COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on "Why quantum dynamics can be formulated as a Markov process"

Daniel T. Gillespie
Research and Technology Division, Naval Air Warfare Center, China Lake, California 93555
(Received 2 June 1995)

In a paper with the title given above, Garbaczewski and Olkiewicz [Phys. Rev. A **51**, 3445 (1995)] claim that a contrarily entitled paper of Gillespie [Phys. Rev. A **49**, 1607 (1994)] is misleading because it is based on an allegedly overly restrictive assumption. In the discussion given here of that claim, the impugned assumption, and some foundational aspects of Markovian and non-Markovian stochastic processes, it is concluded that the thesis of the later title has not been established. [S1050-2947(96)01006-2]

PACS number(s): 03.65.−w, 02.50.Ga

In a recent paper entitled "Why quantum mechanics cannot be formulated as a Markov process" [1], I considered a simple dynamical system whose state vector is predicted by quantum theory to oscillate between two states $|1\rangle$ and $|2\rangle$ according to a formula of the form

$$|\Psi(t)\rangle = e^{-ict}\cos\omega t|1\rangle - ie^{-ict}\sin\omega t|2\rangle. \quad (1)$$

I argued that this quantum system cannot be mathematically modeled as a memoryless stochastic process that randomly jumps back and forth between the two states, while being at any instant definitely in one state or the other, essentially for the following reason: Quantum mechanics requires that the probability for the system to leave a state in the next infinitesimally small time dt be proportional to $(dt)^2$, whereas that probability for any self-consistent memoryless jumping process can only be proportional to dt. The goal of characterizing the time evolutions of all discrete state quantum systems as classically understandable *jump Markov processes* on their respective discrete state spaces is thus shown to be unattainable. But the siren attraction of that goal, and, in particular, the temptation to assume, *contrary* to the tenets of orthodox quantum mechanics, that the state vector in Eq. (1) describes a system that is always "either in state $|1\rangle$ or in state $|2\rangle$," is easy to understand: The probabilities for a measurement to find the system in states $|1\rangle$ and $|2\rangle$ at any time $t>0$ are $\cos^2\omega t$ and $\sin^2\omega t$, respectively, and for any t those probabilities add up to 1 [2].

A more recent paper by Garbaczewski and Olkiewicz [3] carries a title *contrary* to that of Ref. [1], and, furthermore, has Ref. [1] as *its* Ref. [1]. One might accordingly expect that Garbaczewski and Olkiewicz have uncovered some error in Ref. [1] which negates its conclusion. But that turns out not to be the case: The authors of Ref. [3] point out no flaws in the reasoning of Ref. [1]; however, they do claim that the conclusions reached therein are largely irrelevant, because they are based on the "very restrictive assumption" that the system is at any instant definitely in one state or the other.

I freely acknowledge the reliance of Ref. [1] on the cited assumption, inasmuch as my chief aim was to demonstrate that no license to hold that traditionally forbidden assumption is granted by the classical theory of univariate jump Markov processes. But I must point out that the authors of Ref. [3] never address the question of how the system could jump back and forth between the two states without landing *and dwelling* in first one state and then the other; this question seems rather critical to formulating any plausible Markovian jumping model. Nor do the authors ever spell out precisely how *reliquishing* the assumption (that the system is always either in state $|1\rangle$ or in state $|2\rangle$) enables them to overcome the quadratic-versus-linear-*dt* dilemma mentioned earlier, so that a valid jump Markovian description can emerge. In fact, after their second paragraph, the authors of Ref. [3] make no further mention of *jump* Markov processes at all. Instead, they concern themselves with *continuous* (or diffusion) Markov processes, which are of quite a different breed. (Whereas for a *jump* Markov process it makes sense to ask "how long will the system remain in the current state?" and "which state will the system occupy next?," those two questions have no meaning for a *continuous* Markov process.) In the main body of their work, Garbaczewski and Olkiewicz are concerned, not with master equations like my Eq. (13) which govern jump Markov processes, but rather with Fokker-Planck and Langevin equations like their Eqs. (4) and (9) which govern continuous Markov processes. A more appropriate target for Garbaczewski and Olkiewicz might therefore have been my more recent paper entitled "Incompatibility of the Schrödinger equation with Langevin and Fokker-Planck equations" [4]: That paper purports to show that the square modulus of the wave function of an initially confined free particle *fails* to satisfy certain conditions that *must* be satisfied by the probability density function of any *continuous* Markov process. If that claim is true (and nothing in Ref. [3] persuades me otherwise), then it would evidently provide a specific counter example to the titled thesis of Ref. [3].

It may be that one dividing issue here is a difference of opinion as to the *definition* of a Markov process. I am prompted to suggest this possibility by Garbaczewski and Olkiewicz's comments following their Eq. (31), which, as they note, is the Chapman-Kolmogorov equation. In the usual ''conditioning'' notation used by most authors [5–6], that equation reads

$$P(x_3,t_3|x_1,t_1)=\int_{-\infty}^{\infty} P(x_3,t_3|x_2,t_2)P(x_2,t_2|x_1,t_1)dx_2 \quad (t_1<t_2<t_3). \quad (2)$$

In connection with this equation and their proposed quantal P function, Garbaczewski and Olkiewicz report some ''bad news for those who would expect that everything goes smoothly as in the classical probabilistic considerations.'' Specifically, they find that this ''almost obvious, seemingly indisputable formula . . . does *not* hold true as a strong identity.'' Well, that seems to me to be very bad news indeed. My understanding is that Eq. (2) is a *sine qua non* of Markov process theory, and its failure to hold can be taken as proof positive that we are *not* dealing with a genuine Markov process. Support for this view can be found in at least two well-known treatises on stochastic process theory: van Kampen [7] calls Eq. (2) ''an identity, which must be obeyed by the transition probability of any Markov process.'' And Gardiner [8] says that Eq. (2) embodies ''the *Markov postulate* and . . . is the central dynamical equation to all Markov processes.'' I therefore do not understand how Garbaczewski and Olkiewicz can maintain that Markov process theory provides a viable framework for quantum dynamics if they believe that *any* quantum system is inconsistent with Eq. (2)—*unless* they subscribe to a different definition of a Markov process.

