

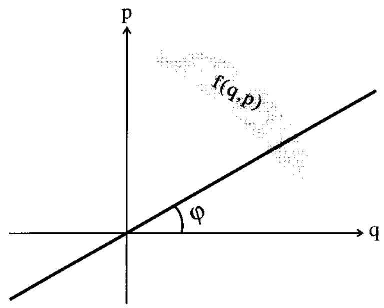

FIG. 3. The shaded region represents a classical state $f(q,p)$. The orientation distribution $P_{\text{orient}}(\varphi)$ is the overlap of $f(q,p)$ with the straight line $\delta(q\sin\varphi - p\cos\varphi)$.

$$P_{\text{orien}}(\varphi) = \int dq\; dp\, \delta(q\;\sin\varphi - p\;\cos\varphi) f_{\text{cl}}(q,p) \tag{4.5a}$$

$$= \int_0^\infty dr [f_{\text{cl}}(r,\varphi) + f_{\text{cl}}(r,\varphi+\pi)], \tag{4.5b}$$

the overlap of $f_{\text{cl}}(q,p)$ with the complete straight line (4.3) (see Fig. 3). This is the value at $x=0$ of the φ-quadrature distribution

$$P_\varphi(x) = \int dq\; dp\, \delta(q\;\sin\varphi - p\;\cos\varphi - x) f_{\text{cl}}(q,p). \tag{4.6}$$

As pointed out by Schleich and his co-workers [19], the "quadrature based" phase distribution ["half" of (4.5)]

$$P_{\text{quad}}(\varphi) = \int_0^\infty dr\; f_{\text{cl}}(r,\varphi) \tag{4.7}$$

suitably renormalized, gives a good idea of the true phase distribution (4.4), in the case of states sufficiently localized *radially* that the wedge radial weight r in (4.4) is of little consequence.

V. QUANTUM PHASE-SPACE DISTRIBUTIONS

Considering (4.4), one naturally thinks of using quantum *phase space* distributions for defining quantum *phase* distributions. We first recall in this section some basic results concerning phase space representations of quantum states and operators.

Consider a wave function

$$\Psi(x) = R(x) e^{iS(x)}, \quad R(x), S(x) \;\; \text{real} \tag{5.1}$$

(we here use, as position variables, x in configuration space, and q in phase space). It will be enlightening to also think of x and q as "time," of p as "frequency," and of $\Psi(x)$ as a time-dependent signal. Another common representation of Ψ is its momentum (frequency) representation

$$\tilde{\Psi}(p) = (2\pi)^{-1/2} \int dx\; e^{-ipx} \Psi(x) \tag{5.2}$$

which gives the momenta (frequencies) comprising Ψ. The absolute square $|\tilde{\Psi}(p)|^2$ is the "power spectrum" of Ψ. One often has, however, a description involving both position *and* momentum (time *and* frequency), that is, a phase-space representation. For instance, in the process of hearing, one does not perceive the detailed time variation of the air pressure, nor its frequency spectrum, but rather a time-varying "local" frequency spectrum. Let us intuit the general phase space appearance of $\Psi(x)$:

We first note that $\langle p\rangle(x) = S'(x) = \partial S/\partial x$ appears as a "local" mean momentum at x. Thus, a crude phase-space representation of $\Psi(x)$ is a ribbon centered on the line $p = S'(q)$. The ribbon has a width $\Delta p(x)$, due to the variations of $R(x)$. The relevant characteristic here is the relative rate of variation $|R'(x)/R(x)|$: Since e^{ipx} produces relative rates of variation of size $\approx p$, we have, roughly [8c],

$$\langle p\rangle(x) \approx S'(x), \quad \Delta p(x) \approx \left|\frac{R'(x)}{R(x)}\right|. \tag{5.3}$$

For instance, if $\Psi(x) = x^\alpha$, then $\Delta p(x) \approx x^{-1}$, so that x^α gets more and more sharply localized in momentum as $|x|$ increases.

The above argument is only qualitative. Can one build *quantitative* phase-space representations? Since hearing naturally provides a time-frequency representation of sound, let us imitate that process: This can be modeled (very crudely) as effected by a bank of damped oscillators of different natural frequencies [20]. The energy imparted by the signal $\Psi(x)$ to the oscillator of frequency p, at time q, is (approximately) a *windowed* Fourier spectrum

$$f(q,p) = (2\pi)^{-1} \left| \int dx\; e^{-ipx} h(q-x) \Psi(x) \right|^2, \tag{5.4}$$

where $h(x)$ is the damping function of the oscillator [assumed independent of its natural frequency; the $(2\pi)^{-1}$ is for normalization]. Of course, $h(x)$ has its own p width, which adds to that of $\Psi(x)$ [21].

The windowed Fourier spectrum provides a quantitative phase-space representation which, however, depends on a choice of window function. Is there an "absolute" phase-space representation? Somewhat surprisingly, there is: Consider, first, the overlap of a real function $f(x)$ with is inversion about q,

$$W(q) = \int dx\; f(q+x) f(q-x). \tag{5.5}$$

This accurately portrays the whereabouts of $f(x)$, but with an added feature: $W(q)$ contains "inner interferences" between different parts of $f(x)$; for instance, if $f(x)$ has two humps, then in addition to imaging the two humps, $W(q)$ also has an extra hump at their midpoint (Fig. 4).

As it stands, (5.5) is of little interest. Things become interesting if we *simultaneously* invert in momentum: To invert a (complex) function $\Psi(x)$ in p, complex conjugate it, as this causes $\langle p\rangle(x) = S'(x)$ to change sign (this is the usual time-reversal operation, better called *motion* reversal). To displace $\Psi(x)$ by p, multiply it be e^{ipx}. Thus, corresponding to $f(x+q)$ in (5.5), we displace $\Psi(x)$ by $-q$, and then by

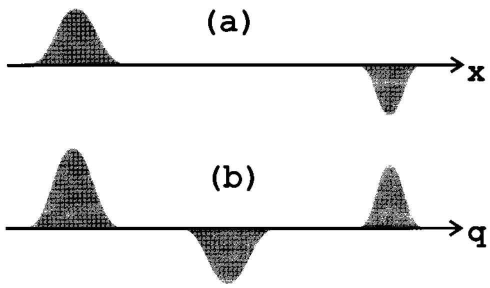

FIG. 4. (a) A real function $f(x)$ with two humps; (b) the corresponding $W(q)$ reproduces the two humps positively, and also has an "interference" structure (of any sign) in the region of their midpoint, because the reflection of each hump about that region overlaps the other hump.

$-p$, that is, $\Psi(x)\rightarrow\Psi(x+q)\rightarrow e^{-ipx}\Psi(x+q)$; corresponding to $f(q-x)$, we displace $\Psi(x)$ by q and invert in x, then displace by p and invert in p, that is $\Psi(x)\rightarrow\Psi(x-q)\rightarrow\Psi(q-x)\rightarrow e^{ipx}\Psi(q-x)\rightarrow e^{-ipx}\Psi^*(q-x)$. The overlap of $\Psi(x)$ with its inversion about (q,p) is then

$$W(q,p)=\pi^{-1}\int dx\; e^{-ipx}\Psi(q+x)e^{-ipx}\Psi^*(q-x), \quad (5.6a)$$

$$\int dq\; dp\; W(q,p)=\int dx|\Psi(x)|^2 \quad (5.6b)$$

[we added π^{-1} to get the normalization (5.6b)]. This is the Wigner function [8]: It accurately portrays the phase-space localization of $\Psi(x)$, as evidenced by the "marginals"

$$\int dp\; W(q,p)=|\Psi(q)|^2, \quad \int dq\; W(q,p)=|\tilde{\Psi}(p)|^2. \quad (5.7)$$

However, inner interference structures superpose on the "true" phase-space features of $\Psi(x)$, as the following example will illustrate.

Let first $\Psi(x)=\langle x|qp\rangle^{\mathrm{coh}}$ be a coherent state (3.3) centered on (q,p): Then

$$\Psi(x)=\langle x|qp\rangle^{\mathrm{coh}}=\pi^{-1/4}e^{-(1/2)iqp}e^{ipx}e^{-(1/2)(x-q)^2}, \quad (5.8)$$

$$W^{\mathrm{coh}}_{\mathrm{qp}}(q',p')=\pi^{-1}e^{-(q'-q)^2-(p'-p)^2}. \quad (5.9)$$

Let now $\Psi(x)$ be a superposition of two coherent states:

$$\Psi_{1+2}(x)=\langle x|q_1p_1\rangle^{\mathrm{coh}}+\langle x|q_2p_2\rangle^{\mathrm{coh}}. \quad (5.10)$$

Denoting $u=(q,p)$ and $u\wedge u'=qp'-pq'$, we have

$$W_{1+2}(q,p)=W^{\mathrm{coh}}_1(q,p)+W^{\mathrm{coh}}_2(q,p) + W^{\mathrm{coh}}_m(q,p)\cos[\Phi+\tfrac{1}{2}u_1\wedge u_2], \quad (5.11a)$$

$$q_m=\frac{q_1+q_2}{2}, \quad p_m=\frac{p_1+p_2}{2}, \quad (5.11b)$$

$$\Phi=(u-u_m)\wedge(u_1-u_2). \quad (5.11c)$$

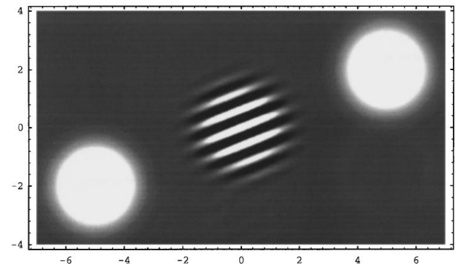

FIG. 5. Density plot showing the Wigner function of a superposition of two coherent states; note the interference structure between the two (positive) "true" phase-space structures (white is positive, black is negative).

The phase Φ is the area of the parallelogram defined by the vectors $(u-u_m)$ and (u_1-u_2). So the interference term in (5.11), localized about the midpoint (q_m,p_m), oscillates in the direction perpendicular to the line (u_1-u_2) with a wavelength $2\pi|u_1-u_2|^{-1}$ (Fig. 5) [22].

The Wigner function of a state operator $\hat{\rho}$ is defined as an obvious extension of (5.6): We write it as

$$W(q,p)=(2\pi)^{-1}\int ds\; e^{-isp}\langle q+\tfrac{1}{2}s|\hat{\rho}|q-\tfrac{1}{2}s\rangle. \quad (5.12)$$

More generally, the *Weyl symbol* of an operator $\hat{A}$, denoted $[\hat{A}]_w(q,p)$ or $A_w(q,p)$, is defined as [8]

$$[\hat{A}]_w(q,p)=\int ds\; e^{-isp}\langle q+\tfrac{1}{2}s|\hat{A}|q-\tfrac{1}{2}s\rangle. \quad (5.13)$$

It is (2π) times the "Wigner function" of the operator $\hat{A}$. Reciprocally, a Wigner function is the Weyl symbol of a state operator $\hat{\rho}$, normalized to $\mathrm{Tr}\{\hat{\rho}\}$. An example of Weyl symbol is

$$[|q_1p_1\rangle^{\mathrm{coh}}\langle q_2p_2|]_w(q,p) =2e^{-(q-q_m)^2-(p-p_m)^2}e^{i\Phi}e^{(1/2)i(q_1p_2-p_1q_2)} \quad (5.14)$$

$[\Phi,q_m,p_m$ as in (5.11)]. The real part of (5.14) is just the interference term in (5.11).

One easily shows the following classical-like results [8]:

$$\mathrm{Tr}\{\hat{A}\hat{B}\}=(2\pi)^{-1}\int dq\; dp\; A_w(q,p)B_w(q,p), \quad (5.15)$$

$$\mathrm{Tr}\{\hat{A}\}=(2\pi)^{-1}\int dq\; dp\; A_w(q,p), \quad (5.16)$$

which we shall use repeatedly. It follows from (5.15) that

$$\int dq\; dp\; W_1(q,p)W_2(q,p)=(2\pi)^{-1}\mathrm{Tr}\{\hat{\rho}_1\hat{\rho}_2\}\geqslant 0, \quad (5.17)$$

that is, the overlap of two Wigner functions is always positive. An example is the windowed spectrum (5.4), which is equal to the convolution of $W_\Psi(q,p)$ with the Wigner function $W_h(q,p)$ of $h(x)$:

$$(2\pi)^{-1}\left|\int dx\, e^{-ipx}h(q-x)\Psi(x)\right|^2$$

$$=\int dq'dp'\; W_\Psi(q-q',p-p')W_h(q',p')\geqslant 0. \quad (5.18)$$

This *may* contain interference structures, but superimposed on a positive background, for a net non-negative result. This means that the smearing of $W_\Psi(q,p)$ with $W_h(q,p)$ (or with any other Wigner function) either essentially wipes out interference structures, or, if only partially, also causes the "true" (positive) features to spread out into the interference regions, to produce a net non-negative result.

As another example of (5.17), we have

$$\int dq'dp'\; W(q',p')\frac{1}{\pi ab}\exp\left[-\frac{(q-q')^2}{a^2}-\frac{(p-p')^2}{b^2}\right]$$

$$\geqslant 0 \quad (ab\geqslant 1), \quad (5.19)$$

since any Gaussian of area $\geqslant 1$ is the Wigner function of a thermal oscillator state [8a]. A limiting case of (5.19) is the integral of $W(q,p)$ over an infinite straight line (an infinitely flattened elliptic Gaussian), such as the marginal [see (5.7)]

$$\int dq\; W(q,p')=\int dq\; dp\;\; \delta(p-p')W(q,p)$$

$$=\langle p'|\hat\rho|p'\rangle\geqslant 0. \quad (5.20)$$

Still another example is

$$\int_0^\infty dr[W(r,\varphi)+W(r,\varphi+\pi)]=\langle p'=0|e^{-i\varphi\hat H}\hat\rho e^{i\varphi\hat H}|p'=0\rangle$$

$$\geqslant 0 \quad (5.21)$$

obtained by replacing the "horizontal" line $\delta(p-p')$ in (5.20) by the slanted line (4.3); this amounts to replacing $|p'\rangle$ by the "rotated momentum state" $e^{i\varphi\hat H}|p'=0\rangle$, whose Weyl symbol is indeed (4.3). (By the Weyl symbol of a vector $|\Psi\rangle$, we understand the Weyl symbol of the operator $|\Psi\rangle\langle\Psi|$.)

Because the marginals of $W(q,p)$ are precisely $\langle q|\hat\rho|q\rangle$ and $\langle p|\hat\rho|p\rangle$, the Wigner-Weyl phase-space representation is as sharp as possible. Coarser phase-space representations of $\hat\rho$ may be obtained by smearing $W(q,p)$ with any functions $G(q,p)$:

$$f_G(q,p)=\int dq'dp'\; W(q-q',p-p')G(q',p'),$$

$$\int dq\; dp\; G(q,p)=1. \quad (5.22)$$

Correspondingly, the marginals of $f_G(q,p)$ are smearings of $\langle q|\hat\rho|q\rangle$ and $\langle p|\hat\rho|p\rangle$.

Putting, in (5.22), $G(q,p)=\pi^{-1}e^{-q^2-p^2}$, which is the Wigner function of $\phi_0(x)=\pi^{-1/4}e^{-(1/2)x^2}$, the ground state of $\hat H=\frac{1}{2}(\hat q^2+\hat p^2)$, we get the (non-negative) Q function

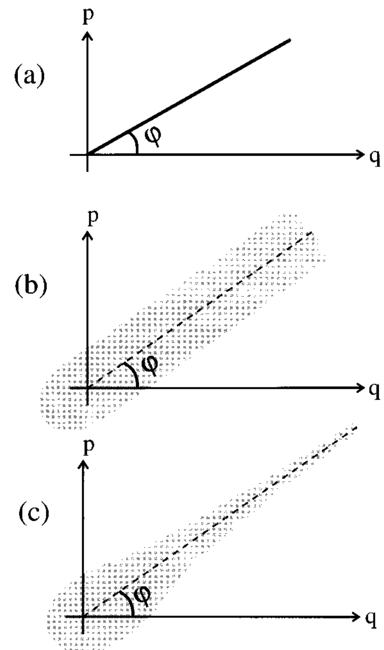

FIG. 6. The "phase distribution" of a quantum state may be expressed as $P(\varphi)=\int dq\; dp\Gamma(q,p;\varphi)W(q,p)$, where $W(q,p)$ is the Wigner function of the state, and the function $\Gamma(q,p;\varphi)=\Gamma(r,\theta;\varphi)\to\delta(\theta-\varphi)$ tends to a zero *angular* width as $r\to\infty$, but not necessarily to a zero "true" width. In (a), $\Gamma(q,p;\varphi)=\delta_\varphi(q,p)$ is a sharp radius at all r; in (b), the radius has a finite true width at all r [for instance, the convolution of $\delta_\varphi(q,p)$ with $\pi^{-1}e^{-q^2-p^2}$]; in (c), the true width is finite near the origin and tends to zero as $r\to\infty$.

$$Q(q,p)=\pi^{-1}\int dq'dp'\; W(q',p')e^{-(q-q')^2-(p-p')^2} \quad (5.23a)$$

$$={}^{\mathrm{coh}}\langle qp|\hat\rho|qp\rangle^{\mathrm{coh}}. \quad (5.23b)$$

If $\hat\rho=|\Psi\rangle\langle\Psi|$, then $Q(q,p)=|{}^{\mathrm{coh}}\langle qp|\Psi\rangle|^2$ is the windowed spectrum (5.4), with $h(x)=\pi^{-1/4}e^{-(1/2)x^2}$.

VI. QUANTUM PHASE DISTRIBUTIONS

Let us now return to *phase* distributions, and apply the classical definitions of Sec. IV to quantum *phase-space* distributions. Since the Wigner function $W(q,p)=W(r,\theta)$ is as sharp as possible, we try, as a phase distribution,

$$P_w(\varphi)=\int dq\; dp\;\; \delta_\varphi(q,p)W(q,p)=\int_0^\infty r\; dr\;\; W(r,\varphi). \quad (6.1)$$

Compare this with (5.21): We note two differences: (i) The integration measure is rdr, instead of dr; (ii) $P_w(\varphi)$ is the integral of $W(q,p)$ over *half* a straight line [Fig. 6(a)]. If $W(q,p)$ is only localized within a sector of angular width less than π, then the half line may be extended to a full line without altering the value of the integral, so that one expects $P_w(\varphi)\geqslant 0$, essentially (not strictly because $rdr\neq dr$). But if $W(r,\theta)$ has an interference structure covering the origin, then *because it starts suddenly at* $r=0$, the integral (6.1) is sensitive to the oscillations of $W(r,\theta)$ there, and may be *negative* [23]. So, not being strictly non-negative, $P_w(\varphi)$ cannot always do as a phase *probability* distribution.

As pointed out in Refs. [19], the "quadrature based" phase distribution

$$P_{\text{quad}}(\varphi)=\int_0^\infty dr\ W(r,\varphi) \tag{6.2}$$

is positive if $W(r,\theta)$ is angularly localized within a π sector [in which case (6.2) is equal to (5.21)], and, suitably renormalized, closely resembles $P_w(\varphi)$ if $W(r,\theta)$ is sufficiently localized radially [see after Eq. (4.6)]. The distribution (6.2) has the advantage of being easily measurable—just measure the expectation (5.21)—but is of restricted applicability.

To cure the nonpositivity of $P_w(\varphi)$, one may use, instead of $W(q,p)$, a phase-space distribution which is strictly non-negative. Such may be obtained by smearing $W(q,p)$ with a Gaussian of area $\geqslant 1$, as in (5.19). To preserve as much as possible the sharpness of $W(q,p)$, we choose a Gaussian of minimum area $=1$, for instance $\pi^{-1}e^{-q^2-p^2}$. There results the Q function (5.23). So consider

$$P_Q(\varphi)=\int dq\ dp\,\delta_\varphi(q,p)Q(q,p)=\int_0^\infty r\ dr\ Q(r,\varphi) \tag{6.3a}$$

$$=\int dq\ dp\ \Gamma_Q(q,p;\varphi)W(q,p) \tag{6.3b}$$

$$=\text{Tr}\{\hat{\Gamma}_Q(\varphi)\hat{\rho}\}. \tag{6.3c}$$

To go from (6.3a) to (6.3b), we transferred the smearing (5.23a) to $\delta_\varphi(q,p)$, defining

$$\Gamma_Q(q,p;\varphi)=\pi^{-1}\int dq'dp'\,\delta_\varphi(q',p')e^{-(q-q')^2-(p-p')^2} \tag{6.4}$$

localized along a smeared radius of width ≈ 1[Fig. 6(b)]. In (6.3c), we expressed $P_Q(\varphi)$ in terms of the POM (probability operator measure),

$$\hat{\Gamma}_Q(\varphi)=\int_0^\infty r\ dr|r\varphi\rangle^{\text{coh}}\langle r\varphi| \tag{6.5}$$

showing that $P_Q(\varphi)$ is directly measurable [unlike $P_w(\varphi)$]. An *actual* quantum phase measurement, performed by Noh, Fougères, and Mandel (NFM) [24], was shown by Freyberger and Schleich [25] to yield $P_Q(\varphi)$. More basically, D'Ariano and Paris [26] show that NFM measure the POM (6.5).

Despite its practical advantages, the phase distribution $P_Q(\varphi)$ seems coarse, being the "overlap" of $W(q,p)$ with the fat radius $\Gamma_Q(q,p;\varphi)$[Fig. 6(b)]. One senses that in the case, for instance, of a squeezed state, for which $W(q,p)\geqslant 0$, or just of any state for which $P_w(\varphi)\geqslant 0$, the phase distribution $P_w(\varphi)$ is much more precise than $P_Q(\varphi)$.

One feels, in fact, that to assure positivity, it would suffice to smear the sharp radius $\delta_\varphi(q,p)$ in (6.1) *only near its abrupt beginning at the origin*, to reduce sensitivity to possible oscillations of $W(q,p)$; the smearing of $\delta_\varphi(q,p)$ *away* from the origin [as in (6.4)] seems unnecessary, and just spoils the finesse of $P_w(\varphi)$ there. So one might hope that a phase distribution finer than $P_Q(\varphi)$, yet non-negative, would be a compromise between (6.1) and (6.3), of the form

$$P_\Gamma(\varphi)=\int_0^\infty r\ dr\int_0^{2\pi}d\theta\ \Gamma(r,\theta;\varphi)W(r,\theta) \tag{6.6}$$

for some function $\Gamma(r,\theta;\varphi)$ tending smoothly to zero over distances of order 1 all around the origin, except in the direction $\theta\approx\varphi$ where it gradually tends to the *sharp* radius $\delta(\theta-\varphi)$ as $r\to\infty$ [Fig. 6(c)]. This is the overall picture that will emerge in this paper: We will find that *pure* quantum "phase states," such as the SG states (3.6), have Wigner functions of this general form; they are the closest possible quantum analogs of the classical "phase eigenstates" $\delta_\varphi(q,p)$.

VII. ORDERING RULES AND QUANTIZATION OF CLASSICAL OBSERVABLES

A third line of attack on the "quantum phase" [besides using number shift operators and phase-space distributions] consists in quantizing the classical phase $\tan^{-1}(p/q)$ by means of *ordering rules* for the noncommuting operators $\hat{q}$ and $\hat{p}$. We recall in this section some basic facts concerning ordering rules (complementary details are given in Appendixes A and B).

Given a (real) classical observable $g(q,p)$, the "natural" way to construct a quantum observable $\hat{g}$, reducing to $g(q,p)$ in the classical limit $\hbar\to 0$, $\hat{q}\to q$, $\hat{p}\to p$, is as [8]

$$\hat{g}=g(\hat{q},\hat{p})_\Omega=\int dq\ dp\ g(q,p)\hat{\Delta}_\Omega(q,p), \tag{7.1}$$

where Ω specifies an *ordering rule* for $\hat{q}$ and $\hat{p}$. The integral is over the whole phase plane, and $\hat{\Delta}_\Omega(q,p)$ are Ω *ordered δ operators*:

$$\hat{\Delta}_\Omega(q,p)=[\delta(\hat{q}-q)\delta(\hat{p}-p)]_\Omega=\hat{D}_{qp}\hat{\Delta}_\Omega\hat{D}_{qp}^{-1}, \tag{7.2a}$$

$$\hat{\Delta}_\Omega=[\delta(\hat{q})\delta(\hat{p})]_\Omega,\quad \hat{D}_{qp}=e^{ip\hat{q}-iq\hat{p}}, \tag{7.2b}$$

where $\hat{D}_{qp}$ are phase-space displacement operators. An ordering rule Ω is completely specified by its "fiducial" operator $\hat{\Delta}_\Omega$. The "δ property" [see (A.11)]

$$\int dq\ dp\ \hat{\Delta}_\Omega(q,p)=(2\pi)\text{Tr}\{\hat{\Delta}_\Omega\}=\hat{1} \tag{7.3}$$

is the only essential property required of $\hat{\Delta}_\Omega$, so that *any* operator $\hat{\Delta}_\Omega$ trace normalized to $(2\pi)^{-1}$ can be used to define a formal ordering rule Ω via (7.1) and (7.2) [27,28].

One usually chooses $\hat{\Delta}_\Omega$ Hermitian, to assure that $g(\hat{q},\hat{p})_\Omega$ also is. Also, although (7.1) obviously goes into $g(q,p)$ as $\hbar\to 0$, one must still select (since $\hbar$ is finite after all) the ordering rule such that one gets the proper results in the high excitation correspondence limit. This still leaves, in general, many possibilities, so that there are in fact many different operators $g(\hat{q},\hat{p})_\Omega$ corresponding to $g(q,p)$. Other (more or less subjective) criteria must be invoked to make a "best" choice.

If $g(q,p)=g(r,\theta)=g(\theta)$ is independent of r, then

$$g(\hat{q},\hat{p})_\Omega=\int_0^{2\pi}d\theta\, g(\theta)\hat{\Gamma}_\Omega(\theta), \tag{7.4a}$$

where we define Ω *ordered angular* δ *operators*

$$\hat{\Gamma}_\Omega(\varphi)=[\delta(\tan^{-1}(\hat{p}/\hat{q})-\varphi)]_\Omega=\int_0^\infty r\, dr\, \hat{\Delta}_\Omega(r,\varphi). \tag{7.4b}$$

The "canonical" ordering is, for a number of reasons, Weyl ordering [8,28]. The Weyl ordered δ operator $\hat{\Delta}_w$ is the parity operator [29]:

$$\hat{\Delta}_w=[\delta(\hat{q})\delta(\hat{p})]_w=\pi^{-1}\hat{\Pi},\quad \hat{\Pi}=\int dq|-q\rangle\langle q|. \tag{7.5}$$

Weyl ordering and Weyl symbols (5.13) are dual notions, in the sense that

$$A_w(q,p)=2\pi\ \mathrm{Tr}\{\hat{\Delta}_w(q,p)\hat{A}\}, \tag{7.6a}$$

$$\hat{A}=[A_w(\hat{q},\hat{p})]_w=\int dq\, dp\, A_w(q,p)\hat{\Delta}_w(q,p). \tag{7.6b}$$

One reason Weyl ordering is basic is that $\hat{\Delta}_w$ is the "sharpest" possible, since

$$\langle q|\hat{\Delta}_w|q\rangle=(2\pi)^{-1}\delta(q),\quad \langle p|\hat{\Delta}_w|p\rangle=(2\pi)^{-1}\delta(p). \tag{7.7}$$

Any ordering rule Ω, for which $\hat{\Delta}_\Omega=[\delta(\hat{q})\delta(\hat{p})]_\Omega$ or its Weyl symbol

$$G_\Omega(q,p)=[\hat{\Delta}_\Omega]_w(q,p),\qquad \int dq\, dp\, G_\Omega(q,p)=1 \tag{7.8}$$

are *well behaved*, is a "coarse graining" of Weyl ordering in the sense that

$$\hat{A}=g(\hat{q},\hat{p})_\Omega$$

$$\Leftrightarrow A_w(q,p)=\int dq'dp'g(q',p')G_\Omega(q-q',p-p'), \tag{7.9a}$$

that is, the Weyl symbol of $g(\hat{q},\hat{p})_\Omega$ is the smearing (convolution) of $g(q,p)$ with the (well behaved) function $G_\Omega(q,p)$[equal to $\delta(q)\delta(p)$ if $\Omega=w$]. Taking the Fourier transform of (7.9a), we get

$$\tilde{G}_\Omega(k,s)=\frac{\tilde{A}_w(k,s)}{\tilde{g}(k,s)},\quad \tilde{G}_\Omega(0,0)=1, \tag{7.9b}$$

where we denote, for any function $f(q,p)$,

$$\tilde{f}(k,s)\equiv\int dq\, dp\, e^{ikq-isp}f(q,p). \tag{7.10}$$

It follows that for $\hat{A}$ to be expressible as $\tilde{A}=g(\hat{q},\hat{p})_\Omega$, for *some* Ω, it suffices that

$$\int dq\, dp\, A_w(q,p)=\int dq\, dp\, g(q,p). \tag{7.11}$$

The resulting $G_\Omega(q,p)$ [given by (7.9b)] may be more or less singular, depending on $\hat{A}$ and $g(q,p)$. An important application of (7.11), for us, is to the case that $A_w(r,\theta)\to g(\theta)$ gets independent of r as $r\to\infty$:

$$[\hat{A}]_w(r,\theta)\to g(\theta)\quad (r\to\infty) \tag{7.12a}$$

implies

$$\hat{A}=g(\tan^{-1}(\hat{p}/\hat{q}))_\Omega, \tag{7.12b}$$

for some Ω, since $e^{-\varepsilon r}A_w(r,\theta)$ and $e^{-\varepsilon r}g(\theta)$ have identical integrals over phase space, as $\varepsilon\to 0$. Reciprocally, (7.12b) satisfies (7.12a) if $G_\Omega(q,p)$ is localized about the origin: Indeed, by (7.9), the Weyl symbol of (7.12b) is the smearing of $g(\theta)$ with $G_\Omega(q,p)$, and a "local" smearing does not affect angular asymptotics, because the *angular* width of $G_\Omega(q-q',p-p')$ vanishes as $r'=(q'^2+p'^2)^{1/2}\to\infty$.

Covariant orderings: If $\hat{\Delta}_\Omega$ satisfies the equivalent conditions

$$[\hat{H},\hat{\Delta}_\Omega]=0\Leftrightarrow i[\hat{H},\hat{\Delta}_\Omega(r,\theta)]=\frac{\partial}{\partial\theta}\hat{\Delta}_\Omega(r,\theta) \tag{7.13a}$$

$$\Leftrightarrow\hat{\Delta}_\Omega(r,\theta)=e^{i\theta\hat{H}}\hat{\Delta}_\Omega(r,0)e^{-i\theta\hat{H}} \tag{7.13b}$$

then we say that the ordering rule Ω is *covariant* for $\hat{H}$. In that case, $\hat{\Delta}_\Omega(r,\theta)$ is generated from $\hat{\Delta}_\Omega(r,0)$ by $e^{i\theta\hat{H}}$. Also, $G_\Omega(r,\theta)=[\hat{\Delta}_\Omega]_w(r,\theta)$ is independent of θ, and the Weyl symbol of $\hat{\Delta}_\Omega(r,\theta)$ is just that of $\hat{\Delta}_\Omega(r,0)$ rotated by an angle θ.

