Traveling-wave states and secondary instabilities in optical parametric oscillators

Stefano Longhi
Centro di Elettronica Quantistica e Strumentazione Elettronica del Consiglio Nazionale delle Ricerche, Dipartimento di Fisica del Politecnico, Piazza Leonardo da Vinci 32, 20133 Milano, Italy

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A complex order-parameter description of pattern formation in large aspect ratio optical parametric oscillators (OPOs) with flat end reflectors and uniform pumping is presented starting from the mean-field model of the OPO equations [G.-L. Oppo *et al.*, Phys. Rev. A **49**, 2028 (1994)]. It is shown that, in the nondegenerate case, the full OPO equations have an exact continuum family of traveling-wave (TW) solutions, which are preferred to standing-wave (SW) states found in the degenerate case. These solutions correspond to an off-axis emission for both signal and idler fields along two symmetric directions to satisfy momentum conservation in the parametric conversion process. Stability of TW versus SW solutions is investigated by deriving two coupled Newell-Whitehead-Segel equations describing the growth of SW or TW close to threshold. Analytical expressions for long-wavelength phase instabilities of the TW states above threshold are obtained from the coefficients of a Cross-Newell phase equation, and are shown to be the same for OPOs with high or low finesse for the pump field. By direct linear stability analysis of the TW solutions, it is also shown that the appearance of amplitude instabilities may reduce the region of stable TW states in the case of OPOs with a high finesse for the pump field. [S1050-2947(96)01406-0]

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I. INTRODUCTION

Pattern formation in nonlinear optical systems has been the object of extensive investigations in recent years [1]. In particular, theoretical and experimental studies of transverse laser dynamics have revealed the appearance of a new kind of instabilities leading to spatial pattern formation, symmetry-breaking bifurcations, and complex space-time patterns [2–8]. Early studies in this field were mainly focused on small aspect ratio systems, in which a small number of transverse modes may be excited [2]. In more recent studies, large aspect ratio systems, usually obtained by considering cavities with plane mirrors of infinite transverse extension and plane-wave pumping, have introduced a host of interesting additional phenomena related to the existence of universal pattern instabilities [3–8]. The central new feature obtained by going to large aspect ratio systems is the possibility of reducing the nonlinear transverse dynamics to a universal form, providing a connection between pattern formation phenomena in optics and in other physical fields, particularly in hydrodynamics [5,6,8]. In the laser case, it was shown that the full Maxwell-Bloch equations admit of exact traveling-wave solutions, and that the nature of the solution above threshold strongly depends on the sign of the detuning between the atomic resonance frequency and the longitudinal cavity one [4]. For positive detunings, threshold lowering associated to an off-axis emission is predicted [4,6]. Because of the rotational symmetry in the transverse plane, a whole annulus of wave vectors may become active near threshold, and pattern selection is governed by the nonlinear competition among these modes. The fact that, above threshold, only a single wave vector [corresponding to a pure traveling-wave (TW) solution] is able to dominate and to suppress all others, at least within local patches [6], is quite remarkable. The process by which a pure TW state is selected is governed by the nonlinear terms in the amplitude equations which describe the growth of standing or traveling waves close to threshold [6,7]. In particular, a combination of two opposing waves, a standing wave (SW), is unstable, collapsing to a single TW. Pattern formation in optical parametric oscillators (OPOs) with flat end reflectors has also been investigated, and threshold lowering due to the appearance of roll patterns has been predicted in the degenerate case and for negative detunings of the signal field [9–11]. Although the phenomenon which makes the threshold independent of the signal detuning in the negative side is quite similar to the appearance of TW states in lasers, off-axis emission in OPOs manifests itself as roll patterns. Physically, the roll solution originates from interference between two symmetric TWs which are simultaneously generated in the parametric conversion of the pump photons due to momentum conservation [9]. From a mathematical point of view, the tendency to yield roll patterns instead of traveling waves is related to the possibility of reducing the OPO dynamics to an order parameter equation which is sensitive to the phase field [11]. However, when operating in the nondegenerate case, or when signal and idler fields are degenerate in frequency but are polarized along two orthogonal directions [12], we expect that momentum conservation can be satisfied without producing interference, which is the basic reason for roll formation. In this paper we show indeed that the full OPO equations with uniform pumping previously considered in Ref. [10] admit of *stable* TW states for both signal and idler fields. A remarkable feature of these equations analogous to the laser case is that, when removing the degeneracy constraint, a continuum of *exact* solutions exists above the neutral stability curve. These solutions are represented by two *symmetric* TWs for signal and idler fields, and by a homogeneous plane wave for the pump field. Note that the fact that idler and signal photons are emitted along two symmetric directions is a consequence of momentum conservation, whereas the uniformity of the pump field also above threshold is due to the absence of interference between these TWs. The preference of TW versus SW states is here addressed by deriving the amplitude equations of the system, an approach of weakly nonlinear analysis that now has become standard [13,14]. These equations are expressed in the form of two coupled Newell-Whitehead equations, and describe the growth of SW or TW close to threshold. Above threshold, the TW states may be destabilized due to the appearance of both phase and amplitude instabilities. The stability domain of the TW solutions corresponds to what has been called the *Busse balloon* for convective systems [15], a terminology recently introduced also in the optical context [8]. The existence of exact TW solutions of the full OPO equations allows for a direct determination of the Busse balloon boundaries by use of standard linear stability methods. This is the most powerful technique as it omits no classes of instability. However, it is inevitably numerical in practice and offers little physical insight into the problem. Further analytical results are obtained, as in the laser case, by deriving a phase diffusion equation of the underlying pattern. This equation permits testing of stability of the TW solution against long-wavelength transverse sideband perturbations, identifying both Eckhaus and zigzag stability boundaries. These instabilities could lead to defect formation and therefore may play an important role in the transition to complex temporal patterns. It is shown that the boundaries of the stability domain as obtained by the phase diffusion equation are *independent* of the cavity decay rate of the pump field. However, a comparison of the Busse balloon domain, as obtained by direct computation of the eigenvalues in the linearized problem and by use of the phase equation, puts into evidence the existence of amplitude instabilities, which may reduce the stability domain to a small region inside the domain of existence of TWs in the case of OPOs with a high finesse for the pump field.