Certainly it is *permissible* for a stochastic process $X(t)$ to have a singly conditioned density function P that does *not* obey the Chapman-Kolmogorov equation (2). In that case, though, P will no longer be sufficient to completely define the process: It will be necessary to specify an *infinite set* of conditioned density functions, $P\equiv P^{(1)},P^{(2)},P^{(3)},\ldots,$ where

$$P^{(n)}(x_{n+1},t_{n+1}|x_n,t_n;\ldots;x_1,t_1) \quad (t_1<\ldots<t_n<t_{n+1})$$

is the density function of the process at time t_{n+1} given the respective process values $x_n,\ldots x_1$ at the n specified earlier times $t_n,\ldots t_1$. The challenge in specifying all these functions $P^{(n)}$ is to ensure that they satisfy the hierarchy of integral equations,

$$\begin{aligned}P^{(n)}(x_{n+2},t_{n+2}|x_n,t_n;\ldots;x_1,t_1) &= \int_{-\infty}^{\infty} P^{(n+1)}(x_{n+2},t_{n+2}|x_{n+1},t_{n+1};\ldots;x_1,t_1)\\ &\quad\times P^{(n)}(x_{n+1},t_{n+1}|x_n,t_n;\ldots;x_1,t_1)dx_{n+1}\\ &\quad (t_1<t_2<\cdots<t_{n+2};n=1,2,\ldots),\end{aligned} \quad (3)$$

which are *required* to hold by the laws of probability theory [9]. In the *usual* definition of a ''Markovian'' stochastic process, $P^{(n)}(x_{n+1},t_{n+1}|x_n,t_n,\ldots,x_1,t_1)$ is presumed to be identically equal to $P^{(1)}(x_{n+1},t_{n+1}|x_n,t_n)$ for all $n\geqslant 2$, and in that case every one of Eqs. (3) collapses to the Chapman-Kolmogorov equation (2). But *regardless* of whether or not that simplifying Markov condition is imposed, the hierarchy of integral Eqs. (3) *must* be satisfied. That requirement imposes *constraints* on the equations that govern the dynamical evolution of a stochastic process. In the case of the traditional *jump Markovian* stochastic process, for instance, those constraints boil down to the requirement that $P^{(1)}(x',t+dt|x,t)$, for $x'\neq x$, be *linear* in the infinitesimal variable *dt* [10], as was mentioned in our first paragraph.

If one proceeds unmindful of the necessity for establishing a set of conditioned density functions $\{P^{(n)}\}$ obeying Eqs. (3), then one can come up with all sorts of intriguing but completely fallacious candidates for the function $P^{(1)}\equiv P$. But I see no indication in Ref. [3] that its authors have been duly attentive to that chore—nor aware of its immensity when the usual Markovian postulate is not invoked: They begin their paper with the sanguine statement: ''It is clear that a stochastic process is *any* conceivable evolution which we can analyze in terms of probability.'' And they subsequently produce a function P that does not satisfy the Chapman-Kolmogorov equation (2), but they neglect to prove that that function can serve as the first in a sequence of density functions $\{P^{(n)}\}$ that obeys the set of consistency relations (3).

Garbaczewski and Olkiewicz claim, in the first paragraph of Ref. [3], that the title of Ref. [1] ''leads the unsuspicious reader much too far'' because of the ''very restrictive assumption'' on which it is based. I suggest instead that, in a paper that appears to ignore the common definition of a Markov process, Garbaczewski and Olkiewicz have neither weakened my conclusions in [1] regarding jump Markov processes, nor provided an adequate basis for their own claims regarding continuous (diffusion) Markov processes. But I am satisfied to leave it to journal readers, the suspicious ones as well as the unsuspicious ones, to decide this issue for themselves.

[1] D. T. Gillespie, Phys. Rev. A **49**, 1607 (1994).

[2] In nonmathematical terms, the urge to assume that the system is always in either state $|1\rangle$ or state $|2\rangle$ springs from the widely accepted ontological premise that ''everybody has got to be someplace,'' reputedly first enunciated by Groucho Marx in response to a question posed by an indignant husband.

[3] P. Garbaczewski and L. Olkiewicz, Phys. Rev. A **51**, 3445 (1995).

[4] D. T. Gillespie, Found. Phys. **25**, 1041 (1995).

[5] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1983).

[6] C. W. Gardiner, *Handbook of Stochastic Methods for Physics,*

Chemistry and the Natural Sciences (Springer-Verlag, Berlin, 1985).
[7] Reference [5], p. 82.
[8] Reference [6], p. 5.
[9] Equations (3) are basically instances of the fundamental probability density function relation $P_Z(z)=\int P_{Z|Y}(z|y)P_Y(y)dy$.
[10] D. T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic, New York, 1991). See Sec. 4.1.