An important family of covariant orderings was introduced by Cahill and Glauber [30]: It is parameterized by a continuous real number s, with $s=-1$, $s=0$, and $s=1$ yielding normal, Weyl, and antinormal orderings (note that our s is equal to minus the s of Cahill-Glauber). The Weyl symbol of $\hat{\Delta}_s=[\delta(\hat{q})\delta(\hat{p})]_s$ for $s\geqslant 0$ is the circular Gaussian

$$G_s(q,p)=[\hat{\Delta}_s]_w(q,p)=\frac{e^{-(q^2+p^2)/s}}{\pi s}\quad (s\geqslant 0). \tag{7.14}$$

The operators $\hat{\Delta}_s$ are thermal oscillator states for $s\geqslant 1$, and that multiplied by the parity operator for $0\leqslant s\leqslant 1$. At $s=1$ (antinormal ordering), the "temperature" $\to 0$, and one gets

$$\hat{\Delta}_{s=1}(q,p)=(2\pi)^{-1}|qp\rangle^{\mathrm{coh}}\langle qp|,\quad |qp\rangle^{\mathrm{coh}}=\hat{D}_{qp}|0\rangle, \tag{7.15}$$

where $|0\rangle$ is the ground state of $\hat{H}$, and $|qp\rangle^{\mathrm{coh}}$ are coherent states. While many ordering rules, for arbitrary $\hat{\Delta}_\Omega$, seem rather artificial, Cahill-Glauber orderings are "natural," as explained in Appendix B.

Ω *phase-space distributions*. An ordering rule Ω naturally associates with any quantum state $\hat{\rho}$ a phase-space distribution

$$f_\Omega(q,p)=\mathrm{Tr}\{\hat{\Delta}_\Omega(q,p)\hat{\rho}\}=\mathrm{Tr}\{[\delta(\hat{q}-q)\delta(\hat{p}-p)]_\Omega\hat{\rho}\}, \tag{7.16a}$$

$$\int dq\ dp\ f_\Omega(q,p)=\mathrm{Tr}\{\hat{\rho}\}. \tag{7.16b}$$

This can be expressed in the form (5.22), if we let $G(q,p)$ be the Weyl symbol of $\hat{\Delta}_\Omega$.

Quantization of the classical phase. The natural quantizations of the classical phase $\tan^{-1}(p/q)$ are Ω *ordered phase operators*

$$\hat{\phi}_\Omega=[\tan^{-1}(\hat{p}/\hat{q})]_\Omega$$
$$=\int dq\ dp\ \tan^{-1}(p/q)[\delta(\hat{q}-q)\delta(\hat{p}-p)]_\Omega. \tag{7.17}$$

The Weyl ordered phase operator $\hat{\phi}_w$ was first introduced, and studied extensively with mathematical rigour, by Smith, Dubin, and Hennings [9]. We shall quote some of their results later on. More general ordering rules were considered independently in [10]. The phase operators we shall define will all be expressible as (7.17), for some Ω.

This completes our overview of some of the existing approaches to the quantum phase. It allowed us to set down a number of results that we shall need, and announce some general features to come out of our own analysis, which we now begin.

VIII. PHASE CORRESPONDENCE

A state operator $\hat{\rho}_{q'p'}=\hat{\rho}_{r'\theta'}$ will be said to be *localized* about $(q',p')=(r',\theta')$ if its position and momentum expectations, $\langle q|\hat{\rho}_{r'\theta'}|q\rangle$ and $\langle p|\hat{\rho}_{r'\theta'}|p\rangle$, are localized within finite intervals Δq and Δp about $q'=r'\cos\theta'$ and $p'=r'\sin\theta'$, respectively. It may be represented by a box centered at (q',p') in the (q,p) phase plane (Fig. 1). As the amplitude $r'\to\infty$, the angle subtended by the box tends to zero, and the phase becomes sharply defined as θ': So the state becomes "classical" insofar as phase is concerned. Such *large amplitude localized states* are the *only* quantum states to which a classical phase can be assigned unambiguously. Accordingly, an operator $\hat{A}$ will be said to *correspond* to the (classical) phase function $A(\theta)$ if its expectation with any such "classical phase" state is $A(\theta')$, that is, if

$$\mathrm{Tr}\{\hat{A}\hat{\rho}_{r'\theta'}\}\to A(\theta') \quad \text{as} \quad r'\to\infty \quad (\text{for any localized } \hat{\rho}_{q'p'}). \tag{8.1}$$

We will now express this condition in terms of Weyl symbols.

In view of (5.15), we may rewrite (8.1) as

$$\int_0^\infty r\ dr\int_0^{2\pi}d\theta\ A_w(r,\theta)W_{r'\theta'}(r,\theta)\to A(\theta') \quad (r'\to\infty), \tag{8.2}$$

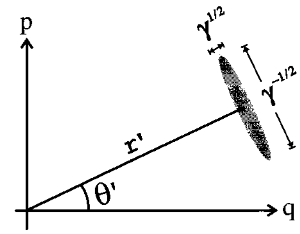

FIG. 7. The shaded ellipse represents an amplitude squeezed state centered at $(q',p')=(r',\theta')$.

where $W_{r'\theta'}(r,\theta)$ is the Wigner function of $\hat{\rho}_{r'\theta'}$. The most sharply localized quantum states are *squeezed states*, such as [see (B9)]

$$W^{sq}_{q'p'}(q,p)=\pi^{-1}e^{-\gamma(q-q')^2-(p-p')^2/\gamma}, \tag{8.3}$$

where γ is a real number (the squeeze parameter). This may be represented by an ellipse in the Wigner phase plane, with semiaxes parallel to the q and p axes. One may rotate the state such than its major axis is perpendicular to the radius r' (Fig. 7). One then has, for r' sufficiently large that the angle subtended by the ellipse is small,

$$W^{sq}_{r'\theta'}(r,\theta)\approx\pi^{-1}e^{-\gamma r'^2(\theta-\theta')^2-(r-r')^2/\gamma}. \tag{8.4}$$

Inserting (8.4) into (8.2), we obtain the requirement

$$(\gamma/\pi)^{1/2}\int_0^\infty r\ dr\ A_w(r,\theta')e^{-(r-r')^2/\gamma}\to A(\theta') \quad (r'\to\infty). \tag{8.5}$$

This must hold for any value of γ, in particular $\gamma\to 0$ [the angular width of (8.4) then increases, but still gets sharp as $r'\to\infty$]; as $\gamma\to 0$, (8.5) becomes

$$[\hat{A}]_w(r,\theta)\to A(\theta) \quad (r\to\infty). \tag{8.6}$$

Thus, (8.1) implies (8.6). Reciprocally, (8.6) implies (8.1) and (8.2), since it implies $A_w(r,\theta)W_{r'\theta'}(r,\theta)\to A(\theta')W_{r'\theta'}(r,\theta)$ [since $W_{r'\theta'}(r,\theta)$ is localized], and $\int r\ dr\ d\theta W(r,\theta)=1$. We thus conclude that (8.1) and (8.6) are equivalent "phase correspondence" principles.

We used, above, the fact that (8.1) implies $\langle r\theta|\hat{A}|r\theta\rangle\to A(\theta)$ for any squeezed state $|r\theta\rangle$. More generally, (8.1)–(8.6) imply that for any two squeezed states $|r_1\theta_1\rangle$ and $|r_2\theta_2\rangle$:

$$\langle r_1\theta_1|\hat{A}|r_2\theta_2\rangle=(2\pi)^{-1}\int_0^\infty r\ dr\int_0^{2\pi}d\theta\ A_w(r,\theta)\times\big[|r_1\theta_1\rangle\langle r_2\theta_2|\big]_w(r,\theta) \tag{8.7a}$$

$$\to A(\theta_1)\langle r_1\theta_1|r_2\theta_2\rangle \quad (r_1\to\infty). \tag{8.7b}$$

This is because $I_{12}(r,\theta)\equiv[|r_1\theta_1\rangle\langle r_2\theta_2|]_w(r,\theta)$, localized about the midpoint $u_m=(r_m,\theta_m)$ between the two states [see (5.14)], oscillates with a wavelength $|u_1-u_2|^{-1}$. The integral (8.7a) is thus sizable only if $|u_2-u_1|$ is not large, so that the points (r_1,θ_1) and (r_2,θ_2), together with the support of $I_{12}(r,\theta)$, will subtend a vanishing angle as $r_1\to\infty$: so we can approximate $A_w(r,\theta)\approx A(\theta_1)$, whence (8.7b).

In general the Weyl symbol of a product of operators is not equal to the product of their Weyl symbols. However, if $A_w(r,\theta)$ and $B_w(r,\theta)$ get independent of r as $r\to\infty$, then we have:

$$\left.\begin{array}{l}A_w(r,\theta)\to A(\theta)\\ B_w(r,\theta)\to B(\theta)\end{array}\right\}\Rightarrow[\hat{A}\hat{B}]_w(r,\theta)\to A(\theta)B(\theta)\quad(r\to\infty)\tag{8.8a}$$

whence, also,

$$[\hat{A}]_w(r,\theta)\to A(\theta)\Rightarrow[f(\hat{A})]_w(r,\theta)\to f(A(\theta))\quad(r\to\infty).\tag{8.8b}$$

To show (8.8), we use the completeness of squeezed states, and (8.7): We get, at large r,

$$\begin{aligned}\langle r\theta|\hat{A}\hat{B}|r\theta\rangle&=\frac{1}{2\pi}\int_0^\infty r'dr'\int_0^{2\pi}d\theta'\langle r\theta|\hat{A}|r'\theta'\rangle\langle r'\theta'|\hat{B}|r\theta\rangle\\&\approx\frac{1}{2\pi}A(\theta)B(\theta)\int_0^\infty r'dr'\int_0^{2\pi}d\theta'\langle r\theta|r'\theta'\rangle\langle r'\theta'|r\theta\rangle\\&=A(\theta)B(\theta).\end{aligned}\tag{8.9}$$

Since this holds for any squeezed state, there follows (8.8). The latter can also be shown in another, less intuitively clear manner, as follows.

The Weyl symbol of a product of operators, $\hat{A}\hat{B}$, is given in terms of the Weyl symbols of each operator by the Moyal formula [31]

$$[\hat{A}\hat{B}]_w(q,p)=e^{i\Lambda(\partial,\partial')}A_w(q',p')B_w(q,p)|_{q'=q,p'=p}\tag{8.10a}$$

$$\begin{aligned}\Lambda(\partial,\partial')&=\frac{\partial}{\partial q}\frac{\partial}{\partial p'}-\frac{\partial}{\partial p}\frac{\partial}{\partial q'}\\&=\sin(\theta'-\theta)\left[\frac{1}{2}\frac{\partial^2}{\partial r\partial r'}+\frac{1}{rr'}\frac{\partial^2}{\partial\theta\partial\theta'}\right]\\&\quad+\cos(\theta'-\theta)\left[\frac{1}{r'}\frac{\partial^2}{\partial r\partial\theta'}-\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r'}\right],\end{aligned}\tag{8.10b}$$

where $r=(q^2+p^2)^{1/2}$. Note that $\Lambda(\partial,\partial')$ reduces powers of r by two units. Thus, if $A_w(r,\theta)\to A(\theta)$ and $B_w(r,\theta)\to B(\theta)$ as $r\to\infty$, then $[\hat{A}\hat{B}]_w(r,\theta)\to A(\theta)B(\theta)+r^{-4}F(r,\theta)$, where $F(r,\theta)$ is a series in inverse powers of r^2, whence (8.8).

If $\hat{A}$ or $\hat{B}$ in (8.10a) is quadratic in $\hat{q}$ and $\hat{p}$, then $\Lambda(\partial,\partial')^nA_w(q,p)B_w(q'p')=0$ for $n\geqslant3$, and

$$-i[\hat{A},\hat{B}]_w(q,p)=\{A_w,B_w\}_{\text{PB}}(q,p)\quad(\hat{A}\text{ or }\hat{B}\text{ quadratic})\tag{8.11}$$

realizing the Dirac correspondence (2.4). In particular, since $\{H,.\}_{\text{PB}}=\partial/\partial\theta$, by (2.2), we have, for any operator $\hat{A}$,

$$-i[\hat{H},\hat{A}]_w(r,\theta)=\frac{\partial}{\partial\theta}A_w(r,\theta),\tag{8.12a}$$

$$[e^{-it\hat{H}}\hat{A}e^{it\hat{H}}]_w(r,\theta)=A_w(r,\theta+t),\tag{8.12b}$$

that is, $e^{-it\hat{H}}$ causes a Weyl symbol to rotate clockwise, i.e., to *evolve classically*.

IX. PHASE δ OPERATORS

An operator $\hat{\Gamma}(\varphi)$ will be called a *phase δ operator* if it *corresponds* to the phase δ function $\delta_{2\pi}(\theta-\varphi)$[defined in (2.9)], that is, if it satisfies the equivalent conditions

$$\text{Tr}\{\hat{\Gamma}(\varphi)\hat{\rho}_{r\theta}\}\to\delta_{2\pi}(\theta-\varphi)\quad(r\to\infty)\tag{9.1a}$$

$$[\hat{\Gamma}(\varphi)]_w(r,\theta)\to\delta_{2\pi}(\theta-\varphi)\quad(r\to\infty)\tag{9.1b}$$

for any localized state $\hat{\rho}_{r\theta}$. Obviously, one may concoct any function $\Gamma(r,\theta;\varphi)\to\delta_{2\pi}(\theta-\varphi)$ as $r\to\infty$, and build a phase δ operator with Weyl symbol $\Gamma(r,\theta;\varphi)$ as

$$\hat{\Gamma}(\varphi)=\int_0^\infty r\,dr\int_0^{2\pi}d\theta\,\Gamma(r,\theta;\varphi)\hat{\Delta}_w(r,\theta).\tag{9.2}$$

Putting in particular $\Gamma(r,\theta;\varphi)=\delta_{2\pi}(\theta-\varphi)$, we get

$$\hat{\Gamma}_w(\varphi)=\int_0^\infty r\,dr\,\hat{\Delta}_w(r,\varphi),\quad[\hat{\Gamma}_w(\varphi)]_w(r,\theta)=\delta_{2\pi}(\theta-\varphi)\tag{9.3}$$

whose Weyl symbol is (by definition) $\delta_{2\pi}(\theta-\varphi)$ at all values of r. It is a special example of Ω ordered phase δ operator [see (7.4b)]

$$\hat{\Gamma}_\Omega(\varphi)=[\delta(\tan^{-1}(\hat{p}/\hat{q})-\varphi)]_\Omega=\int_0^\infty r\,dr\,\hat{\Delta}_\Omega(r,\varphi).\tag{9.4}$$

This satisfies (9.1) provided the Weyl symbol $G_\Omega(q,p)=[\hat{\Delta}_\Omega]_w(q,p)$ is localized about the origin [see after Eq. (7.12)]: By (7.9a) [with $g(r,\theta)=\delta(\theta-\varphi)$], the Weyl symbol $[\hat{\Gamma}_\Omega(\varphi)]_w(r,\theta)$ is then a "local" smearing of the sharp radius $\delta(\theta-\varphi)$, which thus acquires a *finite* width [as in Fig. 6(b)], but still gets infinitely sharp in *angular* width as $r\to\infty$. In view of (7.12), *any* $\hat{\Gamma}(\varphi)$ can, in fact, be expressed as

$$\hat{\Gamma}(\varphi)=\hat{\Gamma}_{\Omega(\varphi)}(\varphi)\tag{9.5}$$

for *some* ordering rule $\Omega(\varphi)$ (which may depend on φ). Putting $\varphi=0$, and noting that the Fourier transform of $\delta(\theta)=q\,\delta(p)\Theta(q)$ is $\tilde{\delta}(k,s)=-(k+i0)^{-2}$, we see that $G_{\Omega(0)}(q,p)$, equal to the Fourier inverse of $-k^2\tilde{\Gamma}(k,s;0)$, is at worst a tempered distribution [32]; and likewise for any value of φ.

We say that $\hat{\Gamma}(\varphi)$ is *φ normalized* if

$$\int_0^{2\pi}d\varphi\hat{\Gamma}(\varphi)=\hat{1}.\tag{9.6}$$

It is *covariant* if it satisfies the following equivalent conditions:

$$i[\hat{H},\hat{\Gamma}(\varphi)]=\frac{\partial}{\partial\varphi}\hat{\Gamma}(\varphi),\quad \hat{\Gamma}(\varphi)=e^{i\varphi\hat{H}}\hat{\Gamma}e^{-i\varphi\hat{H}}, \tag{9.7a}$$

$$[\hat{\Gamma}(\varphi)]_w(r,\theta)=[\hat{\Gamma}]_w(r,\theta-\varphi), \tag{9.7b}$$

where $\hat{\Gamma}\equiv\hat{\Gamma}(0)$, and (9.7b) follows from (9.7a) by (8.12). Thus, a covariant $\hat{\Gamma}(\varphi)$ is generated from $\hat{\Gamma}(0)$ by $e^{i\varphi\hat{H}}$, and its Weyl symbol is just that of $\hat{\Gamma}(0)$ rotated by φ. Using $e^{\pm 2\pi i\hat{H}}=e^{\pm\pi i}$ (since the eigenvalues of $\hat{H}$ are $n+\frac{1}{2}$), we deduce from (9.7a) that

$$e^{-it\hat{H}}\hat{\Gamma}(\varphi)e^{it\hat{H}}=\hat{\Gamma}(\varphi-t\ \mathrm{mod}2\pi)\equiv\hat{\Gamma}(\varphi-t). \tag{9.8}$$

In view of (9.1b) and (8.12), any $\hat{\Gamma}(\varphi)$ is *asymptotically covariant*, in the sense that

$$[e^{-it\hat{H}}\hat{\Gamma}(\varphi)e^{it\hat{H}}]_w(r,\theta)\rightarrow\delta(\theta+t-\varphi)\quad(r\rightarrow\infty). \tag{9.9}$$

The Ω ordered δ operator (9.4) is always φ normalized, since $\int r\ dr\ d\varphi\ \hat{\Delta}_\Omega(r,\varphi)=\hat{1}$ [see (7.3)]; it is covariant if $\hat{\Delta}_\Omega$ is covariant (7.13). So the Cahill-Glauber s-ordered phase δ operators

$$\hat{\Gamma}_s(\varphi)=[\delta(\tan(\hat{p}/\hat{q})-\varphi)]_s=\int_0^\infty r\ dr\ \hat{\Delta}_s(r,\varphi) \tag{9.10}$$

are φ normalized and covariant. The Weyl symbol of $\hat{\Gamma}_s(\varphi)$ is the convolution of the sharp radius $\delta(\theta-\varphi)$ with the Gaussian (7.14) of width s, hence it is a radius of width $\approx s$, similar to that shown in Fig. 6(b). Examples of noncovariant $\hat{\Gamma}(\varphi)$ are

$$\hat{\Gamma}_{s,\gamma}(\varphi)=[\delta(\tan(\hat{p}/\hat{q})-\varphi)]_{s,\gamma}=\int_0^\infty r\ dr\ \hat{\Delta}_{s\gamma}(r,\varphi), \tag{9.11}$$

where (s,γ) are "squeezed s ordering," defined in (B19), and $\hat{\Delta}_{s,\gamma}$ are squeezed thermal states.

X. PHASE STATES AND PHASE POM'S

If $\hat{\Gamma}(\varphi)\geqslant 0$ is non-negative, we call it a *phase state*. This is in general un-normalizable, similarly to position eigenkets $|q'\rangle\langle q'|=\delta(\hat{q}-q')$ [whose Weyl symbols are $\delta(q-q')$]. Even though such "states" can never be realized physically (which would require infinite energies), but can only be approached arbitrarily, it is customary to call them "states" anyway. They may be viewed as the $\varepsilon\rightarrow 0$ limit of normalized *approximate* phase states $\hat{\Gamma}^\varepsilon(\varphi)$, defined by [we here invoke (5.16)]

$$\hat{\Gamma}^\varepsilon(\varphi)=\frac{\hat{\Gamma}^{\varepsilon}{}'(\varphi)}{\mathrm{Tr}\{\hat{\Gamma}^{\varepsilon}{}'(\varphi)\}},\quad \mathrm{Tr}\{\hat{\Gamma}^\varepsilon(\varphi)\}=1, \tag{10.1a}$$

$$[\hat{\Gamma}^{\varepsilon}{}'(\varphi)]_w(r,\theta)=2\pi\varepsilon^2e^{-\varepsilon r}[\hat{\Gamma}(\varphi)]_w(r,\theta), \tag{10.1b}$$

where we defined $\hat{\Gamma}^{\varepsilon}{}'(\varphi)$ such that $\mathrm{Tr}\{\hat{\Gamma}^{\varepsilon}{}'(\varphi)\}\rightarrow 1$ as $\varepsilon\rightarrow 0$ (see Appendix H).

If $\hat{\Gamma}(\varphi)$ has the defining properties of a POM (probability operator measure) [26], namely, it is *non-negative* and φ *normalized*,

$$\hat{\Gamma}(\varphi)\geqslant 0,\quad \int_0^{2\pi}d\varphi\ \hat{\Gamma}(\varphi)=\hat{1}, \tag{10.2}$$

then we call it a *phase POM*. For instance, $\hat{\Gamma}_\Omega(\varphi)$ in (9.4) is always φ normalized, by (7.3), and is non-negative if $\hat{\Delta}_\Omega\geqslant 0$. Thus, the Cahill-Glauber $\hat{\Gamma}_s(\varphi)$, for $s\geqslant 1$, are phase POM's.

A phase state $\hat{\Gamma}(\varphi)$ of the form

$$\hat{\Gamma}(\varphi)=|\varphi\rangle\langle\varphi| \tag{10.3}$$

is called *projective*, or *pure*. Again, a pure phase state $|\varphi\rangle$ should be viewed as the limit of a normalizable *approximate* phase state. With $\hat{\Gamma}^\varepsilon(\varphi)$ defined as in (10.1), we have (see Appendix H)

$$\hat{\Gamma}^\varepsilon(\varphi)=|\varphi\rangle^\varepsilon\langle\varphi|,\quad |\varphi\rangle^\varepsilon=\frac{|\varphi\rangle^{\varepsilon'}}{|{}'^{\varepsilon}\langle\varphi|\varphi\rangle^{\varepsilon'}|^{1/2}}, \tag{10.4a}$$

$$\langle q|\varphi\rangle^{\varepsilon'}=(2\pi)^{1/2}\varepsilon e^{-(1/2)\varepsilon q/\cos\varphi}\langle q|\varphi\rangle. \tag{10.4b}$$

An important covariant and pure phase POM is the SG-POM

$$\hat{\Gamma}_{\mathrm{SG}}(\varphi)=|\varphi\rangle_{\mathrm{SG}}\langle\varphi|,\quad |\varphi\rangle_{\mathrm{SG}}=(2\pi)^{-1/2}\sum_{n=0}^\infty e^{in\varphi}|n\rangle, \tag{10.5a}$$

where $|\varphi\rangle_{\mathrm{SG}}$ are the Susskind-Glogower states (3.6). That $\hat{\Gamma}_{\mathrm{SG}}(\varphi)$ indeed satisfies (9.1b) is shown by Herzog, Paul, and Richter [33] [see also (E12) and (E13)]. They show that asymptotically

$$[\hat{\Gamma}_{\mathrm{SG}}(\varphi)]_w(r,\theta)\approx 2r^2\left\{\frac{J_1(4r^2(\theta-\varphi))}{4r^2(\theta-\varphi)}\right\}\rightarrow\delta(\theta-\varphi)\quad(r\rightarrow\infty), \tag{10.5b}$$

where J_1 is a Bessel function. Since the curly bracket { } is a function of $r^2(\theta-\varphi)$, the Weyl symbol (10.5b) has an angular width $\Delta\theta\sim r^{-2}$, whence we get a true width $r\Delta\theta\sim r^{-1}$. Its "height" $\sim r^2$. Thus, the Weyl symbol of $\hat{\Gamma}_{\mathrm{SG}}(\varphi)$ in fact tends to a *sharp* radius of zero true width (not just zero *angular* width) as $r\rightarrow\infty$, hence is of the form that we had surmised in (6.6) and Fig. 6(c).

XI. PHASE OPERATORS

Since definite phases can be assigned only to *large amplitude localized states*, we demand of a (Hermitian) phase operator $\hat{\phi}$ only that the phase distribution

$$P(\varphi)=\mathrm{Tr}\{\delta(\hat{\phi}-\varphi)\hat{\rho}\} \tag{11.1}$$

attribute the correct sharp phase θ to any state $\hat{\rho}_{r\theta}$ localized about (r,θ), as $r\to\infty$: Hence, we must have, in view of (8.1)–(8.6),

$$\mathrm{Tr}\{\delta(\hat{\phi}-\varphi)\hat{\rho}_{r\theta}\}\to\delta_{2\pi}(\varphi-\theta) \quad (r\to\infty) \tag{11.2a}$$

$$[\delta(\hat{\phi}-\varphi)]_w(r,\theta)\to\delta_{2\pi}(\theta-\varphi), \tag{11.2b}$$

that is, $\delta(\hat{\phi}-\varphi)$ must be a phase δ operator, as defined in (9.1). It follows from (11.2), and the trivial identify $\hat{\phi}=\int_0^{2\pi}\varphi\, d\varphi\;\delta(\hat{\phi}-\varphi)$ that

$$[\hat{\phi}]_w(r,\theta)\to\int_0^{2\pi}\varphi\, d\varphi\;\delta_{2\pi}(\theta-\varphi)=\theta \bmod 2\pi \quad (r\to\infty). \tag{11.3}$$

Thus, the Weyl symbol of a phase operator tends to the *single-valued* classical phase

$$\phi_{\mathrm{cl}}=\phi_{\mathrm{cl}}(r,\theta)=\theta \bmod 2\pi \tag{11.4}$$

as $r\to\infty$, that is, $\hat{\phi}$ *corresponds* to $\phi_{\mathrm{cl}}(r,\theta)$. Reciprocally, (11.3) implies (11.2) by (8.8): More precisely, we expect from (8.8) and (11.3) that $[\delta(\hat{\phi}-\varphi)]_w(r,\theta)$ tends to a sharp radius of zero *true* width [as in the case of $\hat{\Gamma}_w(\varphi)$ and $\hat{\Gamma}_{\mathrm{SG}}(\varphi)$], not just zero *angular* width [as in the case of $\hat{\Gamma}_s(\varphi), s>0$].

One may concoct any function $\phi_w(r,\theta)\to\theta \bmod 2\pi$ as $r\to\infty$, and build the phase operator

$$\hat{\phi}=\int_0^{\infty}r\, dr\int_0^{2\pi}\theta\, d\theta\;\phi_w(r,\theta)\hat{\Delta}_w(r,\theta). \tag{11.5}$$

In view of (7.12), $\hat{\phi}=\hat{\phi}_\Omega$ for *some* ordering rule Ω, where we define the Ω *ordered phase operator*

$$\hat{\phi}_\Omega=[\tan^{-1}(\hat{p}/\hat{q})]_\Omega=\int_0^{\infty}r\, dr\int_0^{2\pi}\theta\, d\theta\;\hat{\Delta}_\Omega(r,\theta). \tag{11.6}$$

Since $\phi_{\mathrm{cl}}(q,p)$ has discontinuities in both q and p, its Fourier transform $\tilde{\phi}_{\mathrm{cl}}(k,s)$ vanishes slowly at infinity (see, e.g., [32]), so that $G_\Omega(q,p)$ is, at worst, a tempered distribution. Reciprocally, $\hat{\phi}_\Omega$ satisfies (11.3) if $G_\Omega(q,p)$ is localized about the origin [by (7.9a), the sharp jump of $\phi_{\mathrm{cl}}(r,\theta)$ at $\theta=2\pi$ gets smoothed over the width of $G_\Omega(q,p)$, hence over an asymptotically vanishing *angular* width].

Any phase state $\hat{\Gamma}(\varphi)$ is an approximate eigenstate of any phase operator $\hat{\phi}$, in the sense that

$$\langle f(\hat{\phi})\rangle_{\Gamma(\varphi)}\equiv\mathrm{Lim}_{\varepsilon\to0}\;\mathrm{Tr}\{f(\hat{\phi})\hat{\Gamma}^{\varepsilon}(\varphi)\}=f(\varphi) \tag{11.7}$$

so that, in particular, $\langle\hat{\phi}\rangle_{\Gamma(\varphi)}=\varphi$, and $\langle(\hat{\phi}-\langle\hat{\phi}\rangle)^2\rangle_{\Gamma(\varphi)}=0$. Equation (11.7) follows from

$$\begin{aligned}\mathrm{Tr}\{f(\hat{\phi})\hat{\Gamma}^{\varepsilon}(\varphi)\}&\\ &=\frac{1}{2\pi}\int_0^{\infty}r\, dr\int_0^{2\pi}d\theta[f(\hat{\phi})]_w(r,\theta)[\hat{\Gamma}^{\varepsilon}(\varphi)]_w(r,\theta)\\ &\to\varepsilon^2\int_0^{\infty}r\, dr\int_0^{2\pi}d\theta\; f(\theta)e^{-\varepsilon r}\delta(\theta-\varphi)=f(\varphi),\end{aligned} \tag{11.8}$$

where we used the fact that as $\varepsilon\to0$, the integral is dominated by large r, at which $f(\hat{\phi})_w(r,\theta)\approx f(\theta)$, by (11.3) and (8.8), and $\hat{\Gamma}^{\varepsilon}(\varphi)_w(r,\theta)\to2\pi\varepsilon^2e^{-\varepsilon r}\delta(\theta-\varphi)$[see (10.1)]. Similar results for the specific case of $\hat{\Gamma}_{\mathrm{SG}}(\varphi)$ and $\hat{\phi}_w$ were obtained by Dubin, Hennings, and Smith [9(c)].

The true eigenstates (eigen POM, strictly speaking) of $\hat{\phi}$ are the pure phase states

$$\delta(\hat{\phi}-\varphi)=|\varphi\rangle\langle\varphi|, \quad \hat{\phi}|\varphi\rangle=\varphi|\varphi\rangle \tag{11.9}$$

No phase eigenstates $|\varphi\rangle$, for any phase operator, have been explicitly calculated yet. Only partial results concerning the spectrum of the Weyl ordered $\hat{\phi}_w$ have been obtained, by Dubin, Hennings, and Smith [9b].

Any phase δ operator $\hat{\Gamma}(\varphi)$ induces a phase operator

$$\hat{\phi}_\Gamma=\int_0^{2\pi}\varphi\, d\varphi\;\hat{\Gamma}(\varphi). \tag{11.10}$$

Reciprocally, *any* phase operator can be expressed in the form (11.10), trivially as

$$\hat{\phi}=\int_0^{2\pi}\varphi\, d\varphi\;\delta(\hat{\phi}-\varphi)=\int_0^{2\pi}d\varphi|\varphi\rangle\varphi\langle\varphi| \tag{11.11}$$

or, in view of (11.6) and (9.4), as

$$\hat{\phi}=\hat{\phi}_\Omega=\int_0^{2\pi}\varphi\, d\varphi\;\hat{\Gamma}_\Omega(\varphi). \tag{11.12}$$

In general, $f(\hat{q},\hat{p})_\Omega g(\hat{q},\hat{p})_\Omega\neq[f(\hat{q},\hat{p})g(\hat{q},\hat{p})]_\Omega$, so that

$$\delta(\hat{\phi}_\Omega-\varphi)=\delta([\tan^{-1}(\hat{p}/\hat{q})]_\Omega-\varphi) \tag{11.13a}$$

$$\neq\hat{\Gamma}_\Omega(\varphi)=[\delta(\tan^{-1}(\hat{p}/\hat{q})-\varphi)]_\Omega. \tag{11.13b}$$

We note that

$$\int_0^{2\pi}d\varphi\; f(\varphi)\delta(\hat{\phi}_\Omega-\varphi)=f(\hat{\phi}_\Omega)=f([\tan^{-1}(\hat{p}/\hat{q})]_\Omega), \tag{11.14a}$$

$$\int_0^{2\pi}d\varphi\; f(\varphi)\hat{\Gamma}_\Omega(\varphi)=f(\tan^{-1}(\hat{p}/\hat{q}))_\Omega. \tag{11.14b}$$

These two integrals are equal if and only if $f(\varphi)=a+b\varphi$. For instance, $\int_0^{2\pi}\varphi\, d\varphi\;\delta(\hat{\phi}_\Omega-\varphi)=(11.12)$.