The paper is organized as follows. In Sec. II we review the model set of equations for the problem of a nondegenerate OPO with infinitely extended plane parallel mirrors [10], presenting an exact family of TW solutions for these equations. In Sec. III we discuss the threshold problem and derive two coupled amplitude equations in the negative detuning case, showing how TW solutions are preferred to SW ones. Finally, in Sec. IV we discuss the stability problem of the TW solutions both by standard linear stability analysis and by deriving a Cross-Newell phase equation. Exact expressions of the phase (Eckhaus and zigzag) stability boundaries are presented, and emergence of amplitude instabilities is also briefly investigated.

II. BASIC OPO EQUATIONS AND TRAVELING-WAVE SOLUTIONS

The starting point of our analysis is provided by a set of three coupled equations describing the dynamics of signal, idler, and pump waves in an optical parametric oscillator with flat end mirrors, generalized to include diffraction effects. These equations were derived in previous papers [9–11] to study pattern formation in OPO systems. With the same notations as in Ref. [10], in the nondegenerate case, they are given by

$$\partial_t B = \gamma_0[-(1+i\Delta_0)B + ia_0\nabla^2 B] - \gamma_0 A_1 A_2, \tag{1a}$$

$$\partial_t A_1 = \gamma_1[-(1+i\Delta_1)A_1 + ia_1\nabla^2 A_1 + \mu A_2^*] + \gamma_1 A_2^* B, \tag{1b}$$

$$\partial_t A_2 = \gamma_2[-(1+i\Delta_2)A_2 + ia_2\nabla^2 A_2 + \mu A_1^*] + \gamma_2 A_1^* B. \tag{1c}$$