A phase POM satisfying, like $\delta(\hat{\phi}-\varphi)=|\varphi\rangle\langle\varphi|$,

$$\hat{\Gamma}(\varphi)\hat{\Gamma}(\varphi')=\delta(\varphi-\varphi')\hat{\Gamma}(\varphi), \tag{11.15a}$$

will be called *pure orthogonal* [to distinguish this kind of orthogonality from another, trace orthogonality, to be introduced in Sec. XVI]. We note that if $\hat{\phi}_\Gamma=\int_0^{2\pi}\varphi\, d\varphi\;\hat{\Gamma}(\varphi)$, then (11.15a) implies

$$\hat{\phi}_\Gamma\hat{\Gamma}(\varphi)=\varphi\hat{\Gamma}(\varphi), \quad \hat{\Gamma}(\varphi)\hat{\phi}_\Gamma=\varphi\hat{\Gamma}(\varphi), \tag{11.15b}$$

whence we get

$$\hat{\Gamma}(\varphi)=\delta(\hat{\phi}_\Gamma-\varphi)=|\varphi\rangle_\Gamma\langle\varphi|. \qquad (11.15c)$$

It follows that any pure-orthogonal phase POM $\hat{\Gamma}(\varphi)$ is pure, is of the form $\delta(\hat{\phi}-\varphi)$, and is the eigenPOM of a phase operator, namely $\hat{\phi}_\Gamma$.

XII. COMMUTATION RELATIONS

Let us now consider the commutator $[\hat{H},\hat{\phi}]$, whose initially supposed impossible value, $[\hat{H},\hat{\phi}]=i$, caused some confusion: By (8.12) and (11.3), we have

$$-i[\hat{H},\hat{\phi}]_w(r,\theta)=(\partial/\partial\theta)[\hat{\phi}]_w(r,\theta)\to 1-2\pi\delta_{2\pi}(\theta) \qquad (12.1a)$$

$$(r\to\infty)$$

$$[e^{-it\hat{H}}\hat{\phi}e^{it\hat{H}}]_w(r,\theta)=[\hat{\phi}]_w(r,\theta+t)\to\theta+t \bmod 2\pi \qquad (12.1b)$$

in correspondence with the classical result (2.8) for the *single-valued* phase. In the special case that $\hat{\Gamma}(\varphi)$ is φ *normalized and covariant* [see (9.6) and (9.7)], we get

$$i[\hat{H},\hat{\phi}_\Gamma]=\int_0^{2\pi}\varphi\, d\varphi(\partial/\partial\varphi)\hat{\Gamma}(\varphi), \qquad (12.2a)$$

where $\hat{\phi}_\Gamma=\int_0^{2\pi}\varphi\, d\varphi\, \hat{\Gamma}(\varphi)$. Integrating (12.2a) by parts, using $\int_0^{2\pi}d\varphi\, \hat{\Gamma}(\varphi)=1$, we obtain

$$-i[\hat{H},\hat{\phi}_\Gamma]=\hat{1}-2\pi\hat{\Gamma}, \quad \hat{\Gamma}\equiv\hat{\Gamma}(0)=\hat{\Gamma}(2\pi) \quad (12.2b)$$

as we had anticipated in (2.10). Also, by integrating the equation

$$\frac{\partial}{\partial t}e^{-it\hat{H}}\hat{\phi}e^{it\hat{H}}=-ie^{-it\hat{H}}[\hat{H},\hat{\phi}]e^{it\hat{H}}=\hat{1}-2\pi\hat{\Gamma}(t) \qquad (12.3a)$$

we find that [34]

$$e^{-it\hat{H}}\hat{\phi}e^{it\hat{H}}=\hat{\phi}+t-2\pi\int_0^t dt'\,\hat{\Gamma}(t') \qquad (12.3b)$$

whose Weyl symbol gives us back (12.1b) in the form

$$\begin{aligned}[e^{-it\hat{H}}\hat{\phi}e^{it\hat{H}}]_w(r,\theta)\\ &=[\hat{\phi}]_w(r,\theta)+t-2\pi\int_0^t dt'[\hat{\Gamma}(t')]_w(r,\theta)\\ &\to\theta \bmod 2\pi+t-2\pi\int_0^t dt'\,\delta_{2\pi}(t'-\theta)\\ &=\theta+t,\ \bmod 2\pi. \end{aligned} \qquad (12.4)$$

We likewise get [compare (2.6)]

$$e^{i\lambda[\hat{\phi},\cdot]}\hat{H}=e^{i\lambda\hat{\phi}}\hat{H}e^{-i\lambda\hat{\phi}}=\hat{H}+\lambda-2\pi\int_0^\lambda d\lambda'\,e^{i\lambda'\hat{\phi}}\hat{\Gamma}e^{-i\lambda'\hat{\phi}} \qquad (12.5a)$$

whose Weyl symbol tends, as $r\to\infty$, to the classical result [see (A7)]

$$e^{-\lambda\{\phi,\cdot\}_{\mathrm{PB}}}H=H+\lambda-2\pi\lambda\,\delta_{2\pi}(\theta). \qquad (12.5b)$$

It follows from (12.3) that any pure-orthogonal phase POM, necessarily of the form $\delta(\hat{\phi}-\varphi)$, by (11.15), can never be covariant, since

$$\begin{aligned}e^{-it\hat{H}}\delta(\hat{\phi}-\varphi)e^{it\hat{H}}&=\delta\left(\hat{\phi}-\varphi+t-2\pi\int_0^t dt'\,\hat{\Gamma}(t')\right)\\ &\neq\delta(\hat{\phi}-\varphi+t).\end{aligned} \qquad (12.6)$$

This is effectively forbidden by the no-go identity (2.5). Of course $\delta(\hat{\phi}-\varphi)$ is, like any other phase δ operator, asymptotically covariant in the sense (9.9) [which here also follows directly from (12.1b) and (8.8)]. Also, the Weyl symbol of (12.6) is just the Weyl symbol of $\delta(\hat{\phi}-\varphi)$ rotated by an angle t, as implied by (8.12). Thus, $e^{-it\hat{H}}\delta(\hat{\phi}-\varphi)e^{it\hat{H}}$ and $\delta(\hat{\phi}-\varphi+t)$ are two *different* phase states, but of *equal* phase $\varphi-t$.

XIII. POSITION REPRESENTATIONS

The Weyl ordered phase δ operator $\hat{\Gamma}_w(\varphi)$, whose Weyl symbol is $\delta(\theta-\varphi)$ by definition, has position matrix elements [see Appendix D, and definition (4.2)]

$$\langle q'|\hat{\Gamma}_w(\varphi)|q\rangle=e^{(1/2)i(q'^2-q^2)\tan\varphi}\frac{1}{\cos^2\varphi}\left|\frac{q+q'}{2}\right|\Theta\left(\frac{q+q'}{\cos\varphi}\right). \qquad (13.1)$$

In the case of a general phase δ operator $\hat{\Gamma}(\varphi)$, whose Weyl symbol $[\hat{\Gamma}(\varphi)]_w(r,\theta)\to\delta(\theta-\varphi)$ only as $r\to\infty$, we have

$$\langle q'|\hat{\Gamma}(\varphi)|q\rangle\to\langle q'|\hat{\Gamma}_w(\varphi)|q\rangle \text{ as } q,q'\to\infty. \qquad (13.2)$$

The result (13.1) was obtained by Hennings, Smith, and Dubin [9(c)]; they also calculated the matrix elements of the Weyl phase operator $\hat{\phi}_w$ (see Appendix D).

In the case of a pure phase state $\hat{\Gamma}(\varphi)=|\varphi\rangle\langle\varphi|$ we deduce from (13.2) that

$$\langle q|\varphi\rangle\to\frac{|q|^{1/2}}{\cos\varphi}e^{(1/2)iq^2\tan\varphi}\Theta\left(\frac{q}{\cos\varphi}\right) \quad (q\to\infty). \qquad (13.3)$$

The "chirp" $e^{(1/2)iq^2\tan\varphi}$ is characteristic of localization along the radius $\theta=\varphi$ in phase space, since it has a "local wave number" [recall (5.3)]

$$\langle p\rangle(q)=\frac{\partial}{\partial q}[\tfrac{1}{2}q^2\tan\varphi]=q\ \tan\varphi. \quad (13.4)$$

The same phase factor appears in (see Appendix C)

$$\langle q|e^{i\varphi\hat{H}}|p=0\rangle=(2\pi i)^{-1/2}(\cos\varphi)^{-1/2}e^{(1/2)iq^2\tan\varphi}. \quad (13.5)$$

The rotated momentum state $e^{i\varphi\hat{H}}|p=0\rangle$, whose Weyl symbol is [recall (4.3) and (5.21)]

$$[e^{i\varphi\hat{H}}|p=0\rangle]_w(q,p)=\delta(q\ \sin\varphi-p\ \cos\varphi)$$
$$=\frac{\delta(\theta-\varphi)+\delta(\theta-\varphi-\pi)}{r} \quad (13.6)$$

is indeed localized along the rotated p axis. The extra factor $(q/\cos\varphi)^{1/2}$ in (13.3), as against (13.5), reflects the extra "wedge" weight $r=q/\cos\theta$ in $[|\varphi\rangle\langle\varphi|]_w(r,\theta)\sim\delta(\theta-\varphi)$, as against (13.6). In view of (5.3), the $q^{1/2}$ dependence implies that the (true) phase-space width of $|\varphi\rangle$ decreases like r^{-1} as $r\to\infty$.

XIV. NUMBER REPRESENTATIONS

We now consider matrix elements between eigenkets $|n\rangle$ of $\hat{H}=\frac{1}{2}(\hat{q}^2+\hat{p}^2)$. In view of (5.15), one has, for any operator $\hat{A}$,

$$2\pi\langle n'|\hat{A}|n\rangle=\int_0^\infty r\ dr\int_0^{2\pi}d\theta[|n\rangle\langle n'|]_w(r,\theta)[\hat{A}]_w(r,\theta). \quad (14.1)$$

When $(n'-n)^2\ll n+n'\gg 1$, the Weyl symbol $[|n\rangle\langle n'|]_w(r,\theta)$ is an angular wave $e^{i(n'-n)\theta}$ *effectively* localized on a ring of radius $\approx(n+n')^{-1/2}$, and thickness ≈ 1, for it vanishes rapidly on the outside, and oscillates rapidly on the inside of that ring (see Appendix F). It follows that if $\hat{A}$ corresponds to $A(\theta)$, that is $[\hat{A}]_w(r,\theta)\to A(\theta)$ as $r\to\infty$, then [see (F6)–(F8)]

$$2\pi\langle n'|\hat{A}|n\rangle\to\int_0^{2\pi}d\theta\ e^{i(n'-n)\theta}A(\theta),$$
$$\text{as}\quad n+n'\to\infty,\quad \frac{(n'-n)^2}{n+n'}\ll 1, \quad (14.2)$$

that is, $\langle n'|\hat{A}|n\rangle$ tends to the $(n'-n)$ Fourier coefficient of $A(\theta)$.

Applying (14.2) to phase δ operators $\hat{\Gamma}(\varphi)$ and phase operators $\hat{\phi}$, which correspond to $\delta(\theta-\varphi)$ and θ mod2π, respectively, we get, using (3.1),

$$\left.\begin{aligned}&2\pi\langle n'|\hat{\Gamma}(\varphi)|n\rangle\to e^{i(n'-n)\varphi} && (14.3a)\\ &\langle n'|\hat{\phi}|n\rangle\to\pi\delta_{nn'}+(1-\delta_{nn'})\frac{i}{n-n'} && (14.3b)\end{aligned}\right\}\ n+n'\to\infty,\quad \frac{(n'-n)^2}{n+n'}\ll 1.$$

The asymptotics (14.3) may be taken as alternative characterizations of phase δ operators and phase operators. The Susskind-Glogower POM (10.5) distinguishes itself by assuming these asymptotic forms at all values of n and n', since we have, from (3.6a) and (3.9b).

$$2\pi\langle n'|\hat{\Gamma}_{\mathrm{SG}}|n\rangle=e^{i(n'-n)\varphi},$$
$$\langle n'|\hat{\phi}_{\mathrm{SG}}|n\rangle=\pi\delta_{nn'}+(1-\delta_{nn'})\frac{i}{n-n'}. \quad (14.4)$$

Let us now specialize to cases where $\hat{\Gamma}(\varphi)$ is, like $\hat{\Gamma}_{\mathrm{SG}}(\varphi)$, φ normalized and covariant [see (9.6) and (9.7)]. The covariance (9.7) implies

$$\langle n'|\hat{\Gamma}(\varphi)|n\rangle=e^{i(n'-n)\varphi}\langle n'|\hat{\Gamma}|n\rangle. \quad (14.5)$$

Integrating this over φ, using the φ normalization $\int_0^{2\pi}d\varphi\ \hat{\Gamma}(\varphi)=\hat{1}$ and (3.1a), yields $2\pi\delta_{nn'}\langle n'|\hat{\Gamma}|n\rangle=\delta_{nn'}$, whence we get

$$\langle n|\hat{\Gamma}|n\rangle=(2\pi)^{-1},\quad \langle n|\hat{\phi}_\Gamma|n\rangle=\int_0^{2\pi}\varphi\ d\varphi\langle n|\hat{\Gamma}|n\rangle=\pi, \quad (14.6)$$

where $\hat{\phi}_\Gamma=\int_0^{2\pi}\varphi d\varphi\hat{\Gamma}(\varphi)$. We also get from the commutation relation (12.2) [compare the no-go (2.5)]

$$-i(n'-n)\langle n'|\hat{\phi}|n\rangle=\delta_{n'n}-2\pi\langle n'|\hat{\Gamma}|n\rangle, \quad (14.7a)$$

whence we deduce, in accord with (14.3),

$$\langle n'|\hat{\phi}_\Gamma|n\rangle=\frac{2\pi i}{n-n'}\langle n'|\hat{\Gamma}|n\rangle,\quad n'\neq n. \quad (14.7b)$$

Note that the above results pertain to our choice of phase "window," namely $(0,2\pi)$; they get slightly modified if a different window is chosen [35].

The matrix elements $\langle n'|\hat{\Gamma}|n\rangle$ for antinormal and Weyl orderings are

$$2\pi\langle n'|\hat{\Gamma}_{s=1}|n\rangle=(n'!n!)^{-1/2}\Gamma\left(\frac{n'+n}{2}+1\right)$$
$$\approx 1-\frac{1}{4}\frac{(n'-n)^2}{n'+n}+\cdots, \tag{14.8}$$

$$2\pi\langle n'|\hat{\Gamma}_w|n\rangle=2^{(n-n')/2}\left[\frac{n'!}{n!}\right]^{1/2}\frac{\Gamma(\frac{1}{2}n+\mu)}{\Gamma(\frac{1}{2}n'+\mu)}$$
$$\approx 1+\tfrac{1}{2}(-)^n\frac{n'-n}{n'+n}+\cdots\quad, \tag{14.9}$$

where $\Gamma(z)$ are gamma functions, and the approximate values are for $|n'-n|\ll n'+n$. [(14.8) is given in [26a]; (14.9), given in [9b] and [10], is for $n'\geqslant n;\mu=1$ for n odd, $\mu=\frac{1}{2}$ for n even.]. The matrix elements of the Cahill-Glauber $\hat{\Gamma}_s$, for any value of s, have been calculated by Tanas, Miranowicz, and Gantsog [36]. We calculate their asymptotic behaviors in Appendix G, and obtain

$$2\pi\langle n'|\hat{\Gamma}_s|n\rangle=1-\frac{1}{4}s\frac{(n'-n)^2}{n'+n}$$
$$+\frac{1}{2}(-)^n\left(\frac{1-s}{1+s}\right)^{(n+n')/2}(1-s)\frac{n'-n}{n'+n}+\cdots$$
$$(n+n'\to\infty) \tag{14.10}$$

[yielding (14.8) at $s=1$, and (14.9) at $s=0$]. Note that $[(1-s)/(1+s)]^{(n+n')/2}\approx e^{-(n+n')s}$ is exponentially small for $s>0$ (exponentially large for $s<0$), so that the first-order correction in $(n+n')^{-1}$ passes from an $(n'-n)$ to an $(n'-n)^2$ dependence *as soon as* $s\neq 0$, i.e., as soon as the asymptotic *true* width s of $[\hat{\Gamma}_s]_w(r\theta)$ departs from zero. Recalling that the Weyl $\hat{\Gamma}_w=\hat{\Gamma}_{s=0}$, and the Susskind-Glogower $\hat{\Gamma}_{SG}$, both have *sharp* radii as $r\to\infty$ [see after Eqs. (10.5)], we surmise that the asymptotic behaviors "sharp angular width," and "sharp true width," correspond to the respective conditions $(n'-n)^2/(n+n')\ll 1$, and the less stringent $|n'-n|/(n+n')\ll 1$, for having $2\pi\langle n'|\hat{\Gamma}|n\rangle\approx 1$.

Pure phase states. Let now $\hat{\Gamma}(\varphi)=|\varphi\rangle\langle\varphi|$ be a pure phase state, and expand

$$|\varphi\rangle=\sum_{n=0}^{\infty}|n\rangle\langle n|\varphi\rangle=(2\pi)^{-1/2}\sum_{n=0}^{\infty}|n\rangle c_n(\varphi). \tag{14.11}$$

The asymptotics (14.3) imply that [37]

$$\langle n|\varphi\rangle\to(2\pi)^{-1/2}e^{in\varphi}\quad\text{as}\quad n\to\infty. \tag{14.12}$$

This may be compared with the corresponding result for the rotated momentum state $e^{i\varphi\hat{H}}|p=0\rangle$, whose Weyl symbol $\delta(q\cos\varphi-p\sin\varphi)$ is a complete line:

$$\langle n|e^{i\varphi\hat{H}}|p=0\rangle=e^{i(n+1/2)\varphi}\langle n|p=0\rangle$$
$$\to(2\pi)^{-1/2}e^{in\varphi}\frac{1+e^{in\pi}}{(2n)^{1/4}}\quad(n\to\infty) \tag{14.13}$$

following from the known value of $\langle p=0|n\rangle$ [see (K21)]. In the next section, we will understand the differences between (14.12) and (14.13) via a phase-space analysis based on coherent states.

According to (14.12), a pure phase state tends to the Susskind-Glogower state $\langle n|\varphi\rangle_{SG}$ at large n [a conclusion reached in a different way by Lerner, Huang, and Walters [16]—see after Eqs. (3.11)]. This, together with the fact that only large n contributes to large r in the Weyl phase plane, implies that a pure phase state tends, like $\hat{\Gamma}_{SG}(\varphi)$, to a sharp radius of zero true width, as we had already concluded after Eqs. (11.4) and (13.6). [If $\langle n|\varphi\rangle\approx e^{in\varphi}$ for $n>n_1$, say, then $\langle n'|\varphi\rangle\langle\varphi|n\rangle\approx e^{i(n'-n)\varphi}$ for $n,n'>n_1$, without the stringent condition $(n'-n)^2\ll(n+n')$ needed if the width stays finite as $r\to\infty$—see after (14.10).]

Let now $\hat{\Gamma}(\varphi)=|\varphi\rangle\langle\varphi|$ be a pure phase POM, so that $\int d\varphi|\varphi\rangle\langle\varphi|=\hat{1}$. This implies

$$(2\pi)^{-1}\int_0^{2\pi}d\varphi c_{n'}(\varphi)^*c_n(\varphi)=\delta_{n'n}. \tag{14.14}$$

If moreover $\hat{\Gamma}(\varphi)$ is covariant, then $|\varphi\rangle=e^{i\varphi\hat{H}}|\varphi=0\rangle$, hence $c_n(\varphi)=c_ne^{in\varphi}$; then (14.14) implies $|c_n|^2=1$, by (3.1a). Thus, a *pure covariant* phase POM is necessarily of the form (χ_n real)

$$|\varphi\rangle=(2\pi)^{-1/2}\sum_{n=0}^{\infty}e^{in\varphi+i\chi_n}|n\rangle,\quad\chi_n\to 0\quad\text{as}\quad n\to\infty, \tag{14.15}$$

that is, it is essentially the Susskind-Glogower POM (10.5): It follows that $\langle\varphi'|\varphi\rangle={}_{SG}\langle\varphi'|\varphi\rangle_{SG}\neq\delta(\varphi'-\varphi)$ [see (3.6c)], showing again that a pure phase POM cannot be both covariant and orthogonal, as was also concluded from (12.6), and from the no-go (2.5).

Phasors. Following Bergou and Englert [7], we may define "phasors" $\hat{e}$ by the requirement

$$[\hat{e}]_w(r,\theta)\to e^{i\theta}\quad(r\to\infty). \tag{14.16}$$

Putting $A(\theta)=e^{i\theta}$ in (14.2) and using (3.1a), we get

$$\langle n'|\hat{e}|n\rangle\to\delta_{n',n-1},\quad n+n'\to\infty,\quad\frac{(n'-n)^2}{n+n'}\ll 1. \tag{14.17}$$

Recalling Eqs. (3.10), we see that the Lerner-Huang-Walter (LHW) number-shift operators $\hat{E}_\lambda$, with $\lambda_n\to 1$ as $n\to\infty$, are phasors. Their eigenkets (3.11) on the unit circle ($\beta=e^{i\varphi}$) satisfy (14.12), so are phase states. This is not surprising: Indeed, $\hat{e}$ and $\exp(i\hat{\phi})$, for any phase operator $\hat{\phi}$, both correspond to $e^{i\theta}$ by [(8.8)], that is they coincide in the correspondence limit; so the eigenkets of $\hat{\phi}$ should correspond to eigenkets of $\hat{E}_\lambda$ [but note that $\hat{E}_\lambda$ has additional eigenkets, $|\beta|<1$ in (3.11), which are not phase states, hence do not correspond to eigenkets of $\hat{\phi}$].

In fact, in view of our assertion following Eq. (3.11), *any pure phase state is an eigenket of some LHW phasor* $\hat{E}_\lambda$. It is thus not surprising that LHW phasors, such as the Susskind-Glogower (SG) phasor, Eq. (3.4), have been used to generate pure phase states.

Again, we may express

$$\hat{e}=[e^{i\,\tan^{-1}(\hat{p}/\hat{q})}]_\Omega=\left[\left(\frac{\hat{q}+i\hat{p}}{\hat{q}-i\hat{p}}\right)^{1/2}\right]_\Omega \tag{14.18}$$

for some ordering rule Ω. For instance, the Susskind-Glogower $\hat{E}=[\hat{N}+1)^{-1/2}\hat{a}=(\hat{a}\hat{a}^\dagger)^{-1/2}\hat{a}=[(\hat{a}/\hat{a}^\dagger)^{1/2}]_\Omega$ $=\{[(\hat{q}+i\hat{p})/(\hat{q}-i\hat{p})]^{1/2}\}_\Omega$, where the ordering rule Ω may be stated as "Express everything in terms of $\hat{a}$ and $\hat{a}^\dagger$, then replace all $\hat{a}^\dagger$ by $(\hat{a}\hat{a}^\dagger)/\hat{a}$, and finally put all $(\hat{a}\hat{a}^\dagger)$ to the left of all $\hat{a}$."

XV. COHERENT-STATE REPRESENTATIONS

We show in Appendix E that at large r, a pure phase state has the coherent-state representation

$${}^{\mathrm{coh}}\langle r\theta|\varphi\rangle\approx\pi^{-1/4}r^{1/2}e^{(1/2)i\varphi}e^{-(1/2)r^2(\theta-\varphi)^2}e^{(1/2)ir^2(\theta-\varphi)} \qquad (r\to\infty). \tag{15.1}$$

This is localized along the radius $\theta\approx\varphi$, with height $\approx r^{1/2}$, reflecting the "wedge" weight r. One may compare (15.1) with the corresponding result for a rotated momentum state:

$${}^{\mathrm{coh}}\langle r\theta|e^{i\varphi\hat{H}}|p=0\rangle\approx\pi^{-1/4}e^{(1/2)i\varphi}[e^{-(1/2)(\theta-\varphi)^2}e^{(1/2)ir^2(\theta-\varphi)} + e^{-(1/2)r^2(\theta-\varphi-\pi)^2}e^{(1/2)ir^2(\theta-\varphi-\pi)}] \tag{15.2}$$

localized along the complete line $\theta\approx\{\varphi$ or $\varphi+\pi\}$, with height ≈ 1.

In view of (8.7) and (11.3), we have, for any phase operator $\hat{\phi}$,

$${}^{\mathrm{coh}}\langle r\theta|\hat{\phi}|r'\theta'\rangle^{\mathrm{coh}}\to\theta'\,{}^{\mathrm{coh}}\langle r\theta|r'\theta'\rangle^{\mathrm{coh}} \quad (r\to\infty). \tag{15.3}$$

It follows from (15.1) and (15.3) that any pure phase state $|\varphi\rangle$ is an "asymptotic eigenstate" of any phase operator $\hat{\phi}$, in the sense that

$${}^{\mathrm{coh}}\langle r\theta|\hat{\phi}|\varphi\rangle\to\varphi\,{}^{\mathrm{coh}}\langle r\theta|\varphi\rangle \quad (r\to\infty). \tag{15.4}$$

Indeed, using the completeness of coherent states [i.e., (7.3) applied to (7.15)], we have

$${}^{\mathrm{coh}}\langle r\theta|\hat{\phi}|\varphi\rangle =(2\pi)^{-1}\int_0^\infty r'dr'\int_0^{2\pi}d\theta'\,{}^{\mathrm{coh}}\langle r\theta|\hat{\phi}|r'\theta'\rangle^{\mathrm{coh}}\langle r'\theta'|\varphi\rangle \tag{15.5a}$$

$$\approx(2\pi)^{-1}\int_0^\infty r'dr'\int_0^{2\pi}d\theta'\,\theta'\,{}^{\mathrm{coh}}\langle r\theta|r'\theta'\rangle^{\mathrm{coh}}\langle r'\theta'|\varphi\rangle$$

$$\approx\varphi\,{}^{\mathrm{coh}}\langle r\theta|\varphi\rangle, \tag{15.5b}$$

where (15.1) allowed us to approximate $\theta'\langle r'\theta'|\varphi\rangle \approx\varphi\langle r'\theta'|\varphi\rangle$.

Let us now understand the differences between the number representations (14.12) and (14.13): Using again the completeness of coherent states, we have

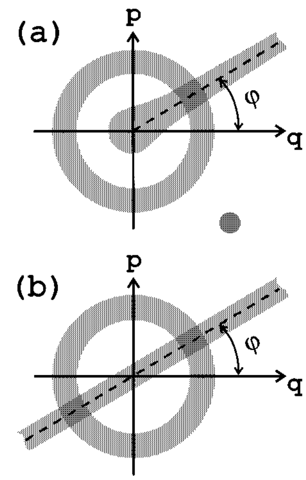

FIG. 8. The shaded ring in the (q,p) phase plane represents ${}^{\mathrm{coh}}\langle qp|n\rangle$, the overlap of a pure number state $n\rangle$ with coherent states ${}^{\mathrm{coh}}\langle qp|$. The small shaded circle represents a coherent state. The fat radius in (a) represents ${}^{\mathrm{coh}}\langle qp|\varphi\rangle$, where $|\varphi\rangle$ is a phase state [of the type shown in Fig. 6(c)]; the fat line in (b) represents ${}^{\mathrm{coh}}\langle qp|e^{i\varphi\hat{H}}|p=0\rangle$, where $e^{i\varphi\hat{H}}|p=0\rangle$ is a rotated momentum state.

$$2\pi\langle n|\varphi\rangle=\int_0^\infty r\,dr\int_0^{2\pi}d\theta\langle n|r\theta\rangle^{\mathrm{coh}}\langle r\theta|\varphi\rangle, \tag{15.6a}$$

$$2\pi\langle n|e^{i\varphi\hat{H}}|p=0\rangle =\int_0^\infty r\,dr\int_0^{2\pi}d\theta\langle n|r\theta\rangle^{\mathrm{coh}}\langle r\theta|e^{i\varphi\hat{H}}|p=0\rangle. \tag{15.6b}$$

Now, at large n [see (E4)]

$${}^{\mathrm{coh}}\langle r\theta|n\rangle\approx(2\pi n)^{-1/4}e^{in\theta}e^{-(r^2-2n)^2/8n} \tag{15.7}$$

is an angular wave $e^{in\theta}$ localized on a ring of radius $\approx(2n)^{1/2}$, and thickness ≈ 1. The integrals (15.6) get contributions only from the regions where the radii (15.1) and (15.2) intercept the ring (15.7) (Fig. 8). [Note that the phases $e^{-in\theta}$ and $e^{(1/2)ir^2\theta}\approx e^{in\theta}$ cancel; this is crucial, because $e^{in\theta}$ oscillates on a scale $\Delta\theta\approx n^{-1}$, which is very fast (as $n\to\infty$) compared to the angular width $\approx r^{-1}\approx(2n)^{-1/2}$ of $e^{-(1/2)r^2(\theta-\varphi)^2}$.] The differences between (14.12) and (14.13) are now easy to comprehend: The extra $(2n)^{1/4}$ in (14.12) reflects the "wedge" weight $r^{1/2}$ in (15.1), equal to $r^{1/2}\approx(2n)^{1/4}$ at the ring (15.7); the $(1+e^{in\pi})$ comes from the fact that (15.2) intercepts the ring at two places, $\theta\approx\varphi$ and $\theta\approx\varphi+\pi$. (Thus, the "+, 0, +, 0, . . . " oscillation of $\langle n|p=0\rangle$ [or of $\langle q=0|n\rangle$], as n varies, arises from interference in phase space, as beautifully analyzed by Schleich and Wheeler [18].) A Weyl phase-space analysis, done at the end of Appendix F, also yields some insight, but not as complete as above.

XVI. TRACE ORTHOGONALITY OF PHASE δ OPERATORS

Many phase δ operators $\hat{\Gamma}(\varphi)$, for instance the Cahill-Glauber $\hat{\Gamma}_s(\varphi)$ and the SG-POM (10.5), are not pure orthogonal (11.15). Consider, however, the following integral, where $\hat{\Gamma}_1(\varphi)$ and $\hat{\Gamma}_2(\varphi)$ may be identical or different phase δ operators, and Δ is some interval:

$$\int_\Delta d\varphi' \,\mathrm{Tr}\{\hat{\Gamma}_2(\varphi')\hat{\Gamma}_1(\varphi)\}$$
$$=(2\pi)^{-1}\int_\Delta d\varphi' \int r\ dr\ d\theta[\hat{\Gamma}_2(\varphi')]_w(r,\theta) \times[\hat{\Gamma}_1(\varphi)]_w(r,\theta), \tag{16.1}$$

where we used (5.15). Because $[\hat{\Gamma}(\varphi)]_w(r,\theta)\to\delta(\theta-\varphi)$, as $r\to\infty$, one sees that (16.1) is infinite if the integration interval Δ (however small) contains φ [so that the radius $\delta(\theta-\varphi')$ sweeps over the radius $\delta(\theta-\varphi)$] and *finite* otherwise (see Appendix I for more details). Thus, if we define $\hat{\Gamma}_1^\varepsilon(\varepsilon)$ as in (10.1), and if $\int d\varphi' \hat{\Gamma}_2(\varphi')=\hat{1}$, then we have

$$\mathrm{Tr}\{\hat{\Gamma}_2(\varphi')\hat{\Gamma}_1^\varepsilon(\varphi)\}\to\delta(\varphi'-\varphi)\quad(\varepsilon\to 0) \tag{16.2}$$

since $\int_\Delta d\varphi' \mathrm{Tr}\{\hat{\Gamma}_2(\varphi')\hat{\Gamma}_1^\varepsilon(\varphi)\}\to 0$ as $\varepsilon\to 0$ if $\varphi\notin\Delta$, and $\int_0^{2\pi} d\varphi' \mathrm{Tr}\{\hat{\Gamma}_2(\varphi')\hat{\Gamma}_1^\varepsilon(\varphi)\}=\mathrm{Tr}\{\hat{\Gamma}_1^\varepsilon(\varphi)\}=1$. We will call this kind of orthogonality *trace orthogonality*.

Let us give some examples: The simplest is if $\hat{\Gamma}_1(\varphi)=\hat{\Gamma}_2(\varphi)=\delta(\hat{\phi}-\varphi)$ are pure-orthogonal phase POM's, in which case

$$\mathrm{Tr}\{\delta(\hat{\phi}-\varphi')\delta(\hat{\phi}-\varphi)\}=\delta(\varphi'-\varphi)\mathrm{Tr}\{\delta(\hat{\phi}-\varphi)\}. \tag{16.3}$$

Another simple example is if $\hat{\Gamma}_1(\varphi)=\hat{\Gamma}_2(\varphi)=\hat{\Gamma}_w(\varphi)$ are Weyl ordered phase δ operators, whose Weyl symbols $[\hat{\Gamma}_w(\varphi)]_w(r,\theta)=\delta(\theta-\varphi)$: Then $[\hat{\Gamma}_w^\varepsilon(\varphi)]_w(r,\theta)=2\pi\varepsilon^2 e^{-\varepsilon r}\delta(\theta-\varepsilon)$, and we get exactly [using (5.15)]

$$\mathrm{Tr}\{\hat{\Gamma}_w(\varphi')\hat{\Gamma}_w^\varepsilon(\varphi)\}=\delta(\varphi'-\varphi). \tag{16.4}$$

Let now $\hat{\Gamma}_1=\hat{\Gamma}_2=|\varphi\rangle\langle\varphi|$ be a pure *covariant* phase POM [e.g., the SG-POM (10.5)], whose scalar product is given by (3.6c) [see after Eq. (14.15)], that is, for $\varphi'\approx\varphi$,

$$\langle\varphi'|\varphi\rangle\approx\tfrac{1}{2}\delta(\varphi'-\varphi)-\frac{i}{2\pi(\varphi'-\varphi)}\quad(\varphi'\approx\varphi). \tag{16.5}$$

Although the integral $\int_\Delta d\varphi'$ of (16.5) is finite for any Δ, that of

$$\mathrm{Tr}\{\hat{\Gamma}(\varphi')\hat{\Gamma}(\varphi)\}=|\langle\varphi'|\varphi\rangle|^2\approx\tfrac{1}{4}\delta(\varphi'-\varphi)^2+\frac{1}{4\pi^2(\varphi'-\varphi)^2} \tag{16.6}$$

is infinite if Δ contains φ, finite otherwise, whence we get (16.2).

Trace orthogonality concerns transition *probabilities*, that is, measured quantities. Pure orthogonality concerns transition *amplitudes*, and decides whether or not $\hat{\Gamma}(\varphi)$ are eigenstates of $\hat{\phi}_\Gamma=\int\varphi\ d\varphi\ \hat{\Gamma}(\varphi)$. Pure implies trace orthogonality, but not the reverse. Trace orthogonality is relevant because one adds transition *probabilities* into different final states, whereas one adds transition amplitudes [such as (16.5)] only into the *same* final state: Indeed, the measurement of a phase POM $|\varphi'\rangle\langle\varphi'|$, on an approximate phase state $|\varphi\rangle^\varepsilon$, yields an outcome φ inside the interval Δ with probability

$$\mathrm{Prob}\{\varphi\in\Delta\}=\int_\Delta d\varphi' |\langle\varphi'|\varphi\rangle^\varepsilon|^2. \tag{16.7}$$

This is different from $|\int_\Delta d\varphi'\langle\varphi'|\varphi\rangle^\varepsilon|^2$, unless $\langle\varphi'|\varphi\rangle=\delta(\varphi'-\varphi)$. According to (16.2), the measurement of any phase POM $\hat{\Gamma}_2(\varphi')$ on any phase state $\hat{\Gamma}_1^\varepsilon(\varphi)$ has the outcome φ with probability 1 as $\varepsilon\to 0$.

XVII. PHASE (QUASI)DISTRIBUTIONS AND UNCERTAINTY RELATIONS

In the preceding sections, we tacitly associated two different phase "distributions" with a phase δ operator $\hat{\Gamma}(\varphi)$, namely

$$Q(\varphi)=\mathrm{Tr}\{\hat{\Gamma}(\varphi)\hat{\rho}\},\quad P(\varphi)=\mathrm{Tr}\{\delta(\hat{\phi}-\varphi)\hat{\rho}\}, \tag{17.1}$$

where $\hat{\phi}=\int_0^{2\pi}\varphi\ d\varphi\ \hat{\Gamma}(\varphi)$. If $\hat{\Gamma}(\varphi)$ is not pure orthogonal, then $\delta(\hat{\phi}-\varphi)\neq\hat{\Gamma}(\varphi)$, and $P(\varphi)\neq Q(\varphi)$. By definition (9.1), both $Q(\varphi)$ and $P(\varphi)$ attribute the correct sharp phase to any large amplitude localized state. In the case of arbitrary states, however, $Q(\varphi)$ may assume negative values; yet it may still be useful as a "quasidistribution" [for instance the Weyl phase distribution (6.1)]. In the case of $\hat{\Gamma}_\Omega(\varphi)$[see (9.4)], $Q(\varphi)$ has the classical-like form

$$Q_\Omega(\varphi)=\mathrm{Tr}\{\hat{\Gamma}_\Omega(\varphi)\hat{\rho}\}=\int_0^\infty r\ dr\ f_\Omega(r,\varphi) \tag{17.2}$$

in terms of the Ω *phase space* (quasi)distribution (7.16). If $\hat{\Gamma}(\varphi)$ is a POM, satisfying (10.2), then $Q(\varphi)\geqslant 0$ and $\int_0^{2\pi}d\varphi\ Q(\varphi)=1$, and $Q(\varphi)$ is a genuine probability distribution. Defining (quasi)averaging operations

$$\langle f(\varphi)\rangle_Q=\int_0^{2\pi}d\varphi\ f(\varphi)Q(\varphi) \tag{17.3}$$

$$\langle f(\varphi)\rangle_P=\int_0^{2\pi}d\varphi\ f(\varphi)P(\varphi)=\mathrm{Tr}\{f(\hat{\phi})\hat{\rho}\}=\langle f(\hat{\phi})\rangle \tag{17.4}$$

[these are the expectations of (11.14)], we have

$$\langle\varphi\rangle_Q=\langle\varphi\rangle_P=\langle\hat{\phi}\rangle. \tag{17.5}$$

But in general, $\langle f(\varphi)\rangle_Q\neq\langle f(\varphi)\rangle_P$. Note that if $\hat{\Gamma}(\varphi)$ is φ normalized and *covariant*, and $\hat{\rho}=|n\rangle\langle n|$, then, in view of (14.5) and (14.6),

$$Q(\varphi)=\langle n|\hat{\Gamma}(\varphi)|n\rangle=(2\pi)^{-1}, \tag{17.6}$$

that is, $Q(\varphi)$ ascribes a random phase to any pure number state $|n\rangle$, while $P(\varphi)$ [and also $Q(\varphi)$ if $\hat{\Gamma}(\varphi)$ is noncovariant] does so only for large n, since $\delta(\hat{\phi}-\varphi)$[and a noncovariant $\hat{\Gamma}(\varphi)$] is only asymptotically covariant.

Number-phase uncertainty relations. Given an observable $\hat{A}$, its dispersion or uncertainty, in the quantum state $\hat{\rho}$, is

$$\Delta A = [\mathrm{Tr}\{(\hat{A} - \langle A \rangle)^2 \hat{\rho}\}]^{1/2}, \quad \langle A \rangle = \mathrm{Tr}\{\hat{A}\hat{\rho}\}. \quad (17.7)$$

We get $\Delta A = 0$ if and only if $\hat{\rho} = |a\rangle\langle a|$ is a pure eigenket of $\hat{A}$. In the case of the harmonic oscillator, the uncertainty ΔH in energy (or photon number) can have any value between 0 and ∞, with $\Delta H = 0$ if $\hat{\rho} = |n\rangle\langle n|$ is a pure number state. In the case of a phase operator $\hat{\phi}$, we have, in view of (17.4), the classical-like expressions

$$(\Delta \phi)^2 = \int_0^{2\pi} d\varphi (\varphi - \langle \phi \rangle)^2 P(\varphi), \quad \langle \phi \rangle = \int_0^{2\pi} \varphi \, d\varphi \, P(\varphi). \quad (17.8)$$

Because φ ranges over a finite window $(0,2\pi)$, the dispersion $\Delta\varphi$ is bounded above, being maximum if $P(\varphi)$ is concentrated at both extremities of the window, such as

$$P(\varphi) = \begin{cases} \varepsilon^{-1} & \text{for } (2\pi - \frac{1}{2}\varepsilon) \leqslant \varphi \leqslant 2\pi \text{ or } 0 \leqslant \varphi \leqslant \frac{1}{2}\varepsilon \\ 0 & \text{otherwise} \end{cases} \quad (\varepsilon \to 0) \quad (17.9)$$

in which case $\Delta\varphi = \langle \varphi \rangle = \pi$ [it does not matter whether (17.9) is realizable or not—all we want is an upper bound on $\Delta\varphi$]. Thus, although it is *sharply localized*, the distribution (17.9) nonetheless has maximum dispersion, because it straddles the "discontinuity cut" at $\varphi = 0$, so that $\varphi = 2\pi$ on one side of the cut, and $\varphi = 0$ on the other side. Note that the same distribution rotated away from the cut has zero dispersion; so the phase uncertainty depends very much on the choice of "cut" or "window," as was observed by Pegg and Barnett [15]. In any case, we have

$$0 \leqslant \Delta H \leqslant \infty, \quad 0 \leqslant \Delta\varphi \leqslant \pi. \quad (17.10)$$

We get $\Delta\varphi = 0$ if $\hat{\rho} = |\varphi\rangle\langle\varphi|$ is a pure eigenstate of $\hat{\phi}$. It follows from (17.10) that, although the commutator $[\hat{H}, \hat{\phi}] \neq 0$, the energy-phase (or number-phase) uncertainty relation is

$$\Delta H \Delta \phi \geqslant 0. \quad (17.11)$$

The lower limit $\Delta H \Delta \phi = 0$ is realized if $\hat{\rho} = |n\rangle\langle n|$, for which $\Delta H = 0$ and $\Delta\phi < \pi$ (at large n, the phase is totally random, $P(\varphi) = (2\pi)^{-1}$ [see after Eq. (17.6)], so that $\Delta\phi = \pi/3^{1/2}$). From a more general perspective, recall that any two observables, such as $\hat{H}$ and $\hat{\phi}$, satisfy the uncertainty relation [38]

$$\Delta H \Delta \phi \geqslant \tfrac{1}{2} |\mathrm{Tr}\{[\hat{H}, \hat{\phi}]\hat{\rho}\}|. \quad (17.12)$$

With $\hat{\rho} = |n\rangle\langle n|$, we get $\mathrm{Tr}\{[\hat{H}, \hat{\phi}]\hat{\rho}\} = \langle n|[\hat{H}, \hat{\phi}]|n\rangle = 0$, and thus recover (17.11).

It has often been argued on phenomenological grounds (or inferred from the impossible commutation relation $[\hat{H}, \hat{\phi}] = i$) that the energy-phase uncertainty relation is

$$\Delta H \Delta \phi \geqslant \tfrac{1}{2}. \quad (17.13)$$

This contradicts the fact that $\Delta H \Delta \phi = 0$ for $\hat{\rho} = |n\rangle\langle n|$—yet it does have a certain validity: Consider the specific case that $\hat{\phi} = \int_0^{2\pi} \varphi \, d\varphi \, \hat{\Gamma}(\varphi)$, where $\hat{\Gamma}(\varphi)$ is normalized and covariant (e.g., the Susskind-Glogower or Cahill-Glauber phase POM's). Then, in view of the commutation relation (12.2b), we get

$$\Delta H \Delta \phi \geqslant \tfrac{1}{2} |1 - 2\pi \, \mathrm{Tr}\{\hat{\Gamma}\hat{\rho}\}| = \tfrac{1}{2} |1 - 2\pi Q(0)|, \quad (17.14)$$

where $Q(\varphi) = \mathrm{Tr}\{\hat{\Gamma}(\varphi)\hat{\rho}\}$ is the quasidistribution defined in (17.1) [recall that it is different from the phase distribution $P(\varphi)$]. If $\hat{\rho} = |n\rangle\langle n|$, then $Q(0) = (2\pi)^{-1}$ by (17.6), and we get again $\Delta H \Delta \phi \geqslant 0$. This shows that the term $Q(0)$ in (17.14) is crucial: It may be expressed as [by (5.15)]

$$Q(0) = \mathrm{Tr}\{\hat{\Gamma}\hat{\rho}\} = \int_0^{2\pi} d\varphi \int_0^{\infty} r \, dr \, \Gamma_w(r, \theta) W_\rho(r, \theta), \quad (17.15)$$

where $W_\rho(r,\theta)$ is the Wigner function of $\hat{\rho}$. Now, $\Gamma_w(r,\theta)$ is concentrated along the radius $\theta = 0$ [recall that $\Gamma_w(r,\theta) \to \delta(\theta)$ as $r \to \infty$]. So, if $W_\rho(r,\theta)$ does not overlap $\Gamma_w(r,\theta)$, that is, if $W_\rho(r,\theta)$ is localized away from the origin and from $\theta = 0$ or 2π, then $\mathrm{Tr}\{\hat{\Gamma}\hat{\rho}\} = 0$, and (17.14) reduces to (17.13). This conclusion, argued for a covariant $\hat{\Gamma}(\varphi)$ also holds in the general case, since, in (17.12),

$$\mathrm{Tr}\{[\hat{H}, \hat{\phi}]\hat{\rho}\} = \int_0^{2\pi} d\varphi \int_0^{\infty} r \, dr [\hat{H}, \hat{\phi}]_w(r, \theta) W_\rho(r, \theta) \quad (17.16)$$

and $-i[\hat{H}, \hat{\phi}]_w(r,\theta) \to 1 - 2\pi\delta(\theta)$ as $r \to \infty$, according to (12.1a). We thus conclude that the energy-phase uncertainty relation is $\Delta H \Delta \phi \geqslant 0$, but that it is effectively $\Delta H \Delta \phi \geqslant \frac{1}{2}$ for states whose Wigner functions are localized away from the origin and from the extremities of the phase window $(0,2\pi)$.

XVIII. SUMMARY OF RESULTS

Let us summarize the results obtained thus far. We addressed the question, how does the classical notion of "phase" apply to a quantum harmonic oscillator, $\hat{H} = \frac{1}{2}(\hat{q}^2 + \hat{p}^2)$, which cannot have sharp position *and* momentum? Our approach was based on the observation that only *large amplitude localized* states can be assigned definite phases. Our only demand on a phase operator was, therefore, that the phase distribution

$$P(\varphi) = \mathrm{Tr}\{\delta(\hat{\phi} - \varphi)\hat{\rho}\} \quad (18.1)$$

attribute the correct sharp phase to any such "classical phase" state. This requires the following asymptotics for Weyl symbols $[\theta=\tan^{-1}(p/q),\ r=(q^2+p^2)^{1/2}]$:

$$[\delta(\hat{\phi}-\varphi)]_w(r,\theta)\rightarrow\delta(\theta-\varphi) \qquad \text{as } r\rightarrow\infty \tag{18.2a}$$

$$[\hat{\phi}]_w(r,\theta)\rightarrow\theta \bmod 2\pi. \tag{18.2b}$$

We defined *phase δ operators* $\hat{\Gamma}(\varphi)$ by the requirement, similar to (18.2a), that

$$[\hat{\Gamma}(\varphi)]_w(r,\theta)\rightarrow\delta(\theta-\varphi) \ \text{ as } \ r\rightarrow\infty. \tag{18.2c}$$

A phase δ operator induces a phase operator

$$\hat{\phi}_\Gamma=\int_0^{2\pi}\varphi\, d\varphi\ \hat{\Gamma}(\varphi) \tag{18.3}$$

and in turn, a phase operator $\hat{\phi}$ induces a phase δ operator $\delta(\hat{\phi}-\varphi)$.

Properties of interest which a phase δ operator can have are

$$e^{-it\hat{H}}\hat{\Gamma}(\varphi)e^{it\hat{H}}=\hat{\Gamma}(\varphi-t) \quad \text{(covariant)}, \tag{18.4a}$$

$$\int_0^{2\pi}d\varphi\ \hat{\Gamma}(\varphi)=\hat{1} \quad \text{(normalized)}, \tag{18.4b}$$

$$\hat{\Gamma}(\varphi)\geqslant 0 \quad \text{(state operator)}, \tag{18.4c}$$

$$\hat{\Gamma}(\varphi)=|\varphi\rangle\langle\varphi| \quad \text{(pure)}, \tag{18.4d}$$

$$\hat{\Gamma}(\varphi)\hat{\Gamma}(\varphi')=\delta(\varphi'-\varphi)\hat{\Gamma}(\varphi) \quad \text{(pure-orthogonal)}. \tag{18.4e}$$

$\hat{\Gamma}(\varphi)$ is a phase POM if it is non-negative and φ normalized. A pure-orthogonal phase POM is necessarily of the form $\delta(\hat{\phi}-\varphi)$, and can never be covariant because $[\hat{H},\hat{\phi}]=i$ is a no-go.

The asymptotics (18.2) imply the following for matrix elements:

$$\left.\begin{aligned}&\langle n'|\hat{\Gamma}(\varphi)|n\rangle\rightarrow(2\pi)^{-1}e^{i(n'-n)\varphi} && (18.5a)\\ &\langle n'|\hat{\phi}_\Gamma|n\rangle\rightarrow\pi\delta_{nn'}-2\pi(1-\delta_{nn'})\frac{i}{n'-n} && (18.5b)\end{aligned}\right\} \quad \frac{(n'-n)^2}{n'+n}\rightarrow 0$$

and

$$\langle q'|\hat{\Gamma}(\varphi)|q\rangle\rightarrow e^{(i/2)(q'^2-q^2)\tan\varphi}\frac{|q+q'|}{2\cos^2\varphi}\Theta\left(\frac{q+q'}{\cos\varphi}\right) \quad (q,q'\rightarrow\infty), \tag{18.5c}$$

where $\Theta(x<0)=0$, $\Theta(x>0)=1$. For pure phase states, we get

$$\langle n|\varphi\rangle\rightarrow(2\pi)^{-1/2}e^{in\varphi} \quad (n\rightarrow\infty), \tag{18.6a}$$

$$\langle q|\varphi\rangle\rightarrow\frac{|q|^{1/2}}{\cos\varphi}e^{(1/2)iq^2\tan\varphi}\Theta\left(\frac{q}{\cos\varphi}\right) \quad (q\rightarrow\infty). \tag{18.6b}$$

The "chirp" $e^{(1/2)iq^2\tan\varphi}$ in (18.6b) is characteristic of localization along the radius $p/q=\tan\varphi$ in phase space, since its "local" wave number is $\langle p\rangle(q)=(\partial/\partial q)(\frac{1}{2}q^2\tan\varphi)=q\ \tan\varphi$; the increasing amplitude $|q|^{1/2}$ reflects the increasing (true) width $r\Delta\theta$ of the "wedge" defined by an angular increment $\Delta\theta$, and implies that the local momentum width of $|\varphi\rangle$ vanishes like r^{-1} as $r\rightarrow\infty$.

A phase state is un-normalizable, and can never be strictly realized, which would require infinite energy. It may be viewed as the limit $\varepsilon\rightarrow 0$ of a normalized *approximate* phase state $\hat{\Gamma}^\varepsilon(\varphi)$, or $|\varphi\rangle^\varepsilon$, defined through its Weyl symbol, or position representation in the pure case, by

$$[\hat{\Gamma}^\varepsilon(\varphi)]_w(r,\theta)=[2\pi/K(\varepsilon)]\varepsilon^2e^{-\varepsilon r}[\hat{\Gamma}(\varphi)]_w(r,\theta), \tag{18.7a}$$

$$\langle q|\varphi\rangle^\varepsilon=[2\pi/K(\varepsilon)]^{1/2}\varepsilon e^{-(1/2)\varepsilon q/\cos\varphi}\langle q|\varphi\rangle, \tag{18.7b}$$

where the normalization constant $K(\varepsilon)\rightarrow 1$ as $\varepsilon\rightarrow 0$. Any phase state is an approximate eigenstate of any phase operator $\hat{\phi}$, in the sense that

$$\langle f(\hat{\phi})\rangle_{\Gamma(\varphi)}\equiv\mathrm{Lim}_{\varepsilon\rightarrow 0}\ \mathrm{Tr}\{f(\hat{\phi})\hat{\Gamma}^\varepsilon(\varphi)\}=f(\varphi), \tag{18.8a}$$

so that, in particular, $\langle\hat{\phi}\rangle_{\Gamma(\varphi)}=\varphi$, and $\langle(\hat{\phi}-\langle\hat{\phi}\rangle)^2\rangle_{\Gamma(\varphi)}=0$. Also, any *pure* phase state $|\varphi\rangle$ is an asymptotic eigenstate of any $\hat{\phi}$, in the sense that

$${}^{\mathrm{coh}}\langle r\theta|\hat{\phi}|\varphi\rangle\rightarrow\varphi\,{}^{\mathrm{coh}}\langle r\theta|\varphi\rangle \quad (r\rightarrow\infty), \tag{18.8b}$$

where $|r\theta\rangle^{\mathrm{coh}}$ are coherent states.

Many phase POM's are not pure orthogonal. However, any phase POM $\hat{\Gamma}_2(\varphi)$ is *trace orthogonal* to any phase state $\hat{\Gamma}_1(\varphi)$, in the sense that

$$\mathrm{Tr}\{\hat{\Gamma}_2(\varphi')\hat{\Gamma}_1^\varepsilon(\varphi)\}\rightarrow\delta(\varphi'-\varphi) \quad \text{as } \varepsilon\rightarrow 0. \tag{18.9}$$

This implies that any phase POM measurement on any phase state $\hat{\Gamma}_1^\varepsilon(\varphi)$ yields the outcome φ with probability 1 as $\varepsilon\rightarrow 0$.

In the special case that $\hat{\Gamma}(\varphi)$ is covariant and φ normalized, $\hat{\phi}_\Gamma=\int_0^{2\pi}\varphi\, d\varphi\ \hat{\Gamma}(\varphi)$ satisfies

$$-i[\hat{H},\hat{\phi}_\Gamma]=\hat{1}-2\pi\hat{\Gamma}(0),$$

$$e^{-it\hat{H}}\hat{\phi}_\Gamma e^{it\hat{H}}=\hat{\phi}_\Gamma+t-2\pi\int_0^t dt'\,\hat{\Gamma}(t') \tag{18.10}$$

corresponding to the Poisson bracket $\{H,\phi_{\rm cl}\}_{\rm PB}=1-2\pi\delta(\theta)$ for the *single-valued* classical phase $\phi_{\rm cl}=\theta \bmod 2\pi$, and to $\phi_{\rm cl}(t)=\phi_{\rm cl}(0)+t \bmod 2\pi$. If $\hat{\Gamma}(\varphi)$ is not both covariant and φ normalized, then the above holds only in the $(r\to\infty)$ correspondence limit. One may compare (18.10) with the "canonical," but impossible, value $[\hat{H},\hat{\phi}]=i$ initially presumed by Dirac [1], causing some confusion.

The "natural" quantization of a classical observable $g(q,p)$ is

$$g(\hat{q},\hat{p})_\Omega=\int dq\, dp\, g(q,p)\hat{\Delta}_\Omega(q,p), \tag{18.11a}$$

where Ω specifies an *ordering rule*, and $\hat{\Delta}_\Omega(q,p)$ are Ω ordered δ operators:

$$\hat{\Delta}_\Omega(q,p)=[\delta(\hat{q}-q)\delta(\hat{p}-p)]_\Omega=\hat{D}_{qp}\hat{\Delta}_\Omega\hat{D}^\dagger_{qp}, \tag{18.11b}$$

$$\hat{\Delta}_\Omega=[\delta(\hat{q})\delta(\hat{p})]_\Omega,\quad \hat{D}_{qp}=e^{ip\hat{q}-iq\hat{p}}, \tag{18.11c}$$

where $\hat{D}_{qp}$ are phase-space displacement operators. Any operator $\hat{\Delta}$, trace normalized to $\mathrm{Tr}\{\hat{\Delta}\}=(2\pi)^{-1}$, satisfies the "$\delta$ property"

$$\int dq\, dp\, \hat{\Delta}(q,p)=\int dq\, dp\, \hat{D}_{qp}\hat{\Delta}\hat{D}^\dagger_{qp}=2\pi\mathrm{Tr}\{\hat{\Delta}\}=\hat{1} \tag{18.11d}$$

and can be used to define a formal ordering rule via (18.11). Any phase operator $\hat{\phi}$ and phase δ operator $\hat{\Gamma}(\varphi)$ can be expressed as

$$\hat{\phi}_\Omega=[\tan^{-1}(\hat{p}/\hat{q})]_\Omega, \tag{18.12a}$$

$$\hat{\Gamma}(\varphi)=[\delta(\tan^{-1}(\hat{p}/\hat{q})-\varphi)]_{\Omega(\phi)} \tag{18.12b}$$

for some ordering rules Ω and $\Omega(\varphi)$. Thereby, these quantum observables are natural quantizations of the classical observables $\theta=\tan^{-1}(p/q)$ and $\delta(\theta-\varphi)=\delta(\tan^{-1}(p/q)-\varphi)$.

Important phase δ operators are (i) the pure and covariant (hence non-pure-orthogonal) Susskind-Glogower phase POM

$$\hat{\Gamma}_{\rm SG}(\varphi)=|\varphi\rangle_{\rm SG}\langle\varphi|,\quad |\varphi\rangle_{\rm SG}=(2\pi)^{-1/2}\sum_{n=0}^{\infty}e^{in\varphi}|n\rangle \tag{18.13}$$

which assume the asymptotic forms (18.5a) and (18.6a) at all values of n. (ii) The Cahill-Glauber s-ordered phase δ operators

$$\hat{\Gamma}_s(\varphi)=[\delta(\tan(\hat{p}/\hat{q})-\varphi)]_s=\int_0^\infty r\, dr\, \hat{\Delta}_s(r,\varphi). \tag{18.14a}$$

This is covariant, and is a POM if $s\geqslant 1$, in which case $\hat{\Delta}_s(q,p)=[\delta(\hat{q}-q)\delta(\hat{p}-p)]_s$ are displaced thermal oscillator states. At $s=0$, we get the Weyl phase δ operator

$$\hat{\Gamma}_w(\varphi)=\hat{\Gamma}_{s=0}(\varphi)=[\delta(\tan(\hat{p}/\hat{q})-\varphi)]_w \tag{18.14b}$$

At $s=1$, we get the coherent phase POM

$$\hat{\Gamma}^{\rm coh}(\varphi)=\hat{\Gamma}_{s=1}(\varphi)=\int_0^\infty r\, dr |r\varphi\rangle^{\rm coh}\langle r\varphi|, \tag{18.14c}$$

where $|r\theta\rangle^{\rm coh}$ are coherent states. The number matrix elements of $\hat{\Gamma}_s(\varphi)$ have the asymptotic forms

$$2\pi\langle n'|\hat{\Gamma}_s|n\rangle=1-\tfrac{1}{4}s\frac{(n'-n)^2}{n'+n}+\tfrac{1}{2}(-)^n\left(\frac{1-s}{1+s}\right)^{(n+n')/2}\times(1-s)\frac{n'-n}{n'+n}+\cdots\quad (n+n'\to\infty) \tag{18.14d}$$

showing the first corrections to (18.5a). Another covariant phase POM is the "number state POM"

$$\hat{\Gamma}_n(\varphi)=(2\pi)^{-1}\int_0^\infty r\, dr\, \hat{D}_{r\varphi}|n\rangle\langle n|\hat{D}^{-1}_{r\varphi} \tag{18.15}$$

considered by Busch, Grabowski, and Lahti [39]; note that $\hat{\Gamma}_{n=0}(\varphi)=\hat{\Gamma}_{s=1}(\varphi)$.

All phase δ operators have, by definition, vanishing *angular* widths in the Weyl phase plane as $r\to\infty$. The Weyl $\hat{\Gamma}_w(\varphi)$ has a null *true* width at all values of r [see Fig. 6(a)], while pure phase states, such as $\hat{\Gamma}_{\rm SG}(\varphi)$, have true widths vanishing like r^{-1} as $r\to\infty$ [Fig. 6(c)]. On the other hand, $\hat{\Gamma}_s(\varphi)$ for $s\neq 0$ and $\hat{\Gamma}_n(\varphi)$ have true widths which stay finite (roughly equal to s and $n^{1/2}$) as $r\to\infty$ [Fig. 6(b)].

Two phase distributions associate with a phase δ operator $\hat{\Gamma}(\varphi)$, namely

$$P(\varphi)=\mathrm{Tr}\{\delta(\hat{\phi}_\Gamma-\varphi)\hat{\rho}\},\quad Q(\varphi)=\mathrm{Tr}\{\hat{\Gamma}(\varphi)\hat{\rho}\}, \tag{18.16}$$

where $\hat{\phi}_\Gamma=\int_0^{2\pi}\varphi\, d\varphi\, \hat{\Gamma}(\varphi)$. If $\hat{\Gamma}(\varphi)$ is not a POM, $Q(\varphi)$ may assume negative values, but may still be useful as a "quasidistribution." If $\hat{\Gamma}(\varphi)=\hat{\Gamma}_\Omega(\varphi)$, then $Q(\varphi)$ is the radial integral of a *phase-space* (quasi)distribution:

$$Q(\varphi)=\int_0^\infty r\, dr\, f_\Omega(r,\varphi),$$

$$f_\Omega(q,p)=\mathrm{Tr}\{[\delta(\hat{q}-q)\delta(\hat{p}-p)]_\Omega\hat{\rho}\}. \tag{18.17}$$

If $\hat{\Gamma}(\varphi)$ is *covariant*, then $Q(\varphi)$ assigns a random phase to any pure number state $|n\rangle$, while $P(\varphi)$ does so only for large n.