In these equations, $B = A_0 - \mu$, A_0, A_1, and A_2 are the normalized slowly varying envelopes for pump, signal, and idler fields, respectively, $\mu = E(1-i\Delta_0)/(1+\Delta_0^2)$ is the parametric gain, and E is the normalized amplitude of the plane-wave pump input field. The detuning parameters for pump, signal, and idler fields are defined by

$$\Delta_0 = \frac{\omega_0 - \omega_L}{\gamma_0}, \quad \Delta_1 = \frac{\omega_1 - \nu_1\omega_L}{\gamma_1}, \quad \Delta_2 = \frac{\omega_2 - \nu_2\omega_L}{\gamma_2},$$

where γ_0, γ_1, and γ_2 are the cavity decay rates of the three fields, ω_0, ω_1, and ω_2 are the three longitudinal cavity frequencies close to the pump frequency ω_L, the signal frequency $\nu_1\omega_L$, and the idler frequency $\nu_2\omega_L$, respectively, and $\nu_1 + \nu_2 = 1$ for energy conservation in the parametric interaction. The diffraction parameters a_0, a_1, and a_2 for the three fields are defined by

$$a_0 = \frac{c}{2k_z\gamma_0}, \quad a_1 = \frac{c}{2\nu_1 k_z \gamma_1}, \quad a_2 = \frac{c}{2\nu_2 k_z\gamma_2},$$

where c is the velocity of light and k_z is the longitudinal wave vector of the field at frequency ω_L. Without loss of generality, we may assume that the parametric gain μ in Eqs. (1), which plays the role of bifurcation parameter, be real and positive. Besides the trivial zero solution, corresponding to the OPO being below threshold, Eqs. (1) have an *exact* continuum family of traveling-wave solutions, dependent on the transverse wave vector $\mathbf{k}$ (the family parameter), given by

$$\begin{pmatrix} B \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \overline{B} \\ \overline{A_1}\exp(i\mathbf{k}\cdot\mathbf{r} + i\omega t) \\ \overline{A_2}\exp(-i\mathbf{k}\cdot\mathbf{r} - i\omega t) \end{pmatrix}, \tag{2}$$

where

$$\overline{B} = -\frac{C\,\exp(i\psi)}{1+i\Delta_0}, \tag{3a}$$

$$\overline{A_1} = \sqrt{C}, \tag{3b}$$

$$\overline{A_2} = \sqrt{C}\,\exp(i\psi). \tag{3c}$$

In Eqs. (3), ψ is the phase difference between idler and signal fields, and $C>0$ represents the intensity of the TW solution; they are given by

$$C = \frac{-(\rho_r - \rho_i\tilde{\Delta}) \pm \sqrt{\mu^2\rho_r - (\rho_i + \rho_r\tilde{\Delta})^2}}{\rho_r}, \tag{4a}$$

$$\cos\psi = (1 + \rho_r C)/\mu, \tag{4b}$$

where we have set

$$\rho_r = (1+\Delta_0^2)^{-1}, \quad \rho_i = \rho_r \Delta_0,$$

$$\tilde{\Delta} = \frac{\gamma_1 \Delta_1 + \gamma_2 \Delta_2 + k^2(\gamma_1 a_1 + \gamma_2 a_2)}{\gamma_1 + \gamma_2},$$

and $k = |\mathbf{k}|$. Finally, in Eq. (2) the frequency ω is given by

$$\omega = \frac{\gamma_1 \gamma_2 [\Delta_2 - \Delta_1 + k^2(a_2 - a_1)]}{\gamma_1 + \gamma_2}, \tag{5}$$