Unless $\hat{\Gamma}_\varphi$ is pure orthogonal, $Q(\varphi)\neq P(\varphi)$. It follows that two different measurements associate with a nonorthogonal phase POM $\hat{\Gamma}(\varphi)$: That of $\hat{\Gamma}(\varphi)$, yielding $Q(\varphi)$, and that of $\hat{\phi}_\Gamma$, that is, of the phase POM $\delta(\hat{\phi}_\Gamma-\varphi)$, yielding $P(\varphi)$. A non-POM phase δ operator $\hat{\Gamma}(\varphi)$, such as

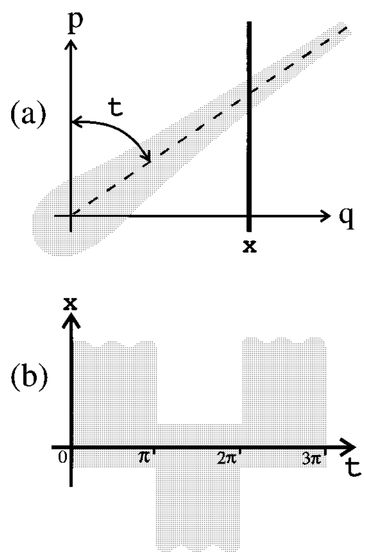

FIG. 9. (a) Under time evolution, the Wigner function of a phase state, represented by the shaded radial "beacon," rotates clockwise in phase space [here, the initial phase was $\pi/2$, i.e., the beacon was vertical at time $t=0$]; the probability $P(x,t)$ of finding the oscillator at position x is the overlap of the beacon with the line $\delta(q-x)$. The expectation $\langle\hat{q}\rangle$ is zero when the beacon is vertical, and $\pm\infty$ otherwise; the dispersion is always infinite, except when the beacon is vertical, in which case it can be (i) zero if the true width $w(r)$ of the beacon vanishes as $r\to\infty$ (the case depicted here), (ii) s if $w(r)\to s$, and (iii) infinite if $w(r)\to\infty$[for instance, if $w(r)\sim r^{\gamma}$ with $0<\gamma<1$, so that the *angular* width $w(r)/r\sim r^{\gamma-1}\to 0$ still]. (b) The shaded regions are those for which $P(x,t)$ is sizable [for the case depicted in (a)]; note the abrupt changes when time passes multiples of π.

$\hat{\Gamma}_w(\varphi)$, or a pure-orthogonal phase POM $\delta(\hat{\phi}_\Gamma-\varphi)$, leads to only one measurement, that of $\hat{\phi}_\Gamma$.

The energy-phase (or number-phase) uncertainty relation is $\Delta H\Delta\phi\geqslant 0$, the lower limit $\Delta H\Delta\phi=0$ being realized by pure number states $|n\rangle\langle n|$; however, in the case of states whose Wigner function is localized away from the origin and from the (single-valued) phase discontinuity at $\theta=0$ or 2π, the uncertainty relation is effectively $\Delta H\Delta\phi\geqslant\frac{1}{2}$.

One may ask, which is the "best" phase δ operator, or phase POM, or phase operator? One has the general feeling that "the sharper is the better." Different $\hat{\Gamma}(\varphi)$ are compared for sharpness through their Weyl symbols. The sharpest phase δ operator is, of course, the Weyl $\hat{\Gamma}_w(\varphi)$, whose Weyl symbol $\delta(\theta-\varphi)$ has zero width at all r [40]. The sharpest Cahill-Glauber phase POM is the $s=1$ coherent phase POM, since $\hat{\Gamma}_s(\varphi)\geqslant 0$ only for $s\geqslant 1$, and its true width $\approx s$. Pure phase POM's, such as the Susskind-Glogower POM, are sharper than $\hat{\Gamma}^{\rm coh}(\varphi)$, at least at large r, since their true widths $\to 0$.

Finally, one may define "phasors" $\hat{e}$ by the requirement $[\hat{e}]_w(r,\theta)\to e^{i\theta}$ as $r\to\infty$ [7]. The modulated number shift operators [16]

$$\hat{E}_\lambda=\sum_{n=1}^{\infty}|n-1\rangle\lambda_n\langle n|,\quad \lambda_n\to 1\quad\text{as } n\to\infty \tag{18.18}$$

are phasors. Among their eigenkets are pure phase states $|\varphi\rangle$ with

$$|\varphi\rangle=(2\pi)^{-1/2}\sum_{n=0}^{\infty}\left(\prod_{m=n+1}^{\infty}\lambda_m\right)e^{in\varphi}|n\rangle,\quad \hat{E}_\lambda|\varphi\rangle=e^{i\varphi}|\varphi\rangle. \tag{18.19}$$

Reciprocally, any pure phase state $|\varphi\rangle$ is an eigenket of *some* phasor of the form (18.18). Such phasors have been widely used to generate phase states. The most famous is the Susskind-Glogower number shift phasor

$$\hat{E}=\sum_{n=1}^{\infty}|n-1\rangle\langle n| \tag{18.20}$$

whose eigen-phase-states are the $|\varphi\rangle_{\rm SG}$ in Eq. (18.13).

XIX. MEASUREMENTS

One may observe a phase state $\hat{\Gamma}(\varphi)$ by measuring, for instance, its time-dependent position probability distribution, that is [by (5.7) and (18.5c)],

$$\begin{aligned}P(q,t)&=\langle q|e^{-it\hat{H}}\hat{\Gamma}(\varphi)e^{it\hat{H}}|q\rangle\\&=\int dp[e^{-it\hat{H}}\hat{\Gamma}(\varphi)e^{it\hat{H}}]_w(q,p)\\&\sim\frac{|q|}{\cos^2(\varphi-t)}\Theta\left(\frac{q}{\cos(\varphi-t)}\right)\quad(q\to\infty).\end{aligned} \tag{19.1}$$

This is the "projection" onto the q axis of a rotating "beacon" pointing in the direction $\varphi+t$ [see Fig. 9(a)]. When $\cos(\varphi-t)>0$, $P(q,t)$ extends infinitely to positive q, but finitely to negative q, so that a $\hat{q}$ measurement yields (almost) certainly a large positive value; this situation reverses *suddenly* as $\cos(\varphi-t)$ changes sign [Fig. 9(b)]. This is indeed characteristic of a sharp phase. The above procedure requires, of course, *many* individual measurements, to be significant.

Let us now envisage a genuine phase measurement: von Neumann [41] showed how to measure, *in principle*, any observable, such as $\hat{\phi}$, pertaining to a system s, say. One makes use of a microscopic "meter" m, with position-momentum operators $[\hat{Q},\hat{P}]=i$, and prepared at time $t=0$ in a position eigenstate $|Q_0\rangle$. Let $\hat{\rho}$ be the initial state of s. Let then s interact with m via an impulsive interaction $\delta(t)\hat{\phi}\hat{P}$ (whether this can be implemented in practice for any phase operator $\hat{\phi}$ is an open question). After the interaction, $s+m$ are in the entangled state

$$\begin{aligned}e^{-i\hat{\phi}\hat{P}}\{|Q_0\rangle\langle Q_0|\otimes\hat{\rho}\}e^{i\hat{\phi}\hat{P}}&=\int d\varphi\, d\varphi'\{|Q_0+\varphi\rangle\langle Q_0+\varphi'|\}_m\\&\quad\otimes\{|\varphi\rangle\langle\varphi|\hat{\rho}|\varphi'\rangle\langle\varphi'|\}_s.\end{aligned} \tag{19.2}$$

We inserted $\int d\varphi|\varphi\rangle\langle\varphi|=\hat{1}_s$, where $|\varphi\rangle$ are eigenkets of $\hat{\phi}$, and used

$$e^{-i\hat{\phi}\hat{P}}|Q_0\rangle_m|\varphi\rangle_s=|Q_0+\varphi\rangle_m|\varphi\rangle_s \tag{19.3}$$

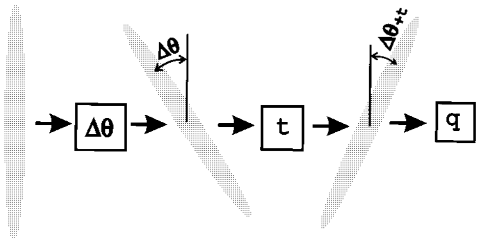

FIG. 10. To determine the (unknown) phase shift $\Delta\theta$ induced by some device, input a known, highly amplitude squeezed state $\hat{\rho}^{\rm in}\approx|q_0\rangle\langle q_0|$, say [in the figure, we put $q_0=0$], and shift the output state, $\hat{\rho}^{\rm out}=e^{-i\Delta\theta\hat{H}}\hat{\rho}^{\rm in}e^{i\Delta\theta\hat{H}}$, by a known phase t; then measure $\hat{q}$, while varying t; the measured distribution $\langle q|e^{-it\hat{H}}\hat{\rho}^{\rm out}e^{it\hat{H}}|q\rangle$ is narrowest, and concentrates at q_0, when the (known) phase t is equal to the (unknown) phase $-\Delta\theta \bmod 2\pi$.

since $\hat{\phi}|\varphi\rangle=\varphi|\varphi\rangle$ and $e^{-i\varphi\hat{P}}|Q_0\rangle=|Q_0+\varphi\rangle$. One then measures the position of the meter m. The outcome is $Q_0+\varphi$ with probability $P(\varphi)=\langle\varphi|\hat{\rho}|\varphi\rangle$, and s gets collapsed into the eigenstate $|\varphi\rangle\langle\varphi|=\delta(\hat{\phi}-\varphi)$. Note that s interacted only with the *microscopic* system m, whose initial state is controllable; m itself must interact strongly with some *macroscopic* apparatus M, in order to induce a macroscopically observable event. Because the *microstate* of M is intrinsically uncontrollable, it perturbs the state of m in a way which we cannot know; but the final state of s itself is precisely known [42]. [Of course, the above is an ideal limit, which can only be approached, since phase "states" $|\phi\rangle$ can never be strictly realized physically, as this would require infinite energy.]

A measurement where, as above, the final state is completely known in terms of the measurement outcome [whatever was the (generally unknown) premeasurement state], is called *complete*, or *preparatory*, since it *prepares* the phase state $\delta(\hat{\phi}-\varphi)$. If s was already in a phase state $\hat{\Gamma}(\varphi_0)$ [not necessarily an eigenstate of $\hat{\phi}$], then the trace orthogonality (18.9) assures that the measurement outcome is φ_0, with probability 1 (ideally), so that the oscillator collapses into the phase state $\delta(\hat{\phi}-\varphi_0)$. Thus, the state gets modified [unless $\hat{\Gamma}(\varphi_0)$ was equal to $\delta(\hat{\phi}-\varphi_0)$], but the *phase* is preserved. So the measurement is "quantum (phase) nondemolishing" (QND) [11].

After a $\hat{\phi}$ measurement prepared the state $\delta(\hat{\phi}-\varphi)$ at time $t=0$, the latter evolves into

$$e^{-it\hat{H}}\delta(\hat{\phi}-\varphi)e^{it\hat{H}}=\delta(\hat{\phi}_t-\varphi),\quad \hat{\phi}_t=e^{-it\hat{H}}\hat{\phi}e^{it\hat{H}}. \tag{19.4}$$

This is a phase state of phase $\varphi-t$, eigenket of the phase operator $\hat{\phi}_t\neq\hat{\phi}+t$ [see (18.10)], but *not* of $\hat{\phi}$. Still, one can accurately monitor the phase by measuring the *same* $\hat{\phi}$ again, because a second $\hat{\phi}$ measurement, after a phase shift t, say, will yield the outcome $\varphi-t$ with probability 1 (because of trace orthogonality), and (being QND) produce the phase state $\delta(\hat{\phi}-\varphi+t)$.

A general quantum measurement may involve a more general initial meter state, and $s-m$ interaction, than above. Also, the final observable measured may pertain to both m and s. [In that case, s interacts *directly* with the macroscopic apparatus M, thereby getting perturbed in an unknowable way, so that the measurement cannot be complete.] Whatever its precise form,a general phase measurement defines a phase POM $\hat{\Gamma}(\varphi)$, and yields an outcome φ with probability [43]

$$\mathrm{Prob}\{\varphi\}=\mathrm{Tr}\{\hat{\Gamma}(\varphi)\hat{\rho}\}. \tag{19.5}$$

In the von Neumann measurement, $\hat{\Gamma}(\varphi)=\delta(\hat{\phi}-\varphi)$ is pure orthogonal [this is required to get (19.3)], but this need not be the case: Indeed, any POM, pure orthogonal or not, can *in principle*, be directly measured [43]. It is even possible, *in principle*, to devise measurements for which $\mathrm{Prob}\{\varphi\}$ is given by (19.5), and which collapse the oscillator into *any* chosen final state [44], e.g., a phase state $\hat{\Gamma}'(\varphi)$ [equal or not to $\hat{\Gamma}(\varphi)$]. This would again allow us to precisely monitor the phase, because of the trace orthogonality of phase states.

Noh, Fougères, and Mandel [24] measured the coherent phase POM (18.14c), yielding $Q_{s=1}(\varphi)$, the radial integral of the Q function. Also, d'Ariano, Macchiavello, and Paris [26b] show that an imperfect NFM measurement measures the Cahill-Glauber "thermal coherent state" phase POM (18.14a), $s\geqslant 1$.

The basic objective is to measure *phase shifts*. A phase-shift measurement need not involve a phase operator, nor a phase POM. Interferometric methods do not. Another phase-POMless way of determining the phase shift induced by some device is to compare input and output *phase-space* distributions: These can be measured by the method of Smithey *et al.*, [46] who measured the Wigner function of a quantum optical field, or otherwise [47]. Still another method is described in Fig. 10. All the above methods require many individual measurements.

Clearly, a most efficient way of measuring phase shifts would be, ideally, by means of genuine *complete* phase measurements (of the first kind or of the non-pure-orthogonal POM kind), which would allow to both prepare sharply phased states, and then monitor their phase evolution by means of *single* "phase nondemolishing" measurements.

APPENDIX A: ROTATIONS AND DISPLACEMENTS IN THE (q,p) PHASE PLANE

We use the following coordinates in the (q,p) plane:

$$\theta=\tan^{-1}(p/q),\quad r=(q^2+p^2)^{1/2},\quad R=H=\tfrac{1}{2}(q^2+p^2). \tag{A1}$$

From (2.2), we get [denoting $\{.,.\}_{\rm PB}=\{.,.\}$ for simplicity]

$$\{H,.\}=\frac{\partial}{\partial\theta},\quad e^{t\{H,.\}}f(r,\theta)=f(r,\theta+t), \tag{A2}$$

$$e^{-t\{H,.\}}\begin{pmatrix} q\\ p\end{pmatrix}=e^{-t\{H,.\}}\begin{pmatrix} r\cos\theta\\ r\sin\theta\end{pmatrix}=\begin{pmatrix} r\cos(\theta-t)\\ r\sin(\theta-t)\end{pmatrix}=\begin{pmatrix} q_t\\ p_t\end{pmatrix}, \tag{A3}$$

where we define

$$\begin{pmatrix} q_\varphi\\ p_\varphi\end{pmatrix}=\mathscr{R}_\varphi\begin{pmatrix} q\\ p\end{pmatrix},\quad \mathscr{R}_\varphi=\begin{pmatrix} \cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}. \tag{A4}$$

Thus, $\{H,.\}=\partial/\partial\theta$ induces rotations of the phase plane, and (q_t,p_t) is the trajectory followed by the oscillator if it was at (q,p) at time $t=0$. Defining $\phi=\phi(R,\theta)=\theta \bmod 2\pi$, we get

$$\{\phi,f\}=-\frac{\partial\phi}{\partial\theta}\frac{\partial f}{\partial R}=-[1-2\pi\delta_{2\pi}(\theta)]\frac{\partial f}{\partial R}, \tag{A5}$$

$$\{\phi,R\}=-[1-2\pi\delta_{2\pi}(\theta)], \quad \{\phi,\theta\}=0. \tag{A6}$$

We deduce that

$$\frac{\partial}{\partial\lambda}e^{-\lambda\{\phi,.\}}R=-e^{-\lambda\{\phi,.\}}\lambda\{\phi,R\}=e^{-\lambda\{\phi,.\}}[1-2\pi\delta_{2\pi}(\theta)]$$
$$=1-2\pi\delta_{2\pi}(\theta) \tag{A7}$$

[since $\{\phi,\theta\}=0$], whence follows (12.5b).

In the quantum case, $e^{-i\varphi\hat{H}}$, where $\hat{H}=\frac{1}{2}(\hat{q}^2+\hat{p}^2)$, induces rotations:

$$e^{i\varphi\hat{H}}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}e^{-i\varphi\hat{H}}=\mathscr{R}_\varphi\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}=\begin{pmatrix}\hat{q}\cos\varphi+\hat{p}\sin\varphi\\ \hat{q}\sin\varphi+\hat{p}\cos\varphi\end{pmatrix}\equiv\begin{pmatrix}\hat{q}_\varphi\\ \hat{p}_\varphi\end{pmatrix} \tag{A8}$$

as is readily shown using $[\hat{q},\hat{p}]=i$. Phase-space displacement operators are defined as

$$\hat{D}_{qp}=e^{ip\hat{q}-iq\hat{p}}=e^{(1/2)iqp}e^{-iq\hat{p}}e^{ip\hat{q}}=e^{-(1/2)iqp}e^{ip\hat{q}}e^{-iq\hat{p}}, \tag{A9a}$$

$$\hat{D}_{qp}\{\hat{q},\hat{p}\}\hat{D}_{qp}^{-1}=\{\hat{q}-q,\hat{p}-p\},$$

$$\hat{D}_{qp}|q'\rangle=e^{(1/2)iqp}e^{ipq'}|q'+q\rangle, \tag{A9b}$$

$$\hat{D}_{qp}\hat{D}_{q'p'}=e^{-(1/2)i(qp'-pq')}\hat{D}_{q+q',p+p'}. \tag{A9c}$$

Denoting $\hat{D}_{qp}=\hat{D}_{r\theta}$, and using (A8), we find that

$$e^{-i\varphi\hat{H}}\hat{D}_{qp}e^{i\varphi\hat{H}}=\hat{D}_{q_\varphi p_\varphi}\Leftrightarrow e^{-i\varphi\hat{H}}\hat{D}_{r\theta}e^{i\varphi\hat{H}}=\hat{D}_{r,\theta-\varphi} \tag{A10a}$$

$$\Leftrightarrow i[\hat{H},\hat{D}_{r\theta}]=\frac{\partial}{\partial\theta}\hat{D}_{r\theta}, \tag{A10b}$$

$$\hat{D}_{r\theta}=\hat{D}_{r\cos\theta,r\sin\theta}=e^{i\theta\hat{H}}\hat{D}_{r,0}e^{i\theta\hat{H}}=e^{-ir\hat{p}_\theta}. \tag{A10c}$$

Using (A9), one shows that, for any operator $\hat{A}$,

$$\int dq dp \hat{D}_{qp}\hat{A}\hat{D}_{qp}^\dagger=2\pi\,\mathrm{Tr}\hat{A}. \tag{A11}$$

APPENDIX B: ORDERING RULES AND WEYL SYMBOLS

Ordered δ operators are defined as $\hat{\Delta}_\Omega(q,p)=\hat{D}_{qp}\hat{\Delta}_\Omega\hat{D}_{qp}^\dagger$, where

$$\hat{\Delta}_\Omega=[\delta(\hat{q})\delta(\hat{p})]_\Omega=(2\pi)^{-2}\int dk\,ds[e^{ik\hat{q}-is\hat{p}}]_\Omega. \tag{B1}$$

In terms of the canonical "action-angle" variables $R=\frac{1}{2}(q^2+p^2)$ and $\theta=\tan^{-1}(p/q)$, one has

$$\delta(q'-q)\delta(p'-p)=\delta(R'-R)\delta_{2\pi}(\theta'-\theta) \tag{B2}$$

[following from $\delta(\sin\theta)=\delta_{2\pi}(\theta)$; see (2.9)], so that

$$\hat{\Delta}_\Omega(q,p)=\hat{\Delta}_\Omega(R,\theta)$$
$$=[\delta(\tfrac{1}{2}(\hat{p}^2+\hat{q}^2)-R)\delta_{2\pi}(\tan^{-1}(\hat{p}/\hat{q})-\theta)]_\Omega, \tag{B3}$$

whence we get the special case (7.4). From (7.5), we get [using $q=r\cos\theta$, $p=r\sin\theta$]

$$2\pi\hat{\Delta}_w(q,p)=2\hat{D}_{qp}\hat{\Pi}\hat{D}_{qp}^\dagger=\int ds e^{-isp}|q-\tfrac{1}{2}s\rangle\langle q+\tfrac{1}{2}s|, \tag{B4}$$

$$2\pi\langle q''|\hat{\Delta}_w(q,p)|q'\rangle=[|q'\rangle\langle q''|]_q(q,p)$$
$$=e^{ip(q''-q')}\delta\left(q-\frac{q'+q''}{2}\right), \tag{B5}$$

$$2\pi\langle q''|\hat{\Delta}_w(r,\theta)|q'\rangle$$
$$=|\cos\theta|^{-1}e^{(1/2)i(q''^2-q'^2)\tan\theta}\delta\left(r-\frac{q'+q''}{2\cos\theta}\right). \tag{B6}$$

We have, reciprocally to (7.6a),

$$\langle q'|\hat{A}|q\rangle=(2\pi)^{-1}\int dp e^{i(q'-q)p}A_w\left(\frac{q+q'}{2},p\right). \tag{B7}$$

Introduce the following Hamiltonian, "squeezed" relative to $\hat{H}=\frac{1}{2}(\hat{q}^2+\hat{p}^2)$:

$$\hat{H}_\gamma=\tfrac{1}{2}(\gamma\hat{q}^2+\hat{p}^2/\gamma)=\hat{a}_\gamma^\dagger\hat{a}_\gamma+\tfrac{1}{2}, \tag{B8a}$$

$$\hat{a}_\gamma=2^{-1/2}(\gamma^{1/2}\hat{q}+i\hat{p}/\gamma^{1/2}), \quad \hat{a}_\gamma^\dagger=2^{-1/2}(\gamma^{1/2}\hat{q}-i\hat{p}/\gamma^{1/2}). \tag{B8b}$$

The Wigner function of the ground state $|0\rangle_\gamma$ of $\hat{H}_\gamma$ is the elliptic Gaussian

$$[|0\rangle_\gamma\langle 0|]_w(q,p)=\pi^{-1}e^{-\gamma q^2-p^2/\gamma}. \tag{B9}$$

The displaced and rotated states $|qp\rangle_{\gamma,\varphi}=\hat{D}_{qp}e^{i\varphi\hat{H}}|0\rangle_\gamma$ are *squeezed states* for the (unsqueezed) Hamiltonian $\hat{H}$. Their Wigner functions are obtained by rotating (B9), and then displacing it.

Applying (7.6b) to the Ω ordered δ operator (7.2), we get

$$\hat{\Delta}_\Omega(q,p)=\int dq'dp'G_\Omega(q'-q,p'-p)\hat{\Delta}_w(q',p'), \tag{B10}$$

where $G_\Omega(q,p)=[\hat{\Delta}]_w(q,p)$. Thus, if $G_\Omega(q,p)$ is well behaved, then $\hat{\Delta}7\Omega(q,p)$ is a "coarse graining" of $\hat{\Delta}_w(q,p)$. Substituting (B10) into (7.1), we get (7.9a). For $\hat{A}$ to be expressible as $\hat{A}=g(\hat{q},\hat{p})_\Omega$, for some Ω, it suffices that $\tilde{A}_w(0,0)=\tilde{g}(0,0)$ [which is equivalent to (7.11)], since we must have $\int dq\,dp\,G_\Omega(q,p)=\tilde{G}_\Omega(0,0)=1$. In general, $G_\Omega(q,p)$ is well behaved if $A_w(q,p)$ is a "smoothing" of $g(q,p)$.

For example, let $g(q,p)=\delta(q)\delta(p)$: Then $\tilde{g}(k,s)=1$, so that any $\hat{A}=\hat{\Delta}$ satisfying

$$2\pi\ \mathrm{Tr}\{\hat{\Delta}\}=\int dq\ dp[\hat{\Delta}]_w(q,p)=1 \qquad \text{(B11)}$$

[we used (A11)] is expressible at $\hat{\Delta}=[\delta(\hat{q})\delta(\hat{p})]_\Omega$ as was effectively asserted after Eq. (7.3). As another example, let $g(q,p)=\pi^{-1}e^{-(q^2+p^2)/a}$, and $A_w(q,p)=\pi^{-1}e^{-(q^2+p^2)/b}$. Then $\tilde{G}_\Omega(k,s)=e^{-(1/4)(b-a)(k^2+s^2)}$. If $b\geqslant a$, then $G_\Omega(q,p)=\pi^{-1}e^{-(q^2+p^2)/(b-a)}$ is well behaved; but if $b<a$, then $G_\Omega(q,p)$ is more singular than a tempered distribution, being defined only on test functions vanishing faster than $e^{-(q^2+p^2)/(a-b)}$.

It follows from (7.2) and (A10b) that

$$i[\hat{H},\hat{\Delta}_\Omega(r,\theta)]=\frac{\partial}{\partial\theta}\hat{\Delta}_\Omega(r,\theta)+i\hat{D}_{r\theta}[\hat{H},\hat{\Delta}_\Omega]\hat{D}_{r\theta}^{-1}, \qquad \text{(B12)}$$

whence we get (7.13). The (covariant) Cahill-Glauber s orderings have

$$\hat{\Delta}_s=\begin{cases}(2\pi s)^{-1}\hat{\Pi}\hat{\rho}_{\mathrm{th}}(2\tanh^{-1}s) & (0\leqslant s\leqslant 1),\\ (2\pi)^{-1}\hat{\rho}_{\mathrm{th}}(2\coth^{-1}s) & (s\geqslant 1),\end{cases} \qquad \text{(B13)}$$

where $\hat{\Pi}=\int dq|-q\rangle\langle q|$ is the parity operator, and

$$\hat{\rho}_{\mathrm{th}}(\beta)=\frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\{e^{-\beta\hat{H}}\}},\quad \hat{H}=\tfrac{1}{2}(\hat{p}^2+\hat{q}^2) \qquad \text{(B14)}$$

are thermal states. At $s=1$ (antinormal ordering), the "temperature" $(2\tanh^{-1}s)^{-1}\to 0$, so that $\hat{\Delta}_{s=1}=(2\pi)^{-1}|0\rangle\langle 0|$, and $\hat{\Delta}_{s=1}(q,p)$ are coherent states. The operators $\hat{\Delta}_s$ for $s<0$ are related to $\hat{\Delta}_w(q,p)$ by

$$(\pi s)^{-1}\int dq\ dp\ e^{-[(q-q')^2+(p-p')^2]/s}\hat{\Delta}_s(q',p')=\hat{\Delta}_w(q,p) \qquad (s\leqslant 0)\quad \text{(B15)}$$

and are highly singular, being *deconvolutions* of $\hat{\Delta}_w(q,p)$, which is already the sharpest possible.

Cahill-Glauber is a "natural" ordering of the creation and annihilation operators $\hat{a}=2^{1/2}(\hat{q}+i\hat{p})$ and $\hat{a}^\dagger$. To see this, let us define "left" and "right" *superoperators* $\hat{A}^{\leftarrow}$ and $\hat{A}^{\rightarrow}$ (i.e., operators acting on operators) by their actions on any operator $\hat{B}$ as

$$\hat{A}^{\leftarrow}\hat{B}=\hat{A}\hat{B},\quad \hat{A}^{\rightarrow}\hat{B}=\hat{B}\hat{A}. \qquad \text{(B16)}$$

Let the commutator $[\hat{A},\hat{B}]$ be a c number. We define s *ordering* of $\hat{A}$ and $\hat{B}$ as follows:

$$f(\hat{A},\hat{B})_s=f(\hat{A}^s,\hat{B}^{-s})\hat{1},\quad \hat{A}^s=\left(\frac{1+s}{2}\right)\hat{A}^{\leftarrow}+\left(\frac{1-s}{2}\right)\hat{A}^{\rightarrow} \qquad \text{(B17)}$$

for any function $f(x,y)$. This makes sense, because the superoperators $\hat{A}^s$ and $\hat{B}^{-s}$ commute (provided $[\hat{A},\hat{B}]$ is a c number). At $s=1$, we get $f(\hat{A},\hat{B})_{s=1}=f(\hat{A}^{\leftarrow},\hat{B}^{\rightarrow})\hat{1}$, that is "$\hat{A}$ to the left of $\hat{B}$" ordering. When $s\neq\pm 1$, each $\hat{A}$ or $\hat{B}$ is put partly to the left, partly to the right. One can show [45] that Cahill-Glauber ordering is s ordering of $\hat{a}^\dagger$ and $\hat{a}$:

$$f(\hat{q},\hat{p})_s=g(\hat{a}^{\dagger s},\hat{a}^{-s})\hat{1}\ \text{ if }\ f(q,p)\equiv g(\alpha^*,\alpha), \qquad \text{(B18)}$$

where $\alpha=2^{-1/2}(q+ip)$. This indeed yields normal ordering at $s=1$, and antinormal at $s=-1$.

An example of ordering which is *noncovariant* for $\hat{H}=\frac{1}{2}(\hat{q}+\hat{p}^2)$ is s ordering of the *squeezed* (relative to $\hat{\mathrm{H}}$) creation/annihilation operators defined in (B8b). Here, $G_{\gamma,s}(q,p)$ is an elliptic Gaussian

$$G_{\gamma,s}(q,p)=[\hat{\Delta}_{\gamma,s}]_w(q,p)=\frac{e^{-i(\gamma q^2+p^2/\gamma)/s}}{\pi s}\quad (s\geqslant 0). \qquad \text{(B19)}$$

For $s\geqslant 1$, $\hat{\Delta}_{\gamma,s}$ are thermal states for $\hat{H}_\gamma$, i.e., "thermal squeezed states" for $\hat{H}$, and that multiplied by $\hat{\Pi}$ for $0\leqslant s\leqslant 1$. Still other noncovariant orderings are obtained by using the rotated $\hat{\Delta}_{\gamma,\varphi,s}=e^{i\varphi\hat{H}}\hat{\Delta}_{\gamma,s}e^{-i\varphi\hat{H}}$. Again, as $s\to 1$, the "temperature" $\to 0$, and $\hat{\Delta}_{\gamma,\varphi,s=1}=|0\rangle_{\gamma,\varphi,s}\langle 0|$ is a squeezed state for $\hat{H}$.

APPENDIX C: POSITION EIGENKETS ROTATED IN PHASE SPACE

In view of (A8), the rotated position eigenket $e^{-i\varphi\hat{H}}|q\rangle$ is an eigenket of $\hat{q}_\varphi$, that is

$$e^{-i\varphi\hat{H}}|q\rangle=e^{i\chi(\varphi)}|q\rangle_{\hat{q}_\varphi}, \qquad \text{(C1)}$$

where an operator subscript on a ket will be used to indicate, when necessary, the operator of which the ket is an eigenket, for instance, $\hat{a}|x\rangle_{\hat{a}}=x|x\rangle_{\hat{a}}$. The phase $\chi(\varphi)$ in (C1) remains to be chosen: Note that at $\varphi=\frac{1}{2}\pi,\pi,3\pi/2,2\pi,\ldots$, etc., we have $\hat{q}_\varphi=\hat{p},-\hat{q},-\hat{p},\hat{q},\ldots$, etc. Now, with the standard choice of phase for $|p\rangle$, i.e., $\langle q|p\rangle=(2\pi)^{-1/2}e^{iqp}$, we have

$$\langle 0|q\rangle_{\hat{q}}=\langle 0|q\rangle_{\hat{p}}=\langle 0|q\rangle_{-\hat{q}}=\langle 0|q\rangle_{-\hat{p}}=\langle 0|q\rangle_{\hat{q}}$$
$$=\pi^{-1/4}e^{-(1/2)q^2}, \qquad \text{(C2)}$$

where $|0\rangle=|0\rangle_{\hat{N}}$ is the ground state of $\hat{H}=\hat{N}+\frac{1}{2}$. The simplest way to let (C1) satisfy (C2) is to choose $\chi(\varphi)$ such that $\langle 0|q\rangle_{\hat{q}_\varphi}=\langle 0|q\rangle$ for all φ [another possibility would be $\langle 0|q\rangle_{\hat{q}_\varphi}=e^{4i\varphi}\langle 0|q\rangle$]. Since we also have $\langle 0|q\rangle_{\hat{q}_\varphi}=e^{-i\chi(\varphi)}\langle 0|e^{-i\varphi\hat{H}}|q\rangle=e^{-i\chi(\varphi)}e^{-(1/2)i\varphi}\langle 0|q\rangle$, we must set $\chi(\varphi)=-\frac{1}{2}i\varphi$. Thus,

$$|q\rangle_{\hat{q}_\varphi}=e^{(1/2)i\varphi}e^{-i\varphi\hat{H}}|q\rangle,\quad |q\rangle_{\hat{p}}=e^{(1/4)i\pi}e^{-(1/2)i\pi\hat{H}}|q\rangle, \qquad \text{(C3)}$$

where we also wrote down the special case $\varphi=\frac{1}{2}\pi$.

Using the definitions (A4), we have (for $0\leqslant\varphi\leqslant 2\pi$)

$$\delta(p_\varphi)=\delta(q\ \sin\varphi-p\ \cos\varphi)=\delta(r\ \sin(\theta-\varphi)) \tag{C4a}$$

$$=\frac{\delta(\sin(\theta-\varphi))}{r}=\frac{\delta(\varphi-\theta)+\delta(\varphi+\pi+\theta)}{r}\quad (r\neq 0). \tag{C4b}$$

The Weyl symbol of the momentum state $|0\rangle_{\hat p}$ is $\delta(p)=\delta(r\sin\theta)$, so that by (8.12),

$$[e^{i\varphi\hat H}|0\rangle_{\hat p}\langle 0|e^{-i\varphi\hat H}]_w(q,p)=\delta(r\ \sin(\theta-\varphi))=\delta(p_\varphi). \tag{C5}$$

Using (B7), we then find that

$$\begin{aligned}\langle q'|e^{i\varphi\hat H}|0\rangle_{\hat p}\langle 0|e^{-i\varphi\hat H}|q\rangle &=(2\pi)^{-1}\int dp\ e^{ip(q'-q)}\delta\left(\frac{q+q'}{2}\sin\varphi-p\ \cos\varphi\right)\\ &=\frac{e^{(1/2)i(q'^2-q^2)\tan\varphi}}{2\pi|\cos\varphi|}.\end{aligned} \tag{C6}$$

It follows that

$$\langle q|e^{i\varphi\hat H}|0\rangle_{\hat p}=\frac{e^{(1/2)iq^2\tan\varphi}}{(2\pi\ \cos\varphi)^{1/2}} \tag{C7}$$

In fact, (C6) implies (C7) only up to a phase $e^{i\zeta(\varphi)}$ independent of q; this was fixed by requiring that ${}_{\hat N}\langle 0|e^{i\varphi\hat H}|0\rangle_{\hat p}=\pi^{1/4}e^{(1/2)i\varphi}$. The choice of phase (C7) indeed gives

$$\begin{aligned}{}_{\hat N}\langle 0|e^{i\varphi\hat H}|0\rangle_{\hat p}&=\int dq\langle 0|q\rangle\langle q|e^{i\varphi\hat H}|0\rangle_{\hat p}\\ &=\pi^{-1/4}(2\pi\ \cos\varphi)^{-1/2}\int dq\ e^{-(1/2)q^2(1-i\ \tan\varphi)}\\ &=\pi^{-1/4}e^{(1/2)i\varphi}(2\pi)^{-1/2}\int dx\ e^{-(1/2)x^2}\\ &=\pi^{-1/4}e^{(1/2)i\varphi},\end{aligned} \tag{C8}$$

where we set $x=q(1-i\ \tan\varphi)^{1/2}=qe^{-(1/2)i\varphi}(\cos\varphi)^{-1/2}$. By using (C3), we find that

$$\langle q|e^{i\varphi\hat H}|0\rangle_{\hat q}=e^{-(1/4)i\pi}\langle q|e^{i(\varphi+\pi/2)\hat H}|0_{\hat p}=\frac{e^{-(1/2)iq^2\cot\varphi}}{(-2\pi i\ \sin\varphi)^{1/2}}. \tag{C9}$$

There follows the standard result [48]

$$\begin{aligned}\langle q|e^{i\varphi\hat H}|q'\rangle&=\langle q|e^{i\varphi\hat H}e^{-iq'\hat p}|0\rangle_{\hat q}=\langle q|e^{-iq'\hat p_\varphi}e^{i\varphi\hat H}|0\rangle_{\hat q}\\ &=(-2\pi i\ \sin\varphi)^{-1/2}\\ &\quad\times\exp\left\{-\frac{i}{2\sin\varphi}[2qq'-(q^2+q'^2)\cos\varphi]\right\}\end{aligned} \tag{C10a}$$

$$=\frac{e^{(1/2)iq^2\tan\varphi}}{(-2\pi i\ \sin\varphi)^{1/2}}\exp\left\{\tfrac{1}{2}i\ \cot\varphi\left(q'-\frac{q}{\cos\varphi}\right)^2\right\}, \tag{C10b}$$

where we used (A10c), (A9b), and (C9). We could, of course, deduce (C6), (C7), (C9) from the known result (C10); but it is instructive to see how (C10) itself can be very simply deduced from Weyl symbols, as above. Using (C10b), we readily show that if $\langle q|\Psi\rangle=\Psi(q)$ is a slowly varying *real* function, hence localized about the axis $p=0$ in phase space [see (5.3)], then

$$\langle q|e^{i\varphi\hat H}|\Psi\rangle\approx\frac{e^{(1/2)iq^2\ \tan\varphi}}{(2\pi\ \cos\varphi)^{1/2}}\Psi(q/\cos\varphi)\quad[\Psi(q)\text{real}] \tag{C11}$$

localized about the line $p\approx q\tan\varphi$ in phase space: This follows from $\langle q|e^{i\varphi\hat H}|\Psi\rangle=\int dq'\langle q|e^{i\varphi\hat H}|q'\rangle\langle q'|\Psi\rangle$ and, provided $\Psi(q')$ varies slowly on the scale of $\tan\varphi$,

$$\begin{aligned}\int dq'\exp\left\{(\tfrac{1}{2}i\ \cot\varphi)\left(q'-\frac{q}{\cos\varphi}\right)^2\right\}\Psi(q')\\ \approx\Psi(q/\cos\varphi)(i\cot\varphi)^{-1/2}.\end{aligned} \tag{C12}$$

Consider now the overlap of $e^{i\varphi\hat H}|0\rangle_{\hat p}$ with $|0\rangle_{\hat p}$:

$${}_{\hat p}\langle 0|e^{i\varphi\hat H}|0\rangle_{\hat p}={}_{\hat q}\langle 0|e^{i\varphi\hat H}|0\rangle_{\hat q}=(-2\pi i\ \sin\varphi)^{-1/2}. \tag{C13}$$

We also have, by use of (5.15),

$$\begin{aligned}|{}_{\hat p}\langle 0|e^{-i\varphi\hat H}|0\rangle_{\hat p}|^2&=\mathrm{Tr}\{[|0\rangle_{\hat p}\langle 0|][e^{i\varphi\hat H}|0\rangle_{\hat p}\langle 0|e^{-i\varphi\hat H}]\}\\ &=\int dq\ dp\,\delta(p)\,\delta(q\ \sin\varphi-p\ \cos\varphi)\\ &=|\sin\varphi|^{-1}\end{aligned} \tag{C14}$$

in accord with (C13). Note that had we used the polar form (C4b), we would have gotten $\int_0^\infty r^{-1}dr[\delta(\varphi)+\delta(\varphi-\pi)]$, which is undefined: This reflects the fact that (C4b) is not valid at $r=0$, which is in fact the only point at which $\delta(p)$ and $\delta(p_\varphi)$ overlap when $\varphi\neq 0$. At $\varphi=\pi/2$, (C13) correctly yields $\langle p=0|q=0\rangle=(2\pi)^{-1/2}$, and (C14) becomes $(2\pi)^{-1}\int dq\ dp\,\delta(q)\,\delta(p)=(2\pi)^{-1}$: Note that this is the overlap of two lines, $\delta(q)$ and $\delta(p)$, which overlap at the *single* point (0,0); but, also, each is infinitely high, leading to a finite net overlap. At $\varphi=0$, (C14) has a nonintegrable singularity, because the two infinitely high lines then completely overlap.

APPENDIX D: POSITION REPRESENTATIONS

1. Phase δ operators

We start with the Weyl ordered $\hat{\Gamma}_w(\varphi)=\int_0^\infty r dr \hat{\Delta}_w(r,\varphi)$: Using (B6), we get

$$2\pi\langle q|\hat{\Gamma}_w(\varphi)|q'\rangle=e^{(1/2)i(q^2-q'^2)\tan\varphi}\frac{|q+q'|}{2\cos^2\varphi}\Theta\left(\frac{q+q'}{\cos\varphi}\right). \tag{D1}$$

One verifies that by substituting (D1) back into

$$[\hat{\Gamma}_w(\varphi)]_w(q,p)=\int ds\ e^{isp}\langle q+\tfrac{1}{2}s|\hat{\Gamma}_w(\varphi)|q-\tfrac{1}{2}s\rangle \tag{D2}$$

one gets back $[\hat{\Gamma}_w(\varphi)]_w(q,p)=\delta(\theta-\varphi)$, with

$$\delta(\theta-\varphi)=\delta(p-q\ \tan\varphi)\frac{|q|}{\cos^2\varphi}\Theta\left(\frac{q}{\cos\varphi}\right). \tag{D3}$$

In the case of a general phase δ operator, whose Weyl symbol $[\hat{\Gamma}(\varphi)]_w(r,\theta)\rightarrow\delta(\theta-\varphi)$ only as $r\rightarrow\infty$, we get, using (B7),

$$2\pi\langle q|\hat{\Gamma}(\varphi)|q'\rangle=\int dp\ e^{i(q-q')p}[\hat{\Gamma}(\varphi)]_w\left(\frac{q+q'}{2},p\right)$$
$$\rightarrow\text{(D1)}\quad\text{as }|\mathrm{q}+\mathrm{q}'|\rightarrow\infty, \tag{D4}$$

where we used $[\hat{\Gamma}(\varphi)]_w(q,p)\rightarrow$ (D3) as $q^2+p^2\rightarrow\infty$, hence also as $|q|\rightarrow\infty$.[One may also infer (D4) from the asymptotic identity of number matrix elements (see Sec. XIV), since $\langle q|n\rangle\neq 0$ only if $n\geqslant\frac{1}{2}q^2$, roughly.]

2. Pure phase states

Let now $\hat{\Gamma}(\varphi)=|\varphi\rangle\langle\varphi|$ be pure: Putting first $q'=q$ in (D4) yields $\langle q|\varphi\rangle\approx(2\pi)^{-1/2}(|q|^{1/2}/\cos\varphi)\Theta(q/\cos\varphi)$; then $q'\neq q$ implies

$$\langle q|\varphi\rangle\approx(2\pi)^{-1/2}\frac{|q|^{1/2}}{\cos\varphi}e^{(1/2)iq^2\tan\varphi}\Theta\left(\frac{q}{\cos\varphi}\right)\quad(q\rightarrow\infty) \tag{D5}$$

up to a phase depending only on φ. This indeed gives back (D4), because $|qq'|^{1/2}=|(X+\delta)(X-\delta)|^{1/2}=X|1-(\delta/X)^2|\approx X$, where $X=\frac{1}{2}(q+q'),\delta=\frac{1}{2}(q-q')$.

3. The Weyl phase operator

We now evaluate the matrix elements of $\hat{\phi}_w$. By (B7),

$$2\pi\langle q|\hat{\phi}_w|q'\rangle=\int_{-\infty}^{\infty}dp\ e^{ip(q-q')}\phi\left(\frac{q+q'}{2},p\right), \tag{D6}$$

where $\phi(q,p)=\tan^{-1}(p/q)$. As p ranges from $-\infty$ to 0 to $+\infty$, $\phi(q,p)$ ranges from $3\pi/2$ to $2\pi=0$ to $\frac{1}{2}\pi$ if $q>0$, and from $3\pi/2$ to π to $\frac{1}{2}\pi$ if $q<0$. We may thus write

$$\phi(q,p)=\begin{cases}\phi_0(q,p)+2\pi\Theta(-p), & q>0 \quad \text{(D7a)}\\ -\phi_0(q,p)+\pi, & q<0, \quad \text{(D7b)}\end{cases}$$

where $\phi_0(q,p)$ is the branch of $\tan^{-1}(p/q)$ which ranges from $-\frac{1}{2}\pi$ to 0 to $\frac{1}{2}\pi$ as p ranges from $-\infty$ to 0 to ∞. We next evaluate

$$\int_{-\infty}^{\infty}dp\ e^{ibp}\phi_0(q,p)=\phi_0(q,p)\frac{e^{ibp}}{ib}\bigg|_{-\infty}^{\infty}$$
$$-\frac{q}{ib}\int_{-\infty}^{\infty}dp\frac{e^{ibp}}{p^2+q^2}, \tag{D8}$$

where we performed an integration by parts, using $(d/dp)\phi_0(q,p)=q^{-1}(1+p^2/q^2)^{-1}$. The first term in (D8) contributes $\mathrm{Lim}_{L\rightarrow\infty}\frac{1}{2}\pi\cos(Lb)/ib=0$ (as a generalized function). The second integral is evaluated by the method of residues (poles at $\pm iq$), yielding

$$\int_{-\infty}^{\infty}dp\ e^{ibp}\phi_0(q,p)=(i\pi/b)e^{-|qb|}\ \mathrm{sgn}(q). \tag{D9}$$

We thus get

$$2\pi\langle q|\hat{\phi}_0|q'\rangle=\frac{i\pi}{x-y}e^{-|q^2-q'^2|}\ \mathrm{sgn}(q+q'). \tag{D10}$$

To get $\langle q|\hat{\phi}_w|q'\rangle$, one must, in view of (D7), add to (D10) the Fourier transforms of $2\pi\Theta(-p)$ or of π, namely $2\pi(q'-q+i\varepsilon)^{-1}$ or $\pi\delta(q'-q)$, depending on the sign of $q+q'$. Smith, Dubin, and Hennings [9(b)] obtain equivalent results (different in some details because they measure angles differently from us).

APPENDIX E: COHERENT-STATE REPRESENTATIONS

The ground state of $\hat{H}=\frac{1}{2}(\hat{q}^2+\hat{p}^2)$ is given by

$$|0\rangle=\pi^{-1/4}|e^{-(1/2)\hat{q}^2}\rangle=\pi^{-1/4}|e^{-(1/2)\hat{p}^2}\rangle, \tag{E1}$$

where, following Dirac, we denote by $|\Psi(\hat{q})\rangle$ the ket whose position representation is $\Psi(q)$: Thus, $\langle q|\Psi(\hat{q})\rangle=\Psi(q)$, and likewise, $\langle p|\phi(\hat{p})\rangle=\phi(p)$. Using (A9), we get, for coherent states,

$$|qp\rangle^{\mathrm{coh}}=\hat{D}_{qp}|0\rangle=\pi^{-1/4}e^{-(1/2)iqp}|e^{ip\hat{q}}e^{-(1/2)(\hat{q}-q)^2}\rangle \tag{E2a}$$

$$=\pi^{-1/4}e^{(1/2)iqp}|e^{-(1/2)(\hat{p}-p)^2}\rangle \tag{E2b}$$

In terms of polar coordinates, we have, using (E2a) and $q=r\cos\theta$, $p=r\ \sin\theta$:

$$\langle q|r\theta\rangle^{\mathrm{coh}}=\pi^{-1/4}e^{(1/2)iq^2\tan\theta}$$
$$\times\exp[-\tfrac{1}{2}(q-r\ \cos\theta)^2(1+i\ \tan\theta)] \tag{E3}$$

with local wave number $\langle p\rangle(q)\approx(\partial/\partial q)\frac{1}{2}[q^2-(q-r\ \cos\theta)^2]\tan\theta=r\ \sin\theta$ characteristic of localization along the line $p=r\ \sin\theta$. We also have, using $R=\frac{1}{2}(q^2+p^2)$,

$$\langle n|R\theta\rangle^{\mathrm{coh}}=(n!)^{-1/2}e^{-R/2}R^{n/2}e^{in\theta} \tag{E4a}$$

$$\approx(2\pi n)^{-1/4}e^{-(R-n)^2/(2n)}e^{in\theta} \tag{E4b}$$

[where we used (K16)]. This is localized on a ring of radius $R \approx n$, or $r \approx (2n)^{1/2}$, and width $\Delta R \approx n^{1/2}$, or $\Delta r \approx 1$ [since $r\Delta r = \Delta R$]. The height $\approx n^{-1/2}$. Following Schleich and Wheeler [18], we may represent the successive "n rings" by nonoverlapping rings of uniform height, each extending from $\rho = n^{1/2}$ to $\rho = (n+1)^{1/2}$ [where $\rho = R^{1/2}$]; their width $\approx n^{-1/2}$, so that we must set their heights ≈ 1.

Let us now consider [by (E2b)]

$$ {}_{\hat{p}}\langle 0|r\theta\rangle^{\mathrm{coh}} = \pi^{-1/4} e^{-(1/2)r^2\sin^2\theta} e^{(1/2)ir^2\sin\theta\cos\theta} \tag{E5a} $$

$$ \approx \pi^{-1/4}[e^{-(1/2)r^2\theta^2} e^{(1/2)ir^2\theta} + e^{-(1/2)r^2(\theta-\pi)^2} \times e^{(1/2)ir^2(\theta-\pi)}] \quad (r \to \infty). \tag{E5b} $$

We also have, in view of (A10c),

$$ e^{-i\varphi\hat{H}}|r\theta\rangle^{\mathrm{coh}} = e^{-i\varphi\hat{H}}\hat{D}_{r\theta}|0\rangle = \hat{D}_{r,\theta-\varphi}e^{-i\varphi\hat{H}}|0\rangle = e^{-(1/2)i\varphi}|r,\theta-\varphi\rangle^{\mathrm{coh}}, \tag{E6} $$

whence we get (15.2). The absolute square of (E5a) is the Q function of $|0\rangle_{\hat{p}}$, and is thus equal to the convolution of its Wigner function $\delta(p)$ with $\pi^{-1}e^{-q^2-p^2}$, which is indeed equal to $\pi^{-1/2}e^{-p^2}$.

Consider now a phase state $|\varphi\rangle$. Let, first, $\varphi = 0$: So at large q, $\langle q|\varphi=0\rangle \approx (2\pi)^{-1/2}q^{1/2}\Theta(q)$, by (D5). Then, at large r:

$$ \begin{aligned} {}^{\mathrm{coh}}\langle r\theta|\varphi=0\rangle &= \int dq\,{}^{\mathrm{coh}}\langle r\theta|q\rangle\langle q|\varphi=0\rangle \\ &\approx (2\pi)^{-1/2}\int dq\,{}^{\mathrm{coh}}\langle r\theta|q\rangle(r\cos\theta)^{1/2} \times\Theta(r\cos\theta) \\ &\approx {}^{\mathrm{coh}}\langle r\theta|0\rangle_{\hat{p}}(r\cos\theta)^{1/2}\Theta(\cos\theta) \\ &\approx \pi^{-1/4}r^{1/2}e^{-(1/2)ir^2\theta}e^{-(1/2)r^2\theta^2} \quad (r\to\infty). \end{aligned} \tag{E7} $$

The second near equality is because ${}^{\mathrm{coh}}\langle r\theta|q\rangle$ is sizable only at $q \approx r\cos\theta$; the third because $(2\pi)^{-1/2}\int dq|q\rangle = |0\rangle_{\hat{p}}$; the fourth because $\Theta(\cos\theta){}^{\mathrm{coh}}\langle r\theta|0\rangle_{\hat{p}}$ is given by the $\theta \approx 0$ term in (E5b). Noting now that $|\varphi=0\rangle = e^{-i\varphi\hat{H}}|\varphi\rangle$ is a phase state of phase $\varphi = 0$, we get, using (E7),

$$ {}^{\mathrm{coh}}\langle r\theta|\varphi\rangle = {}^{\mathrm{coh}}\langle r\theta|e^{i\varphi\hat{H}}|\varphi=0\rangle = {}^{\mathrm{coh}}\langle r,\theta-\varphi|\varphi=0\rangle e^{(1/2)i\varphi} \approx \pi^{-1/4}r^{1/2}e^{(1/2)i\varphi}e^{-(1/2)r^2[(\theta-\varphi)^2+i(\theta-\varphi)]} \quad (r\to\infty). \tag{E8} $$

Note that as $r\to\infty$, the phase $e^{(1/2)ir^2(\theta-\varphi)}$ oscillates rapidly, as a function of θ, inside the Gaussian $e^{-(1/2)r^2(\theta-\varphi)^2}$. The asymptotic form (E8) was obtained by Barnett and Pegg [Ref. [15], Eq. (43)] for the Susskind-Glogower states $|\varphi\rangle_{\mathrm{SG}}$ [they differ by a phase $e^{(1/2)\varphi}$ because they use $|\varphi\rangle_{\mathrm{SG}} = e^{i\varphi\hat{N}}|\varphi=0\rangle_{\mathrm{SG}}$, where $\hat{N} = \hat{H} - \frac{1}{2}$].

We now show that, reciprocally, (E8) implies (D5) for wave functions: Using the completeness of coherent states [or (A11)], we have

$$ \langle q|\varphi\rangle = (2\pi)^{-1}\int_0^{2\pi}d\theta\int_0^\infty r\,dr\langle q|r\theta\rangle^{\mathrm{coh}}\langle r\theta|\varphi\rangle. \tag{E9} $$

For large $q/\cos\varphi$, we can use (E8) for ${}^{\mathrm{coh}}\langle r\theta|\varphi\rangle$, since the real Gaussians in (E3) and (E8) force $r \approx q/\cos\varphi$. Putting $s = r - q/\cos\varphi$, we write $r^2 = q^2/\cos^2\varphi + 2(q/\cos\varphi)s + s^2$, and put $r^\alpha dr \approx (q/\cos\varphi)^\alpha\Theta(q/\cos\varphi)ds$. Now, because of $e^{-(1/2)r^2(\theta-\varphi)^2}$ in (E8), $(\theta-\varphi)$ is of order q^{-1}, which is small at large q: Expanding all functions of θ in powers of $(\theta-\varphi)$, and retaining only terms of order $\geqslant 0$ in q [treating $(\theta-\varphi)$ as q^{-1}], we get, noting that $\cos\varphi(1+i\tan\varphi) = e^{i\varphi}$,

$$ \langle q|r\theta\rangle^{\mathrm{coh}} \approx \pi^{-1/4}\exp\left\{\tfrac{1}{2}iq^2\left[\tan\varphi + \frac{\theta-\varphi}{\cos^2\varphi} + \frac{(\theta-\varphi)^2}{\cos^2\varphi}\tan\varphi\right]\right\}e^{-(1/2)s^2\cos\varphi e^{i\varphi}} \tag{E10a} $$

$$ {}^{\mathrm{coh}}\langle r\theta|\varphi\rangle \approx \frac{e^{(1/2)i\varphi}}{\pi^{1/4}}\left(\frac{q}{\cos\varphi}\right)^{1/2} \times\exp\left\{-\frac{1}{2}\frac{q^2}{\cos^2\varphi}(\theta-\varphi)^2(1+2i\tan\varphi) - \tfrac{1}{2}i\frac{q^2}{\cos^2\varphi}(\theta-\varphi) - i\frac{qs}{\cos\varphi}(\theta-\varphi)\right\}, \tag{E10b} $$

where we used $1/\cos^2\theta = 1/\cos^2\varphi + 2(\theta-\varphi)\tan\varphi/\cos^2\varphi + \cdots$. Note that the rapidly oscillating phases $e^{\pm(1/2)i(q/\cos\varphi)^2(\theta-\varphi)}$ in (E10a) and (E10b) cancel one another in the integral (E9). We get

$$ \begin{aligned} \langle q|\varphi\rangle &\approx \frac{e^{(1/2)i\varphi}}{2\pi^{3/2}}\left(\frac{q}{\cos\varphi}\right)^{3/2}\Theta\left(\frac{q}{\cos\varphi}\right)\int_0^{2\pi}d\theta\int_{-\infty}^\infty ds\,\exp\left\{-\frac{1}{2}\left(\frac{q^2}{\cos^3\varphi}\right)(\theta-\varphi)^2e^{i\varphi} - \tfrac{1}{2}s^2e^{i\varphi}\cos\varphi - i\frac{qs}{\cos\varphi}(\theta-\varphi)\right\} \\ &= (2\pi^2)^{-1/2}\Theta\left(\frac{q}{\cos\varphi}\right)\frac{q^{3/2}}{\cos^2\varphi}\int_0^{2\pi}d\theta\,\exp\left\{-\frac{1}{2}\left(\frac{q^2}{\cos^3\varphi}\right)(\theta-\varphi)^2(e^{i\varphi}+e^{-i\varphi})\right\} = (\mathrm{D5}). \end{aligned} \tag{E11} $$

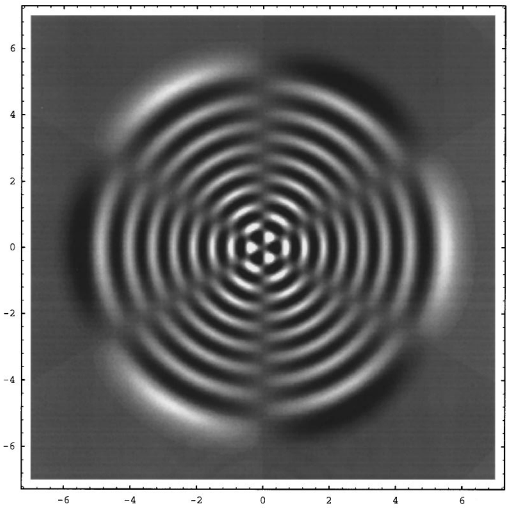

FIG. 11. Density plot showing the real part of $\langle n'|\hat{\Delta}_w(r,\theta)|n\rangle$, Eq. (F4), for $n=15$ and $n'=18$ (white is positive, black is negative).

One could also show the above, more simply, first for $\varphi=0$ and then for any φ by using (C11).

From (E8), we deduce that

$$|{}^{\mathrm{coh}}\langle r\theta|\varphi\rangle|^2 \approx \pi^{-1/2} r e^{-r^2(\theta-\varphi)^2} \quad (r\to\infty) \qquad \text{(E12)}$$

We also have, in view of (5.15), and the fact that the Wigner function of a large amplitude coherent state is given by (8.4) with $\gamma=1$,

$$|{}^{\mathrm{coh}}\langle r\theta|\varphi\rangle|^2 \approx \pi^{-1}\int_0^{2\pi} d\theta' \int_0^{\infty} r'dr' [|\varphi\rangle\langle\varphi|]_w(r',\theta') \times e^{-r'^2(\theta'-\theta)^2} e^{-(r'-r)^2} \qquad \text{(E13a)}$$

$$\approx \pi^{-1/2} r \int_0^{2\pi} d\theta' [|\varphi\rangle\langle\varphi|]_w(r,\theta') e^{-r^2(\theta'-\theta)^2}. \qquad \text{(E13b)}$$

Comparing (E13b) with (E12), we see that as $r\to\infty$, the angular function $e^{-r^2(\theta-\theta')^2}$ does not get broadened by convolution with $[|\varphi\rangle\langle\varphi|]_w(r,\theta')$; this again implies that the latter tends to a *sharp* radius of zero width (not just zero *angular* width) as $r\to\infty$.

APPENDIX F: ASYMPTOTICS OF NUMBER MATRIX ELEMENTS

In view of (E4a), the number matrix elements of coherent states $\hat{\Delta}^{\mathrm{coh}}(r,\theta)=(2\pi)^{-1}|r\theta\rangle^{\mathrm{coh}}\langle r\theta|$ are given by [recall that $R=\frac{1}{2}(q^2+p^2)=\frac{1}{2}r^2$]

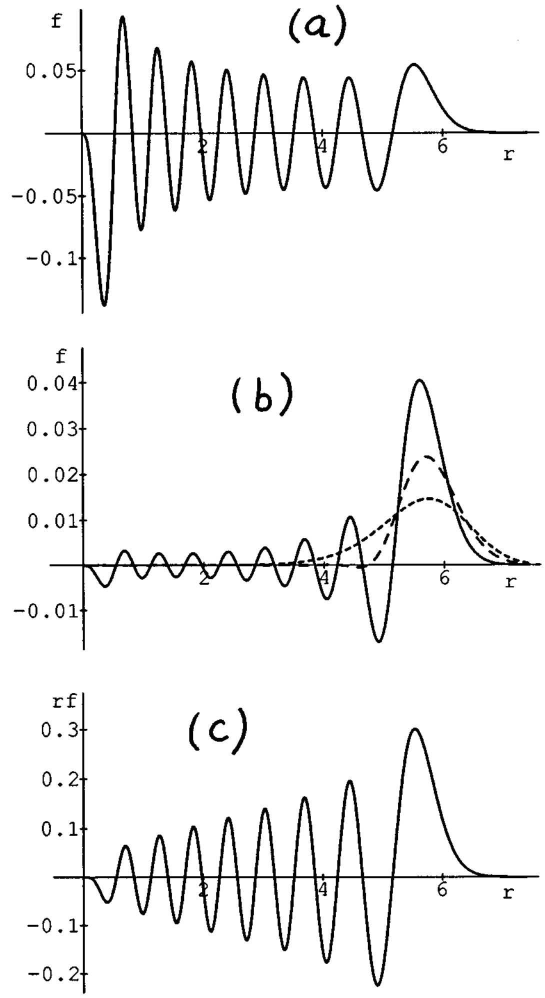

FIG. 12. (a) The function $f_{nn'}(r)$, Eq. (F4c), and (c) $rf_{nn'}(r)$, for $n=15$, $n'=18$. (b) The radial part of $\langle n'|\hat{\Delta}_s(r,\theta)|n\rangle$, Eq. (G1) $[R=\frac{1}{2}r^2]$, for $n=15$, $n'=18$, and three different values of s: $s=0.1$ (full line), $s=0.4$ (long dashes), and the approximation (F2) to $s=1$ (short dashes).

$$2\pi\langle n'|\hat{\Delta}_{s=1}(r,\theta)|n\rangle = \frac{e^{i(n'-n)\theta}}{(n!n'!)^{1/2}} e^{-R} R^{(n+n')/2} \qquad \text{(F1)}$$

$$\approx \frac{[(n+n')/2]!}{(n!n'!)^{1/2}} \frac{e^{i(n'-n)\theta}}{\pi^{1/2}(n+n')^{1/2}} \times \exp\left\{-\frac{[R-\frac{1}{2}(n+n')]^2}{n+n'}\right\} \quad (n+n'\gg 1) \qquad \text{(F2)}$$

[by (K16)]. At $(n+n')\gg 1$, this is radially localized about $R\approx\frac{1}{2}(n+n')$, or $r\approx(n+n)^{1/2}$, with a dispersion $\Delta R\approx(n+n')^{1/2}$, or $\Delta r\approx 1$ [since $\Delta R\approx r\Delta r$]. Note that if

$$n+n'\to\infty,\quad \frac{(n'-n)^2}{n+n'}\ll 1, \tag{F3}$$

then $[\frac{1}{2}(n+n')]!\approx(n!n'!)^{1/2}$, by (K8b), so that the radial Gaussian in (F2) is normalized [also, if (F3) holds, then (F2) follows directly from (E4b)].