which defines the relation dispersion $\omega = \omega(k)$ for TW solutions. The domain of existence of TWs in the plane (k,μ) can be determined from Eq. (4a) by making $C>0$. It should be noted that, for $\rho_r - \rho_i \tilde{\Delta} > 0$, only the positive sign in Eq. (4a) is acceptable, and the TW exists for $\mu > \mu_0$, where $\mu_0 = \sqrt{1+\tilde{\Delta}^2}$. On the other hand, for wave vectors k such that $\rho_r - \rho_i \tilde{\Delta}$ becomes negative, the TW solution exists for $\mu > \mu_e$, where $\mu_e = \sqrt{1+\tilde{\Delta}^2 - (\rho_r - \rho_i \tilde{\Delta})^2/\rho_r}$; in this case, when μ varies in the range $\mu_e < \mu < \mu_0$, there are two acceptable solutions of Eq. (4a), indicating the emergence of bistability. This is a generalization of the bistable behavior previously studied for a degenerate OPO model without diffraction [16]. As is briefly shown in Appendix A, the TW solution corresponding to the lower branch of the bistable loop is unstable, and therefore we will neglect this solution. Physically, the TW solution given by Eq. (2) describes off-axis emission of idler and signal photons along two opposite directions and at frequencies equally detuned, in opposite sides, from the reference frequencies. These features are a consequence of energy and momentum conservation in the parametric conversion of pump photons. It should be noted that, when idler and signal fields are *degenerate both in frequency and polarization*, Eqs. (1) must be solved with the further condition $A_1 = A_2$. In this case, the TW states given by Eq. (2) are no longer good solutions of the problem, and SW states, whose expression can be calculated only by approximate methods [9], are in fact selected.

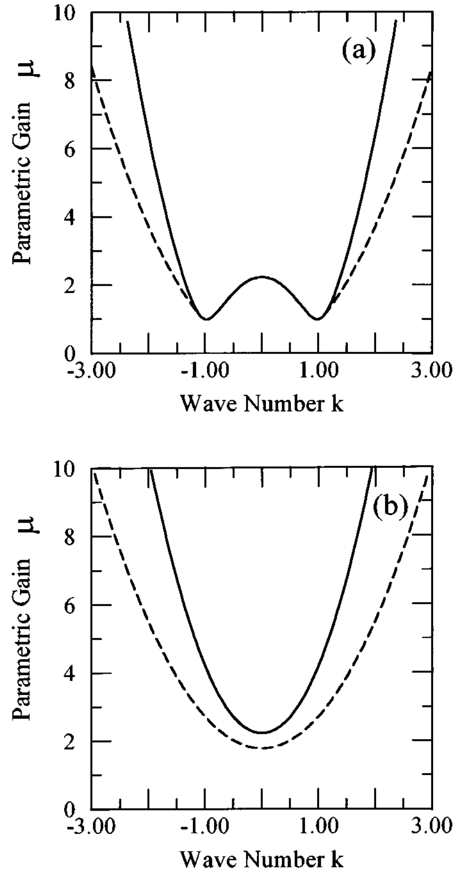

FIG. 1. Neutral stability curve (solid line) and boundary of existence of traveling waves (dashed line) for (a) $\Delta_1 = \Delta_2 = -2$, and (b) $\Delta_1 = \Delta_2 = 2$. For wave numbers where the dashed line is not obscured by the solid one, traveling-wave solutions may exist below threshold for oscillation. The other parameters are $\gamma_0 = \gamma_1 = \gamma_2 = 1$, $v_1 = 0.4$, and $\Delta_0 = 2$. In this figure, and in other figures throughout this work as well, the wave vector k has been normalized to $1/(\gamma_0 a_0)^{1/2}$.

III. LINEAR STABILITY ANALYSIS OF NONLASING SOLUTION AND WEAKLY NONLINEAR ANALYSIS NEAR THRESHOLD

A. Neutral stability curve and threshold condition

The threshold for oscillation is determined by linearizing Eqs. (1) around the trivial zero solution and looking for exponential growth of the perturbations. The linearized equations are

$$\partial_t B = \gamma_0[-(1+i\Delta_0)B + i a_0 \nabla^2 B], \tag{6a}$$

$$\partial_t A_1 = \gamma_1[-(1+i\Delta_1)A_1 + i a_1 \nabla^2 A_1 + \mu A_2^*], \tag{6b}$$

$$\partial_t A_2^* = \gamma_2[-(1-i\Delta_2)A_2^* - i a_2 \nabla^2 A_2^* + \mu A_1], \tag{6c}$$