We will suppose in the following that $n'\geqslant n$ [to get the reverse, simply use $\langle n'|\hat{\Delta}|n\rangle=\langle n|\hat{\Delta}|n'\rangle^*$, since $\hat{\Delta}^\dagger=\hat{\Delta}$]. The matrix elements of the Weyl ordered $\hat{\Delta}_w(q,p)$ are given by (G1) with $s=0$:

$$\langle n'|\hat{\Delta}_w(r,\theta)|n\rangle=[|n\rangle\langle n'|]_w(r,\theta) \tag{F4a}$$

$$=e^{i(n'-n)\theta}f_{nn'}(r) \tag{F4b}$$

$$f_{nn'}(r)=(2\pi)^{-1}2^{(1/2)(n'-n)+1}\times(-)^n\left(\frac{n!}{n'!}\right)^{1/2}r^{n'-n}e^{-r^2}L_n^{n'-n}(2r^2). \tag{F4c}$$

Here, $L_n^{n'-n}(2r^2)$ is a generalized Laguerre polynomial, of order r^{2n}, having n zeros roughly spaced by $\Delta r\approx n^{-1/2}$. The function (F4) is shown in Fig. 11. For $n\gg 1$, $f_{nn'}(r)$ has the general shape shown in Fig. 12(a): It consists of a hump around $r\approx(n'+n)^{1/2}$; farther out, it vanishes like $r^{2n}e^{-r^2}$; on the inside, it oscillates rapidly with a wavelength $n^{-1/2}$, and is approximately given, for $r\gg n^{-1/2}$, by [from (K19)]

$$f_{nn'}(r)\approx(-)^n2^{-1/4}\pi^{-3/2}n^{-1/4}r^{-1/2}\times\cos[(8n)^{1/2}r-\tfrac{1}{2}(n'-n)\pi-\tfrac{1}{4}\pi]+O(n^{-1/2}). \tag{F5}$$

Thus, if $n\gg 1$, then (F4) is *effectively* localized on a ring of radius $r\approx(n'+n)^{1/2}$ and of width ≈ 1, since it vanishes rapidly outside it, and oscillates rapidly inside it. We note that the oscillations of (F4) average to zero when smeared with $\pi^{-1}e^{-q^2-p^2}$: Indeed, we then get (F1), since $\hat{\Delta}^{\mathrm{coh}}(q,p)$ is the convolution of $\hat{\Delta}_w(q,p)$ with $\pi^{-1}e^{-q^2-p^2}$[see (B10) and (B9) $(\gamma=1)$].

We have, for any operator $\hat{A}$,

$$2\pi\langle n'|\hat{A}|n\rangle=\int_0^\infty rdr\int_0^{2\pi}d\theta\langle n'|\hat{\Delta}_w(r,\theta)|n\rangle A_w(r,\theta) \tag{F6a}$$

$$=\int_0^\infty dr\int_0^{2\pi}d\theta\; e^{i(n'-n)\theta}rf_{nn'}(r)A_w(r,\theta), \tag{F6b}$$

where we used $\langle n'|\hat{A}|n\rangle=\mathrm{Tr}\{|n\rangle\langle n'|\hat{A}\}$, and then (5.15) and (F4a). The function $rf_{nn'}(r)$ has the allure shown in Fig. 12(c). Let $\hat{A}$ *correspond* to $A(\theta)$, so that $A_w(r,\theta)\approx A(\theta)$ for $r>r_a$, say. It follows that if $r>r_a$, then smoothing of $A_w(r,\theta)\approx A(\theta)$ with $\pi^{-1}e^{-q^2-p^2}$ affects only its angular behavior; this will not be felt in the part of the integral (F6b) coming from regions where $e^{i(n'-n)\theta}$ varies slowly on the scale of the angular width $\Delta\theta\approx r^{-1}$ of $\pi^{-1}e^{-q^2-p^2}$ (see Fig. 11), that is, from regions $r>r_b\approx 10|n'-n|$, say. Let then $r_c=\mathrm{Max}\{r_a,r_b\}$, and let $(n+n')>r_c+5$, say, so that the final hump of $f_{nn'}(r)$ lies beyond r_c: This requires in particular that $(n+n')^{1/2}>r_b$, hence $(n+n')^{1/2}\gg|n'-n|$, that is, (F3) must be satisfied. Let us now break up the integral (F6) into two pieces, $r<r_c$ and $r>r_c$. In the outer $(r>r_c)$ part of the integral, we can smooth $A_w(r,\theta)\approx A(\theta)$ with $\pi^{-1}e^{-q^2-p^2}$ without changing the value of the integral, as just argued; transferring the smoothing to $\hat{\Delta}_w(r,\theta)$ changes the latter into $\hat{\Delta}^{\mathrm{coh}}(r,\theta)$, so that we get a contribution [provided (F3) is satisfied]

$$I_{\mathrm{out}}\approx\int_{r_c}^\infty r\;dr\int_0^{2\pi}d\theta\langle n'|\hat{\Delta}^{\mathrm{coh}}(r,\theta)|n\rangle A(\theta) \tag{F7a}$$

$$\approx\int_0^{2\pi}d\theta\; e^{i(n'-n)\theta}A(\theta), \tag{F7b}$$

where we used (F2) with $(n+n')!\approx(n!n'!)^{1/2}$ [under (F3)]. The inner part $(r\leqslant r_c)$ of the integral (F6) contributes, in view of (F5) (we neglect unessential factors)

$$I_{\mathrm{in}}\sim n^{-1/4}\int_0^{r_c}drA_w(r,\theta)r^{1/2}\times\cos[(8n)^{1/2}r-\tfrac{1}{2}(n'-n)\pi-\tfrac{1}{4}\pi]. \tag{F8}$$

As n gets large, the oscillations of the cos, of wavelength $\approx n^{-1/2}$, get increasingly rapid: So, provided only that $A_w(r,\theta)$ is bounded, (F8)$\leqslant$ $\mathrm{Max}|A_w(r,\theta)|r_c^{1/2}n^{-1/4}n^{-1/2}\sim n^{-3/4}$, which gets negligible as $n\to\infty$, compared to the outer contribution (F7), or order n^0. There follows (14.2).

We will now use a Weyl phase-space analysis to understand the differences between (14.3a), i.e., $2\pi\langle n'|\hat{\Gamma}(\varphi)|n\rangle\to e^{i(n'-n)\varphi}$, and the corresponding result for a rotated momentum state

$$\hat{B}(\varphi)=e^{i\varphi\hat{H}}|p=0\rangle\langle p=0|e^{-i\varphi\hat{H}} \tag{F9a}$$

whose Weyl symbol is

$$[\hat{B}(\varphi)]_w(r,\theta)=\frac{\delta(\theta-\varphi)+\delta(\theta-\varphi-\pi)}{r}. \tag{F9b}$$

We have

$$2\pi\langle n'|\hat{B}(\varphi)|n\rangle\approx e^{i\varphi(n'-n)}\frac{4}{(n+n')^{1/2}},$$

$$\left(n+n'\to\infty,\;\frac{|n'-n|}{n+n'}\ll 1\right), \tag{F10}$$

if $(n,n')=$(even, even), and zero otherwise, where we used (14.13) with $nn'=(N+\nu)(N-\nu)=N^2(1-\nu^2/N^2)\approx\frac{1}{4}(n+n')^2$, if $N=\frac{1}{2}(n+n')$, $\nu=\frac{1}{2}(n'-n)$. Putting $\hat{A}=\hat{B}(\varphi)$ in (F6) and using (F9b), we get

$$\langle n'|\hat{B}(\varphi)|n\rangle = e^{i(n'-n)\varphi}[1+e^{i\pi(n'-n)}]I_{nn'} \quad \text{(F11a)}$$

$$I_{nn'} = \int_0^\infty dr\ f_{nn'}(r). \quad \text{(F11b)}$$

Because $1+e^{i\pi(n'-n)}=0$ if $n'-n$ is odd, (F11a)=0 if (n,n')=(even, odd) or (odd, even), as it must. If (n,n')=(odd, odd), then (F11a) must vanish, so that $I_{nn'}$ has to vanish. Let us analyze (F6) with $\hat{A}$ replaced by $\hat{B}(\varphi)$, similarly as we did above for the case that $A_w(r,\theta)\to A(\theta)$. Here, we have, rather, $B_w(r,\theta)\to r^{-1}B(\theta)$. This varies slowly radially provided $r\gg 1$. So for $r>r_c=\text{Max}\{10,\ r_b\}$, say [$r_b$ as defined after (F6)], we can again smooth $B_w(r,\theta)$ with $\pi^{-1}e^{-q^2-p^2}$ without altering the value of the $r>r_c$ part of the integral (F6), which thus contributes

$$I_{\text{out}}=(n+n')^{-1/2}, \quad \text{(F12)}$$

where we approximated $r^{-1}\approx(n+n')^{-1/2}$ in the region of the $(n+n')$ ring. Consider now the inner integral $r<r_c$: Due to the factor $r^{-1/2}$ in (F5), it is dominated by the region $(0,\ r_0)$, where $r_0\approx n^{-1/2}$ is the first zero of $f_{nn'}(r)$[see Fig. 12(a)]; this contributes

$$I_{\text{in}}\sim n^{-1/4}\int_0^{r_0}dr\ r^{-1/2}\sim n^{-1/4}r_0^{1/2}\sim n^{-1/2}. \quad \text{(F13)}$$

Later oscillations in $f_{nn'}(r)$ modify the exact value of I_{in}, but not its order in n, nor its overall sign. Here, I_{in} is of the same size as I_{out}. Also, we easily see that the sign of I_{in} [which is the same as that of $f_{nn'}(r<r_0)$], and the sign of I_{out} [always "plus" since the final hump $\sim r^{2n}e^{-r^2}$] are the same if n is even, and opposite if n is odd [since $f_{nn'}(r)$ has n zeros between 0 and the final hump]. For n odd, I_{in} and I_{out} must cancel exactly. We may then surmise that they have the same absolute value also when n is even, and thus expect $I_{nn'}=2I_{\text{out}}=2(n+n')^{-1/2}$, as is indeed the case, from comparing with (F10). Thus, the extra factor $4(n+n')^{-1/2}$ in (F10), as compared to $2\pi\langle n'|\hat{\Gamma}(\varphi)|n\rangle\to e^{i(n'-n)\varphi}$, has the following origin: $(n+n')^{-1/2}$ from the extra r^{-1} in (F9b), roughly equal to $(n+n')^{-1/2}$ in the region of the $(n+n')$ ring; a factor 2 because the inside region contributes the same as the outside one when n is even [the opposite when n is odd]; another 2 because (F9) intercepts the ring twice (see Fig. 8).

APPENDIX G: NUMBER MATRIX ELEMENTS

The matrix elements $\langle n'|\hat{\Delta}_s(R,\theta)|n\rangle$ were given by Cahill and Glauber in the last equation on page 1881 of Ref. [30] (a square root is missing there; their s is minus our s, their α is our $2^{-1/2}(q+ip)$, and their $T(\alpha,s)=\pi\delta^{(2)}(\hat{a}-\alpha)$ [their Eq. (6.13)] is our $2\pi\hat{\Delta}_s(q,p)=2\pi[\delta(\hat{q}-q)\delta(\hat{p}-p)]_s$): For $n'\geqslant n$,

$$2\pi\langle n'|\hat{\Delta}_s(R,\theta)|n\rangle=\left(\frac{n!}{n'!}\right)^{1/2}\left(\frac{2}{1+s}\right)^{n'-n+1}\left(\frac{s-1}{s+1}\right)^n \times e^{-i(n'-n)\theta}R^{(n'-n)/2}e^{-2R/(1+s)} \times L_n^{n'-n}\left(\frac{4R}{1-s^2}\right), \quad \text{(G1)}$$

where $L_n^{n'-n}(x)$ are generalized Laguerre polynomials (K17). At $s=1$, only the term of order n in the Laguerre polynomial survives [because of the prefactor $(s-1)^n$], and we get (F1). At $s=0$, we get (F4). Figure 12(b) shows the radial part of (G1) for three values of s, showing the transition from (F4) at $s=0$ [Fig. 12(a)] to (F1) at $s=1$. Denote

$$N=\tfrac{1}{2}(n+n'),\quad \nu=\tfrac{1}{2}(n'-n). \quad \text{(G2)}$$

We now have [recall that $R=\rho^2$]

$$2\pi\langle n'|\hat{\Gamma}_s|n\rangle=\int_0^\infty 2\rho\ d\rho\langle n'|\hat{\Delta}_s(\rho,0)|n\rangle \quad \text{(G3a)}$$

$$=2(-)^n\left(\frac{n!}{n'!}\right)^{1/2}\left(\frac{2}{1+s}\right)^{n'-n+1}\left(\frac{1-S}{1+s}\right)^n \times\int_0^\infty\rho^{n'-n+1}d\rho\ \exp\left(-\frac{2\rho^2}{1+s}\right) \times L_n^{n'-n}\left(\frac{4\rho^2}{1-s^2}\right) \quad \text{(G3b)}$$

$$=\tfrac{1}{2}(-)^n\left(\frac{n!}{n'!}\right)^{1/2}(1-s) \times\left(\frac{1-s}{1+s}\right)^{(1/2)(n'+n)}F_n(\nu,s), \quad \text{(G3c)}$$

where we denoted

$$F_n(\nu,s)=\int_0^\infty dx\ x^\nu e^{-(1/2)(1-s)x}L_n^{2\nu}(x). \quad \text{(G4)}$$

This is evaluated in Appendix J. Putting $s=0$, and using (J3)–(J4c), we get (recall that $n'\geqslant n$)

$$2\pi\langle n'|\hat{\Gamma}_{s=0}|n\rangle=2^{(n-n')}\left(\frac{n'!}{n!}\right)^{1/2}\frac{\Gamma(\frac{1}{2}n+\frac{1}{2}+\frac{1}{2}\mu_n)}{\Gamma(\frac{1}{2}n'+\frac{1}{2}+\frac{1}{2}\mu_n)} \quad (\mu_n=n\ \text{mod}2). \quad \text{(G5)}$$

Putting $s=1$, and using (J2a) and (K10b), we get

$$\langle n'|\hat{\Gamma}_{s=1}|n\rangle=(n'!n!)^{-1/2}\Gamma(N+1) \quad \text{(G6)}$$

as is also immediate from (F1), whereby $2\pi\langle n'|\hat{\Gamma}_{s=1}|n\rangle=(n'!n!)^{-1/2}\int_0^\infty dRR^Ne^{-R}=$(G6), by (K1). Let us now obtain large N approximations: To first order in N^{-1}, we have, by (K8),

$$\left(\frac{m!}{n!}\right)^{1/2}=\left(\frac{(N+\nu)!}{(N-\nu)!}\right)^{1/2}\approx N^\nu(1+\tfrac{1}{2}\nu/N). \quad \text{(G7)}$$

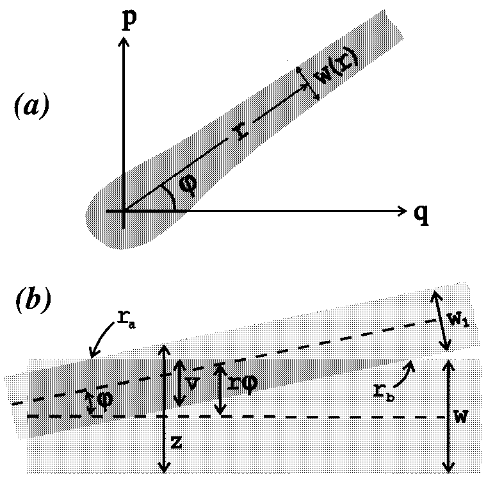

FIG. 13. (a) The Wigner function of a phase state; (b) the overlap of two phase states.

Using this together with (J6), noting that $n^{-\nu} = (N-\nu)^{-\nu} \approx N^{-\nu}(1+\nu^2/N)$, and $n^{-1} \approx N^{-1}(1+\nu/N)$, we get

$$2\pi\langle n'|\hat{\Gamma}_s|n\rangle = 1 - \frac{s\nu^2}{2N} + \frac{(-)^n}{2N}\left(\frac{1-s}{1+s}\right)^N (1-s)\nu + \cdots. \tag{G8}$$

Equation (G8) for $s=0$ and $s=1$ can also be obtained directly from (G5) and (G6) by using (K8).

APPENDIX H: NORMALIZED PHASE STATES

Let $\hat{\Gamma}^{\varepsilon}(\varphi)$ be defined as in (10.1), and consider [by (5.16)]

$$K(\varepsilon) = \mathrm{Tr}\{\hat{\Gamma}^{\varepsilon\prime}(\varphi)\} = \varepsilon^2 \int_0^\infty r\, dr \int_0^{2\pi} d\theta\, e^{-\varepsilon r}[\hat{\Gamma}(\varphi)]_w(r,\theta). \tag{H1}$$

As $\varepsilon \to 0$, the integral (H1) is dominated by large r, at which $[\hat{\Gamma}(\varphi)]_w(r,\theta) \approx \delta(\theta-\varphi)$, so that we get

$$K(\varepsilon) \to \varepsilon^2 \int_0^\infty r\, dr\, e^{-\varepsilon r} = 1 \quad (\varepsilon \to 0). \tag{H2}$$

We may rewrite (10.1a) as

$$[\hat{\Gamma}^{\varepsilon}(q,p)]_w(q,p) = [2\pi/K(\varepsilon)]\varepsilon^2 e^{-\varepsilon q/\cos\varphi}[\hat{\Gamma}(\varphi)]_w(q,p)$$

$$(\varepsilon \to 0) \tag{H3}$$

since, as $\varepsilon \to 0$, the factor $e^{-\varepsilon r} = e^{-\varepsilon q/\cos\theta}$ differs from 1 only at large r, where $\cos\theta \approx \cos\varphi$ because $[\hat{\Gamma}(\varphi)]_w(r,\theta) \approx \delta(\theta-\varphi)$. We obtain, using (B7),

$$\langle q|\hat{\Gamma}^{\varepsilon}(\varphi)|q'\rangle = [q\pi/K(\varepsilon)]\varepsilon^2 e^{-(1/2)(q+q')/\cos\varphi}\langle q|\hat{\Gamma}(\varphi)|q'\rangle \tag{H4}$$

and $K(\varepsilon)$ can be expressed as

$$K(\varepsilon) = \mathrm{Tr}\{\hat{\Gamma}^{\varepsilon\prime}(\varphi)\} = 2\pi\varepsilon^2 \int dq\, e^{-\varepsilon q/\cos\varphi}\langle q|\hat{\Gamma}(\varphi)|q\rangle. \tag{H5}$$

If $\hat{\Gamma}(\varphi) = |\varphi\rangle\langle\varphi|$, then the above imply (10.4) and (18.7b) with

$$K(\varepsilon) = 2\pi\varepsilon^2 \int dq \varepsilon^{-\varepsilon q/\cos\varphi}|\langle q|\varphi\rangle|^2. \tag{H6}$$

Using the asymptotic form (D5), one verifies again that $K(\varepsilon) \to 1$ as $\varepsilon \to 0$, and we have

$$\langle q|\varphi\rangle^{\varepsilon} \approx (2\pi)^{1/2}\frac{q^{1/2}}{\cos\varphi}\varepsilon e^{-(1/2)\varepsilon q/\cos\varphi} e^{-(1/2)iq^2\tan\varphi}\Theta\left(\frac{q}{\cos\varphi}\right)$$

$$(q \to \infty, \varepsilon \to 0). \tag{H7}$$

APPENDIX I: TRACE SCALAR PRODUCTS OF PHASE STATES

We here estimate in an intuitive manner "scalar products"

$$S(\varphi) = \mathrm{Tr}\{\hat{\Gamma}_2(\varphi)\hat{\Gamma}_1(0)\}$$

$$= \frac{1}{2\pi}\int dq\, dp[\hat{\Gamma}_2(\varphi)]_w(q,p)[\hat{\Gamma}_1(0)]_w(q,p), \tag{I1}$$

where $\hat{\Gamma}(\varphi)$ are phase δ operators. Their Weyl symbols satisfy

$$h(r;\theta;\varphi) \equiv [\hat{\Gamma}(\varphi)]_w(r,\theta) \to \delta(\theta-\varphi) \quad \text{as} \quad r \to \infty. \tag{I2}$$

We represent $h(r,\theta;\varphi)$ by a ribbon centered on the radius $\theta=\varphi$, with an angular width $\omega(r)$, hence true width $w(r) = r\omega(r)$, depending on r (Fig. 13); the value of $h(r,\theta;\varphi)$, i.e., its "height," is taken uniform across the width of the ribbon, and given by

$$h(r) = \omega(r)^{-1} = rw(r)^{-1} \tag{I3}$$

in order that the integral of the height over a cross section of the ribbon be equal to $\int_0^{2\pi} d\theta\,\delta(\theta-\varphi) = 1$. We suppose that $w_2(r) \geq w_1(r)$ for all r. Referring to Fig. 13(b), we have

$$2\pi S(\varphi) \approx \int_0^\infty dr\, v(r)h_1(r)h_2(r), \tag{I4}$$

where the overlap increment $v(r)dr$ is given by

$$v(r) = \begin{cases} w_1(r), \quad 0 \leq r \leq r_a, & \text{(I5a)} \\ \dfrac{w_1(r)+w_2(r)}{2} - r\varphi, \quad r_a \leq r \leq r_b\,, & \text{(I5b)} \\ 0, \quad r \geq r_b\,, & \text{(I5c)} \end{cases}$$

where $r_a(\varphi)$ and $r_b(\varphi)$ are the solutions of

$$r_a\varphi = \frac{w_2(r_a) - w_1(r_a)}{2}, \quad r_b\varphi = \frac{w_2(r_a) + w_1(r_a)}{2}, \tag{I6}$$

[a simple way to get (I5b) is to note that the length marked z on Fig. 13(b) has the two obvious expressions $z=w_1+w_2-v$ and $z=r\varphi+\frac{1}{2}w_1+\frac{1}{2}w_2$].

Example 1. Let the (asymptotic) widths $w_1(r)=w_1$ and $w_2(r)=w_2$ be independent of r [this corresponds, e.g., to the Cahill-Glauber $\hat{\Gamma}_s(\varphi)$, whose width $\approx s$]. We here have $r_a=(w_2-w_1)/\varphi$, $r_b=(w_1+w_2)/\varphi$, $h_i=r/w_i$, so that

$$2\pi S(\varphi)\approx\int_0^{r_a}dr\ w_1\frac{r^2}{w_1w_2}+\int_{r_a}^{r_b}dr\left[\frac{w_1+w_2}{2}-r\varphi\right]\frac{r^2}{w_1w_2} \tag{I7a}$$

$$=f(w_1,w_2)|\varphi|^{-1}\quad(\varphi\neq 0), \tag{I7b}$$

where $f(w_1,w_2)$ is an algebraic function of little interest. If we put $w_1=w_2=\varepsilon$, we have $r_a=0$, $r_b=\varepsilon/\varphi$, and

$$2\pi S(\varphi)\approx\int_0^{\varepsilon/\varphi}dr(\varepsilon-r\varphi)(r/\varepsilon)^2=\tfrac{1}{12}\varepsilon^2|\varphi|^{-3}\quad(\varphi\neq 0). \tag{I8}$$

The zero width Weyl case was treated in (16.4). Note that if we replace $\hat{\Gamma}_w^{\varepsilon}(\varphi)$ in (16.4) by a *normalized* (approximate) rotated momentum state

$$[\hat{\rho}^{\varepsilon}(\varphi)]_w(r,\theta)=2\pi\varepsilon e^{-\varepsilon r}[\delta(\theta-\varphi)+\delta(\theta-\varphi-\pi)]/r, \tag{I9a}$$

$$\mathrm{Tr}\{\hat{\rho}^{\varepsilon}(\varphi)\}=(2\pi)^{-1}\int_0^{\infty}r\ dr\ d\theta[\hat{\rho}(\varphi)]_w(r,\theta)=1, \tag{I9b}$$

then we get

$$\mathrm{Tr}\{\hat{\Gamma}_w(\varphi')\hat{\rho}^{\varepsilon}(\varphi)\}=\delta(\varphi'-\varphi)+\delta(\varphi'-\varphi-\pi). \tag{I10}$$

Also, if we replace $\hat{\Gamma}_1$ in (I7) by a *smeared* rotated momentum state of constant true width, then the height $h_1(r)=w_1^{-1}$ (instead of r/w_1), and we get $S(\varphi)\approx|\varphi|^{-2}$.

Example 2. $\omega_1(r)=\omega_2(r)=\gamma r^{-y}$. Here $r_a=0$, while r_b is the solution of $r\varphi=r\omega(r)=\gamma r^{1-y}$, that is, $r_b=|\gamma/\varphi|^{1/y}$. Then

$$2\pi S(\varphi)=\int_c^{r_b}dr\ v(r)h(r)^2=\int_c^{r_b}dr(\gamma r^{1-y}-r\varphi)r^{2y}/\gamma^2 \tag{I11a}$$

$$=\frac{y}{2\gamma(y+1)(y+2)}|\varphi/\gamma|^{-(y+2)/y}\quad(\varphi\neq 0), \tag{I11b}$$

where we started the integration at some arbitrary distance $c\geqslant 1$[the form $\omega(r)=\gamma r^{-y}$, divergent as $r\to 0$, only applies to larger r values], and neglected the (finite) integration constant coming from the lower integration limit c [this can be lumped with the (finite) contribution to $S(\varphi)$ from regions near the origin]. As $y\to\infty$ (zero width), we get $S(\varphi)\approx(1/\gamma y)|\varphi/\gamma|^{-1}$. As $y\to 0$, we get

$$2\pi S(\varphi)\to(y/4\gamma)|\varphi/\gamma|^{-\infty}=\begin{cases}\infty & \text{if}\quad|\varphi|\leqslant\gamma\\ 0 & \text{if}\quad|\varphi|>\gamma\end{cases}\quad(y\to 0,\ y\neq 0,\ \varphi\neq 0). \tag{I12}$$

At $y=0$, corresponding to a wedge of constant *angular* width γ, of uniform height $1/\gamma$, the value of $S(\varphi)$ is obviously that given on the right of (I12).

The case $y=1$, that is $w(r)\approx\gamma/r$, corresponds to pure phase states [see after Eq. (13.6)], such as the Susskind-Glogower POM [see (10.5)]. We here get

$$2\pi S(\varphi)\approx\frac{1}{12\gamma}|\varphi/\gamma|^{-2}\quad(\varphi\neq 0) \tag{I13}$$

in accord with (16.6). [For $y>0$, i.e., *increasing* angular width, our approximate calculation loses its validity; but it is obvious that since $\omega_1(r)$ and $\omega_2(r)$ then eventually overlap as $r\to\infty$, we have $S(\varphi)=\infty$ for all values of φ.]

APPENDIX J: A SPECIAL INTEGRAL

This appendix concerns the integral

$$F_n(\nu,s)=\int_0^{\infty}dx\ e^{-(1/2)(1-s)x}x^{\nu}L_n^{2\nu}(x), \tag{J1}$$

where $L_n^{\alpha}(x)$ are generalized Laguerre polynomials. Since this integral is likely to come up in various problems (besides phase) concerning the quantum harmonic oscillator, we give it a more detailed treatment than is strictly required for our present needs. We use abbreviations such as (GR 8.445) to refer to Eq. 8.445 of Gradshteyn and Ryzhik [49].

1. Results obtained

We obtain the expressions

$$F_n(\nu,s)=\left(\frac{2}{1-s}\right)^{\nu+1}\nu!\sum_{j=0}^{n}\binom{2\nu+n}{n-j}\binom{-\nu-1}{j}\left(\frac{2}{1-s}\right)^{j} \tag{J2a}$$

$$=\left(\frac{2}{1-s}\right)^{\nu+1}\nu!\sum_{j=0}^{n}\binom{-\nu}{n-j}\binom{-\nu-1}{j}(-)^{n-j}\times\left(\frac{1+s}{1-s}\right)^{j} \tag{J2b}$$

$$=(-)^n2^{-\nu+1}\left(\frac{1}{1-s}\right)^{n+\nu+1}\frac{(n+2\nu)!}{n!(\nu-1)!}K_n(\nu,s), \tag{J3}$$

where $K_n(\nu,s)$ has the following equivalent expressions:

$$K_n(\nu,s)=2^{2\nu+n}\sum_{k=0}^{n}\binom{n}{k}B(\nu,\nu+k+1)\left(\frac{s-1}{2}\right)^{n-k} \tag{J4a}$$

$$=(-)^n 2^{2\nu}(1-s)^n\sum_{k=0}^{n}\binom{n}{k}$$

$$\times B(n+\nu-k,\nu+k+1)\left(\frac{s+1}{s-1}\right)^k \tag{J4b}$$

$$=\sum_{k=0}^{n}\binom{n}{k}B(\nu,\sigma_k)s^{n-k} \tag{J4c}$$

$$=\int_0^\pi d\varphi(\sin\varphi)^{2\nu-1}(1+\cos\varphi)[\cos\varphi+s]^n, \tag{J4d}$$

where $B(x,y)$ is the beta function (K2), and we denote

$$\mu_k=k \bmod 2,\quad \sigma_k=\tfrac{1}{2}(k+1+\mu_k)=\begin{cases}\tfrac{1}{2}k+\tfrac{1}{2} & (k \text{ even})\\ \tfrac{1}{2}k+1 & (k \text{ odd}).\end{cases} \tag{J5}$$

We also get the large n approximation

$$F_n(\nu,s)=2(-)^n\frac{1}{1-s}\left(\frac{1+s}{1-s}\right)^{n+\nu}n^{-\nu}\frac{(n+2\nu)!}{n!}$$
$$\times\left[1-\frac{\nu(2\nu+1+s\nu)}{2n}\right.$$
$$\left.+\frac{(-)^n}{2n}\left(\frac{1-s}{1+s}\right)^{n+\nu}(1-s)\nu+\cdots\right]. \tag{J6}$$

The form (J2a) is that obtained by directly substituting (K17) into (J1) and integrating the term by term using (K4) and (K10); it is also obtained by Tanas, Miranowicz, and Gantsog [36]. The forms (J4a) and (J4b) follow for (J2a) and (J2b) by use of (K11) and (K2). It is not trivial to pass from one to the other of the forms (a), (b), (c), except in special cases: For instance, one verifies that putting $s=1$ in (J2a) and in (J2b) yields the same result [in (J2a), we get $\binom{\nu+n-1}{n}$ by (K12), while in (J2b), only the term $j=0$ survives, yielding $(-)^n\binom{-\nu}{n}=\binom{n+\nu-1}{n}$ by (K10b)]. Also, multiplying both (J2a) and (J2b) by $(1-s)^n$, we get terms $(1-s)^{n-j}$ in both sums, so that at $s=1$, only the terms $j=n$ survive, yielding in both cases $\binom{-\nu-1}{n}$.

2. Proofs

We denote by $\mathscr{C}\{z^n\}A(z)$ the coefficient of z^n in $A(z)$. Using the generating functions (K.18), we have

$$F_n(\nu,s)=\mathscr{C}\{z^n\}(1+z)^{2\nu+n}\int_0^\infty dx\ e^{-(1/2)(1-s)x-zx}x^\nu \tag{J7a}$$

$$=\mathscr{C}\{z^n\}(1-z)^{-2\nu-1}\int_0^\infty dx\ e^{-(1/2)(1-s)x-xz/(1-z)}x^\nu \tag{J7b}$$

$$=\mathscr{C}\{z^n\}(2\nu+n)!e^z z^{-\nu}$$
$$\times\int_0^\infty dx\ e^{-(1/2)(1-s)x}J_{2\nu}(2(xz)^{1/2}). \tag{J7c}$$

(a) From (J7a) we get, using (K3),

$$F_n(\nu,s)=\left(\frac{2}{1-s}\right)^{\nu+1}\nu!\,\mathscr{C}\{z^n\}(1+z)^{2\nu+n}\left[1+\frac{2z}{1-s}\right]^{-\nu-1} \tag{J8a}$$

whence we get (J2a) on expanding, using (K9) and (K13). (b) From (J7b), we get, using (K3),

$$F_n(\nu,s)=\left(\frac{2}{1-s}\right)^{\nu+1}\nu!\,\mathscr{C}\{z^n\}(1-z)^{-\nu}\left[1+\frac{1+s}{1-s}\,z\right]^{-\nu-1}, \tag{J8b}$$

whence we get (J2b) on using (K9) and (K13). (c) From (J7c), we get, using (J9b) below with $\beta=2z^{1/2}$, $\gamma=\frac{1}{2}(1-s)$, $u=\beta^2/8\gamma=z/(1-s)$:

$$F_n(\nu,s)=2^{-\nu+1}(1-s)^{-\nu-1}\frac{(n+2\nu)!}{(\nu-1)!}$$
$$\times\mathscr{C}\{z^n\}e^{-zs/(1-s)}\sum_{k=0}^{\infty}\frac{1}{k!}B(\nu,\sigma_k)\left(\frac{-z}{1-s}\right)^k \tag{J8c}$$

whence we get (J3)–(J4c) on expanding the exponential, and using (K13). We next show that (J4a) and (J4b) are both equal to (J4d): Following Smith, Dubin, and Hennings [9(b)], we use the representation (K2b) of the beta function, and get

$$(\text{J4a})=2^{2\nu+n+1}\int_0^{\pi/2}d\phi(\cos\phi\ \sin\phi)^{2\nu-1}\cos^2\phi$$
$$\times\sum_{k=0}^{n}\binom{n}{k}(\cos\phi)^{2k}\left(\frac{s-1}{2}\right)^{n-k}$$
$$=2^n\int_0^{\pi/2}2\,d\phi(\sin 2\phi)^{2\nu-1}(1+\cos 2\phi)$$
$$\times[\cos^2\phi-\tfrac{1}{2}(1-s)]^n=(\text{J4d}),$$

where we used $\cos(2\phi)=2(\cos\phi)^2-1$, and put $\varphi=2\phi$. Similarly, we get

$$(\text{J4b})=(-)^n(1-s)^n 2^{2\nu+1}\int_0^{\pi/2} d\phi(\cos\phi\ \sin\phi)^{2\nu-1}\cos^2\phi$$
$$\times\sum_{k=0}^{n}\binom{n}{k}(\sin\phi)^{2(n-k)}(\cos\phi)^{2k}\left(\frac{s+1}{s-1}\right)^k$$
$$=(1-s)^n\int_0^{\pi} 2d\phi(\sin2\phi)^{2\nu-1}(1+\cos2\phi)$$
$$\times\left[\sin^2\phi+\left(\frac{s+1}{s-1}\right)\cos^2\phi\right]^n=(\text{J4d}).$$

Finally, expanding $[\ ]^n$ in (J4d), and using (K2), we get (J4c).

3. An integral with a Bessel function

$$\int_0^\infty dx\ e^{-\gamma x}J_{2\nu}(\beta x^{1/2})$$
$$=\tfrac{1}{4}\beta\left(\frac{\pi}{\gamma^3}\right)^{1/2}e^{-u}[I_{\nu-1/2}(u)-I_{\nu+1/2}(u)] \tag{J9a}$$
$$=\beta(2\gamma)^{-3/2}\frac{2^{-\nu}}{\Gamma(\nu)}e^{-u}u^{\nu-1/2}\sum_{k=0}^{n}B(\nu,\sigma_k)\frac{(-u)^k}{k!}$$
$$\left(u=\frac{\beta^2}{8\gamma}\right), \tag{J9b}$$

where (GR 8.445)

$$I_\alpha(u)=e^{-(1/2)i\pi\alpha}J_\alpha(e^{(1/2)i\pi}u)$$
$$=(\tfrac{1}{2}u)^\alpha\sum_{j=0}^{\infty}\frac{(u/2)^{2j}}{j!(\alpha+j)!}\quad[-\pi<\arg(u)\leqslant\tfrac{1}{2}\pi]. \tag{J10}$$

The result (J9a) is given in (GR 6.614.1). To get (J9b), we write

$$[I_{\nu-1/2}(u)-I_{\nu+1/2}(u)]$$
$$=(\tfrac{1}{2}u)^{\nu-1/2}\sum_{j=0}^{\infty}\left[\frac{(u/2)^{2j}}{j!(\nu+j-\frac{1}{2})!}-\frac{(u/2)^{2j+1}}{j!(\nu+j+\frac{1}{2})!}\right]$$
$$=(\tfrac{1}{2}u)^{\nu-1/2}\sum_{k=0}^{\infty}(-u/2)^kA_k(\nu),$$

$$A_k(\nu)=\frac{1}{(\frac{1}{2}k-\frac{1}{2}\mu_k)!\,\Gamma(\nu+\sigma_k)}$$
$$=\frac{\pi^{-1/2}2^k}{k!}\frac{\Gamma(\sigma_k)}{\Gamma(\nu+\sigma_k)}$$
$$=\pi^{-1/2}2^k\frac{1}{k!\,\Gamma(\nu)}B(\nu,\sigma_k),$$

where we used (E5) to transform $(\frac{1}{2}k-\frac{1}{2}\mu_k)!=\pi^{1/2}k!\,\Gamma(\sigma_k)^{-1}$.

4. Asymptotic form of $K_n(\nu,s)$

$$K_n(\nu,s)=\int_0^\pi d\varphi(\sin\varphi)^{2\nu-1}(1+\cos\varphi)(\cos\varphi+s)^n. \tag{J11}$$

As $n\to\infty$, only the regions $\varphi\approx0$ and $\varphi\approx\pi$, where $|\cos\varphi+s|$ may be locally maximum (depending on the value of s), contribute to low orders in n^{-1}. Thus

$$K_n(\nu,s)\approx\int_0^\pi d\varphi(\sin\varphi)^{2\nu-1}(1+\cos\varphi)(\cos\varphi+s)^n+(-)^n$$
$$\times\int_0^\pi d\theta(\sin\theta)^{2\nu-1}(1-\cos\theta)(\cos\theta-s)^n, \tag{J12}$$

where we put $\theta=\pi-\varphi$, and used $\int_0^\pi d\varphi=\int_0^\pi d\theta$, $\sin\varphi=\sin\theta$, $\cos\theta=-\cos\theta$. Then

$$K_n(\nu,s)\approx D_1(s)-D_2(s)+(-)^nD_2(-s), \tag{J13}$$

$$D_i(s)=2\int_0^\infty\varphi\ d\varphi(\varphi^2)^{\nu-1}f_i(\varphi^2)[\cos\varphi+s]^n \tag{J14a}$$

$$f_1(\varphi^2)=\left(\frac{\sin\varphi}{\varphi}\right)^{2\nu-1}=1-\frac{2\nu-1}{6}\varphi^2+\cdots, \tag{J14b}$$

$$f_2(\varphi^2)=\frac{1}{2}\left(\frac{\sin\varphi}{\varphi}\right)^{2\nu-1}(1-\cos\varphi)=\tfrac{1}{4}\varphi^2+\cdots. \tag{J14c}$$

We now expand

$$n\ \log(\cos\varphi+s)=n\ \log(1+s)-t-ng(t/n), \tag{J15a}$$

$$t=\frac{n\varphi^2}{2(1+s)},$$

$$g(t/n)=\left(\frac{2-s}{6}\right)(t/n)^2+\left(\frac{s^2-13s+16}{90}\right)(t/n)^3+\cdots. \tag{J15b}$$

We then have

$$D_i(s)=(1+s)^n\left(\frac{2+2s}{n}\right)^\nu$$
$$\times\int_0^\infty dt\ e^{-t}t^{\nu-1}f_i\left(\frac{2+2s}{n}\ t\right)e^{-ng(t/n)}$$
$$=(1+s)^{n+\nu}2^\nu n^{-\nu}\Gamma(\nu)f_i\left(\frac{2+2s}{n}\nu^*\right)e^{-ng(\nu^*/n)}, \tag{J16}$$

where we define $(\nu^*)^k=\Gamma(\nu+k)/\Gamma(\nu)=\nu(\nu+1)\cdots(\nu+k)$. Expanding, we get, to order n^{-1},

$$f_1\left(\frac{2+2s}{n}\nu^*\right)e^{-ng(\nu^*/n)}$$

$$=\left[1-\frac{2\nu-1}{6}\frac{2+2s}{n}\nu\right]\left[1-n\frac{2-s}{6}\frac{\nu(\nu+1)}{n^2}\right]$$

$$=\left[1-\frac{2\nu^2-s\nu(1-\nu)}{2n}\right] \tag{J17a}$$

$$f_2\left(\frac{2+2s}{n}\nu^*\right)e^{-ng(\nu^*/n)}$$

$$=\left[\frac{(1+s)\nu}{2n}+O(n^{-2})\right][1+O(n^{-1})]$$

$$=\frac{(1+s)\nu}{2n} \tag{J17b}$$

whence we get

$$K_n(\nu,s)=(1+s)^{n+\nu}2^{\nu}n^{-\nu}\Gamma(\nu)\left[1-\frac{\nu(2\nu+1+s\nu)}{2n}+\frac{(-)^n}{2n}\left(\frac{1-s}{1+s}\right)^{n+\nu}(1-s)\nu+\cdots\right]. \tag{J18}$$

Substituting this onto (J3), we get (J6).

APPENDIX K: USEFUL FORMULAS

In the following, GR 3.621.5 refers to Eq. 3.621.5 in Gradshteyn and Ryzhik [49].

1. Gamma and beta functions

These are given by (GR 3.621.5, 8.384.1, 8.384.6)

$$\Gamma(z)=\int_0^{\infty}dt\ t^{z-1}e^{-t}=(z-1)\Gamma(z-1),\quad \Gamma(z+1)=z! \tag{K1}$$

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{K2a}$$

$$=2\int_0^{\pi/2}d\phi(\sin\phi)^{2x-1}(\cos\phi)^{2y-1}$$

$$(\mathrm{Re}x>0,\ \mathrm{Re}y>0). \tag{K2b}$$

We will use (K1) in the form (GR 3.381.4)

$$\int_0^{\infty}dx\ x^{(1/2)\alpha}e^{-\gamma x}=\gamma^{-[(1/2)\alpha+1]}\Gamma(\tfrac{1}{2}\alpha+1)$$

$$=\gamma^{-[(1/2)\alpha+1]}(\tfrac{1}{2}\alpha)!. \tag{K3}$$

Denoting $\mu_n=n$ mod2 [so that $\mu_{\mathrm{even}}=0$, $\mu_{\mathrm{odd}}=1$], we have

$$\int_0^{\pi}d\varphi(\sin\varphi)^x(\cos\varphi)^n=\mu_{n+1}B\left(\frac{x+1}{2},\frac{n+1}{2}\right),$$

$$\mu_n=n\ \mathrm{mod}2 \tag{K4a}$$

since $\sin\varphi$ is symmetric, $\cos\varphi$ antisymmetric about $\varphi=\pi/2$. It follows that

$$\int_0^{\pi}d\varphi(\sin\varphi)^x[(\cos\varphi)^n+(\cos\varphi)^{n+1}]$$

$$=B\left(\frac{x+1}{2},\frac{n+1+\mu_n}{2}\right). \tag{K4b}$$

"Duplication formula" (GR 8.335.1),

$$\Gamma(2x)=\pi^{-1/2}2^{2x-1}\Gamma(x)\Gamma(x+\tfrac{1}{2})$$

$$\Leftrightarrow k!=\pi^{-1/2}2^k\left(\frac{k}{2}\right)!\left(\frac{k-1}{2}\right)! \tag{K5}$$

Stirling's formula,

$$\Gamma(z)=(2\pi)^{1/2}z^{z-1/2}e^{-z}\left(1+\frac{1}{12z}+\cdots\right), \tag{K6a}$$

$$\ln(n!)=\tfrac{1}{2}\ln2\pi+(n+\tfrac{1}{2})\ln(n+1)-(n+1)+O(n^{-1}), \tag{K6b}$$

$$n!\approx(2\pi)^{1/2}(n+1)^{n+1/2}e^{-(n-1)}\approx(2\pi)^{1/2}n^{n+1/2}e^{-n}, \tag{K6c}$$

where the last equality follows from $(n+1)^{n+1/2}=n^{n+1/2}(1+1/n)^{n+1/2}\approx n^{n+1/2}e$. For ε fixed, we have, to first order in z^{-1},

$$\Gamma(z+\varepsilon)=\Gamma(z)z^{\varepsilon}\left(1+\frac{\varepsilon^2-\varepsilon}{2z}+\cdots\right), \tag{K7}$$

where we expanded in powers of z^{-1}, noting that $(1+\varepsilon/z)^z=e^{z\ \ln(1+\varepsilon/z)}=e^{\varepsilon-(1/2)\varepsilon^2/z}$. Using then $N!=\Gamma(N+1)$, we get, to first order in N^{-1},

$$(N+\varepsilon)!=N!N^{\varepsilon}\left(1+\frac{\varepsilon+\varepsilon^2}{2N}\right), \tag{K8a}$$

$$\frac{(N+\varepsilon)!}{(N-\varepsilon)!}=N^{2\varepsilon}e^{\varepsilon/N},\quad (N+\varepsilon)!(N-\varepsilon)!=(N!)^2e^{\varepsilon^2/N}, \tag{K8b}$$

where we noted that $(N+1)^{\varepsilon}=N^{\varepsilon}(1+\varepsilon/N)$.

2. Binomial expansion

$$(1+z)^x=\sum_{n=0}^{\infty}\binom{x}{n}z^n, \tag{K9}$$

where the binomial coefficients satisfy

$$\binom{x}{k}=\frac{x!}{k!(x-k)!},\quad x!=\Gamma(x+1), \tag{K10a}$$

$$\binom{-x}{j}=\frac{(-x)!}{j!(-x-j)!}=\frac{(-)^j(x-1+j)!}{j!(x-1)!}=(-)^j\binom{x-1+j}{j}, \tag{K10b}$$

$$B(x,y)^{-1}=x\binom{x+y-1}{y-1}=y\binom{x+y-1}{x-1}, \quad \text{(K11)}$$

$$\sum_{j=0}^{n}\binom{x}{j}\binom{y}{n-j}=\binom{x+y}{n}, \quad \text{(K12)}$$

where (K12) follows from expanding $(1+z)^{x+y}=(1+z)^x(1+z)^y$, and using

$$\left(\sum_{i=0}^{\infty} a_i z^i\right)\left(\sum_{k=0}^{\infty} b_k z^k\right)=\sum_{n=0}^{\infty} z^n \sum_{j=0}^{\infty} a_j b_{n-j}. \quad \text{(K13)}$$

3. Poisson distribution

We obtain a large n approximation for

$$P_n(x)=\frac{1}{n!}x^n e^{-x}. \quad \text{(K14)}$$

This is maximum at $0=dP_n/dx=(nx^{n-1}-x^n)e^{-x}$, i.e., at $x=n$. We now expand $\ln P_n(x)$ in powers of n^{-1}, neglecting terms of order n^{-1} and smaller. We first expand

$$n \ln x=n \ln n+(x-n)-\frac{(x-n)^2}{2n}+\frac{(x-n)^3}{3n^2}+\cdots. \quad \text{(K15)}$$

Since $P_n(x)\sim\exp[-\frac{1}{2}(x-n)^2/n]$, we see that $(x-n)$ has a dispersion $\sim n^{1/2}$, so is of order $n^{1/2}$. We therefore neglect, in the expansion (K15), terms $(x-n)^{k+1}/n^k\sim n^{(1-k)/2}$ for $k\geqslant 3$, which are $O(n^{-1})$. Using then Stirling's formula (K6b), we obtain

$$P_n(x)\approx(2\pi n)^{-1/2}\exp\left[-\frac{(x-n)^2}{2n}+\frac{(x-n)^3}{3n^2}+O(n^{-1})\right] \quad \text{(K16a)}$$

$$\approx(2\pi n)^{-1/2}[e^{-(1/2)(x-n)^2/n}+O(n^{-1/2})]. \quad \text{(K16b)}$$

4. Laguerre polynomials

The Laguerre polynomials are given by

$$L_n^{\alpha}(x)=\sum_{j=0}^{n}\frac{(-)^j}{j!}\binom{\alpha+n}{n-j}x^j. \quad \text{(K17)}$$

Generating functions (GR 8.975),

$$(1+z)^{\gamma}e^{-xz}=\sum_{n=0}^{\infty} z^n L_n^{\gamma-n}(x) \quad (|z|<1), \quad \text{(K18a)}$$

$$(1-z)^{-\alpha-1}e^{-xz/(1-z)}=\sum_{n=0}^{\infty} z^n L_n^{\alpha}(x) \quad (|z|<1), \quad \text{(K18b)}$$

$$J_{\alpha}(2(xz)^{1/2})e^z(xz)^{-\alpha/2}=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+\alpha+1)}L_n^{\alpha}(x) \quad (\alpha>-1), \quad \text{(K18c)}$$

where J_α is a Bessel function. Asymptotic form (GR 8.978),

$$L_n^{\alpha}(x)\approx\pi^{-1/2}n^{(1/2)\alpha-1/4}e^{(1/2)x}x^{-(1/2)\alpha-1/4}\cos[2(nx)^{1/2}-\tfrac{1}{2}\alpha\pi-\tfrac{1}{4}\pi]+O(n^{(1/2)\alpha-3/4}) \quad \text{(K19)}$$

for $n^{-1}\ll x<x_m$, where x_m is given by $2(nx_m)^{1/2}\approx n\pi$, i.e., $x_m\approx\frac{1}{4}n\pi^2$.

5. Harmonic oscillator eigenkets

We have the generating function [50]

$$\pi^{-1/4}\exp[-\tfrac{1}{2}x^2+2^{1/2}\lambda x-\tfrac{1}{2}\lambda^2]=\sum_{n=0}^{\infty}\lambda^n(n!)^{-1/2}\varphi_n(x), \quad \text{(K20)}$$

where $\varphi_n=\langle x|n\rangle$ are the eigenfunctions of $\hat{H}=\frac{1}{2}[x^2-(d/dx)^2]$. Putting $x=0$, we find

$$\varphi_n(0)=\begin{cases}0 & (n \text{ odd})\\ \pi^{-1/4}\dfrac{(-)^{n/2}(n!)^{1/2}}{2^{n/2}(\frac{1}{2}n)!}\approx\pi^{-1/2}(-)^{n/2}(2/n)^{1/4} & (n \text{ even}, \ n\to\infty),\end{cases} \quad \text{(K21)}$$

where we used (K6). The momentum eigenfunctions are deduced by using (C3): We thus have $\langle p=0|n\rangle\approx\pi^{-1/2}(2/n)^{1/4}$ for n even and large.

[1] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **114**, 243 (1927).

[2] W. H. Louisell, Phys. Lett. **7**, 60 (1963).

[3] L. Susskind and J. Glogower, Physics **1**, 49 (1964).

[4] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1994).

[5] J. H. Shapiro and S. R. Shepard, Phys. Rev. A **43**, 3795 (1991).

[6] M. Ban, Phys. Lett. A **176**, 47 (1993); **199**, 275 (1995); Phys. Rev. A **51**, 2469 (1995).

[7] J. Bergou and B. G. Englert, Ann. Phys. **209**, 479 (1991).

[8] (a) M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984); (b) N. L. Balazs and B. K. Jennings, Phys. Rep. **104**, 347 (1984); (c) L. Cohen, *Time-Frequency Analysis* (Prentice-Hall, Englewood Cliffs, NJ, 1995).

[9] (a) T. B. Smith, D. A. Dubin, and M. A. Hennings, J. Mod. Opt. **39**, 1603 (1992); (b) D. A. Dubin, M. A. Hennings, and T. B. Smith, *Publications of the Research Institute for Mathematical Sciences*, (Kyoto University Press, Kyoto, 1994), Vol. 30, No. 3, pp. 479–532; (c) M. A. Hennings, T. B. Smith, and D. A. Dubin (unpublished).

[10] A. Royer (unpublished).

[11] See, e.g., V. B. Braginski, Y. I. Vorontsov, and K. S. Thorne, in *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, 1983); V. B. Braginski and F. Ya. Khalili, *Quantum Measurements* (Cambridge University Press, Cambridge, 1992).

[12] See, e.g., H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, 1980).

[13] P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Oxford University Press, Oxford, 1958).

[14] See the special issue Phys. Scr. **T48** (1993), and the excellent review by R. Lynch, Phys. Rep. **256**, 367 (1995).

[15] D. T. Pegg and S. M. Barnett, J. Mod. Opt. **36**, 7 (1989) (my r is equal to their $2^{1/2}r$); Phys. Rev. A **39**, 1665 (1989).

[16] E. C. Lerner, H. W. Huang, and G. E. Walters, J. Math. Phys. **11**, 1679 (1970).

[17] U. Leonhardt, J. A. Vaccaro, B. Bohmer, and H. Paul, Phys. Rev. A **51**, 84 (1995).

[18] W. Schleich and J. A. Wheeler, Nature **326**, 574 (1987); J. A. Wheeler, Lett. Math. Phys. **10**, 201 (1985); J. P. Dowling, W. Schleich, and J. A. Wheeler, Ann. Phys. (Leipzig) **48**, 423 (1991); W. Schleich, H. Walther, and J. A. Wheeler, Found. Phys. **18**, 953 (1989).

[19] W. Vogel and W. Schleich, Phys. Rev. A **44**, 7642 (1991); M. Hillery, M. Freyberger, and W. Schleich, *ibid.* **51**, 1792 (1995).

[20] See, e.g., D. Gabor, J. Inst. Electr. Eng. **93**, 429 (1946); J. Backus, *The Acoustical Foundations of Music*, 2nd ed. (Norton, New York, 1977).

[21] As an example, let $h(x)=e^{-(1/2)x^2/\sigma^2}$, with σ small enough that $R(q+x)\approx R(q)+(x-q)R'(q)$ and $S(x)\approx S(q)+(x-q)S'(q)$ within the window $(q-\sigma,q+\sigma)$. We then get

$$f(q,p)\approx 1/(2\pi)\{|R(q)|^2+\sigma^4|R'(q)|^2[p-S'(q)]^2\}e^{-\sigma^2[p-S'(q)]^2}.$$

Define the "average at q" of any function $g(p)$ by

$$\langle g(p)\rangle_q=\frac{\int dp\, g(p)f(q,p)}{\int dq\, dp\, f(q,p)}.$$

We find that for σ small enough that $\Delta R=R'\sigma\ll R$, the mean momentum $\langle p\rangle_q$ and the momentum uncertainty $\Delta p(q)=\langle(p^2-\langle p\rangle)^2\rangle_q^{1/2}$ are given by

$$\langle p\rangle_q=S'(q),\quad \Delta p(q)=\left\{\sigma^{-2}+\frac{9}{4}\frac{|R'(q)|^2}{|R(q)|^2}\right\}^{1/2},$$

where $\Delta p(q)$ indeed combines the p width σ^{-1} of the window and the local p width (5.3) of $\Psi(x)$.

[22] As another example, let $\Psi(x)=\Theta(x)x^\alpha$. Then $\Psi(q+x)\Psi^*(q-x)=(q^2-x^2)^\alpha\Theta(q+x)\Theta(q-x)$ is maximum at $x=0$, and decreases monotonously, vanishing at $|x|\geqslant q$; being of width $2q$, its Fourier transform over x, namely $W(q,p)$, is of p width $\approx q^{-1}$: Thus, the local p width of x^α decreases like q^{-1}, as was already argued from Eq. (5.3).

[23] Consider the integral of an oscillating function multiplied by another function $f(x)$: If $f(x)$ vanishes smoothly over a distance of several wavelengths, then the oscillations average out and a small value results; but if $f(x)$ vanishes abruptly, then the integral can be sizable and of any sign. For instance, $\int dx\, \cos(\varphi+x/\lambda)e^{-(1/2)(x/a)^2}=(\frac{1}{2}\pi)^{1/2}a\, \cos\varphi e^{-(1/2)(a/\lambda)^2}\approx 0$ if $a\gg\lambda$, whereas $\int_{-a}^{a}dx\, \cos(\varphi+x/\lambda)=2\lambda\cos\varphi\sin(a/\lambda)$ oscillates between -1 and $+1$ as a function of φ and a.

[24] J. W. Noh, A. Fougères, and L. Mandel, Phys. Rev. A **46**, 2840 (1992).

[25] M. Freyberger and W. Schleich, Phys. Rev. A **47**, R30 (1993); M. Freyberger, K. Vogel, and W. P. Schleich, Phys. Lett. A **176**, 41 (1993).

[26] (a) G. M. D'Ariano and M. G. Paris, Phys. Rev. A **49**, 3022 (1994); (b) G. M. d'Ariano, C. Macchiavello, and M. G. A. Paris, Phys. Lett. A **198**, 286 (1995).

[27] G. S. Agarwal and E. Wolf, Phys. Rev. D **2**, 2161 (1970); **2**, 2187 (1970); **2**, 2206 (1970).

[28] A. Royer, Phys. Rev. A **44**, 44 (1991); **45**, 793 (1992).

[29] B. R. Mollow, Phys. Rev. **162**, 1256 (1967); A. Grossmann, Commun. Math. Phys. **48**, 191 (1976); A. Royer, Phys. Rev. A **15**, 449 (1977).

[30] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969); **177**, 1882 (1969).

[31] J. E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).

[32] M. J. Lighthill, *Fourier Analysis and Generalized Functions* (Cambridge University Press, Cambridge, 1964).

[33] U. Herzog, H. Paul, and Th. Richter, Phys. Scr. **T48**, 61 (1993); their ρ is the same as ours.

[34] We can also derive this result in the following way: We have

$$e^{-it\hat{H}}\hat{\phi}e^{it\hat{H}}=\int_0^{2\pi}\theta d\theta\hat{\Gamma}(\theta+t).$$

Noting that $\hat{\Gamma}(\theta)$ is invariant under 2π shifts of its argument, and letting $\underline{t}=t \bmod 2\pi$, we get

$$\begin{aligned}\hat{\phi}(t)&=\int_0^{2\pi}d\theta\theta\hat{\Gamma}(\theta+\underline{t})=\int_0^{2\pi-\underline{t}}d\theta\theta\hat{\Gamma}(\theta+\underline{t})\\&\quad+\int_{2\pi-\underline{t}}^{2\pi}d\theta\hat{\Gamma}(\theta+\underline{t}-2\pi)\\&=\int_{\underline{t}}^{2\pi}d\theta(\theta-\underline{t})\hat{\Gamma}(\theta)+\int_0^{\underline{t}}d\theta(\theta-\underline{t}+2\pi)\hat{\Gamma}(\theta)\\&=\hat{\phi}-\underline{t}+2\pi\int_0^{\underline{t}}d\theta\hat{\Gamma}(\theta)=\hat{\phi}-t+2\pi\int_0^{t}d\theta\hat{\Gamma}(\theta),\end{aligned}$$

where we broke the first integral in two pieces such that the argument of $\hat{\Gamma}()$ always lies in the interval $(0,2\pi)$, then made obvious changes of integration variables, and used $\int_0^{2\pi}d\varphi\, \hat{\Gamma}(\varphi)=\hat{1}$.

[35] If we choose the phase window $(\alpha,\alpha+2\pi)$ instead of $(0,2\pi)$, and define

$$\hat{\phi}_\alpha=\int_\alpha^{\alpha+2\pi}\varphi d\varphi\hat{\Gamma}(\varphi)=e^{i\alpha\hat{H}}\hat{\phi}e^{-i\alpha\hat{H}}+\alpha$$

then we get

$$\langle n'|\hat{\phi}_\alpha|n\rangle=e^{i\alpha(n'-n)}\langle n'|\hat{\phi}|n\rangle,\quad \langle n|\hat{\phi}_\alpha|n\rangle=\alpha+\pi.$$

Equation (14.3b) is similarly modified.

[36] R. Tanas, A. Miranowicz, and Ts. Gantsog, Phys. Sri. **T48**, 53 (1993).

[37] Let us verify explicitly that the eigenequation $\langle n'|\hat{\phi}|\varphi\rangle=\Sigma_{n\geqslant 0}\langle n'|\hat{\phi}|n\rangle\langle n|\varphi\rangle=\varphi\langle n'|\varphi\rangle$ is satisfied for large n', given the asymptotic forms (14.3b) and (14.12) for $\langle n'|\hat{\phi}|n\rangle$ and $\langle n|\varphi\rangle$. We get

$$\langle n'|\hat{\phi}|\varphi\rangle=\pi\langle n'|\varphi\rangle+i(2\pi)^{-1/2}\sum_{n\geqslant 0(n\neq n')}\frac{e^{in\varphi}}{n-n'}.$$

At $\varphi\neq 0$, the sum over n converges, so that for large enough n', we can extend the sum to all $n\neq n'$ without appreciably altering it. Invoking (3.1b), we rewrite the resulting expression as

$$\langle n'|\hat{\phi}|\varphi\rangle=(2\pi)^{-1}\sum_n\int_0^{2\pi}\varphi' d\varphi' e^{i(n'-n)\varphi'}e^{in\varphi}=\varphi e^{in'\varphi}$$

$$=\varphi\langle n'|\varphi\rangle,$$

where we used $(2\pi)^{-1}\Sigma_n e^{in(\varphi-\varphi')}=\delta_{2\pi}(\varphi-\varphi')$.

[38] See, e.g., A Bohm, *Quantum Mechanics*, 3rd ed. (Springer, New York, 1993), pp. 82–84.

[39] P. Busch, M. Grabowski, and P. J. Lahti, Phys. Rev. A **50**, 2881 (1994).

[40] Let us mention that the highly singular Cahill-Glauber $\hat{\Gamma}_{s<0}(\varphi)$ (in particular the normally ordered $s=-1$, corresponding to the Glauber-Sudarshan P function [30]) are actually sharper than Weyl, as discussed in Appendix B. They are in fact *too sharp*: For instance, $s=-1$ attributes a sharp phase to *any* coherent state (highly excited or not)—whereas Weyl gives just the right phase distribution for coherent and squeezed states.

[41] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).

[42] A. Royer, Phys. Rev. Lett. **73**, 913 (1994); **74**, 1040 (1995).

[43] (a) C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976); (b) A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982); (c) K. Kraus, *States, Effects, and Operations, Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).

[44] M. Ozawa, J. Math. Phys. **25**, 79 (1984).

[45] A. Royer (unpublished).

[46] D. T. Smithey, M. Beck, A. Faridani, and M. G. Raymer, Phys. Rev. Lett. **70**, 1244 (1993); D. T. Smithey, M. Beck, J. Cooper, M. G. Raymer, and A. Faridani, Phys. Scr. **T48**, 35 (1993).

[47] A. Royer, Phys. Rev. Lett. **55**, 2745 (1885); Found. Phys. **19**, 3 (1989).

[48] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[49] I. S. Gradshteyn and I. M. Ryzhik, *Tables and Integrals, Series and Products*, 5th ed. (Academic, New York, 1994).

[50] C. Cohen-Tannoudji, B. Diu, and F. Laloe, *Mécanique Quantique* (Hermann, Paris, 1973), Eqs. (31) and (32) in Complément B_V.