where, for convenience, we have written Eq. (6c) in terms of A_2^*. As can be seen, the dynamics of the pump perturbation B is decoupled from that of the signal and idler fields, and does not lead to instability for any value of the parametric gain μ. We concentrate therefore on the last two equations of the linearized system. We assume that the transverse plane is sufficiently large so that we can neglect boundary conditions and transverse modes form a continuum. Then the most general solution of Eqs. (6b) and (6c) is given by a linear combination of solutions of the form

$$\begin{pmatrix} A_1 \\ A_2^* \end{pmatrix} \propto \exp(\lambda t + i\mathbf{k}\cdot\mathbf{r}),$$

where $\mathbf{k}$ is the transverse wave vector of the perturbation and λ is an eigenvalue of the linear problem. For a given wave number k, the real part of one of the two eigenvalues, say $\mathrm{Re}(\lambda_1)$, crosses zero from negative as μ is increased. Thus $\mathrm{Re}(\lambda_1) = 0$ gives the neutral stability curve $\mu = \mu_0(k)$, and $\omega = \mathrm{Im}(\lambda_1)$ defines the frequency of the Hopf bifurcation. We find

$$\mu_0(k) = \left(1 + \left[\frac{\gamma_1 \Delta_1 + \gamma_2 \Delta_2 + k^2(\gamma_1 a_1 + \gamma_2 a_2)}{\gamma_1 + \gamma_2}\right]^2\right)^{1/2} = \sqrt{1+\tilde{\Delta}^2}, \tag{7}$$

$$\omega = \frac{\gamma_1 \gamma_2 [\Delta_2 - \Delta_1 + k^2(a_2 - a_1)]}{\gamma_1 + \gamma_2}. \tag{8}$$

Note that the neutral stability curve coincides with the curve of existence of TW solutions when $\rho_r - \rho_i \tilde{\Delta} > 0$, and the frequency of the Hopf bifurcation is exactly that given by Eq. (5). Minimizing $\mu_0(k)$ with respect to k gives the threshold for oscillation $\mu_{\rm th} = \mu_0(k_C)$ with critical wave number k_C and frequency $\omega_C = \omega(k_C)$. The behavior of the neutral stability curve, shown in Fig. 1, strongly depends on the sign of the ''effective'' detuning $\Delta = (\gamma_1\Delta_1 + \gamma_2\Delta_2)/(\gamma_1 + \gamma_2)$. As previously reported in Refs. [9, 10], for $\Delta > 0$ the threshold is given by $\mu_{\rm th} = \sqrt{1+\Delta^2}$ and corresponds to the critical wave number $k_C = 0$ and frequency $\omega_C = \gamma_1\gamma_2(\Delta_2 - \Delta_1)/(\gamma_1 + \gamma_2)$. On the other hand, for $\Delta < 0$, threshold lowering to the value $\mu_{\rm th} = 1$ is predicted, corresponding to a critical wave number $k_C = \sqrt{-\Delta(\gamma_1 + \gamma_2)/(a_1\gamma_1 + a_2\gamma_2)}$. It should be noted that, in the latter case, threshold condition for oscillation coincides with that for existence of the TW solution with wave number k_C, and bifurcation of the trivial zero solution is supercritical. On the contrary, in the former case the homogeneous TW state may exist below threshold for oscillation whenever the bistable condition $\rho_r - \rho_i \Delta < 0$ is satisfied. Therefore the bifurcation to the homogeneous TW solution may be subcritical or supercritical. Restricting now our attention to the case of negative detunings ($\Delta < 0$), the dependence of the neutral stability curve only on modulus of the transverse wave vector (not its direction), introduces a rotational degeneracy, so that a whole annulus of wave vectors can become active. The same degeneracy problem was found for the laser equations [6]. In that case, both numerical simulations of the equations and weakly nonlinear analysis close to threshold showed that a pure TW state is able to dominate and to suppress all others [6]. By use of amplitude equations, we will show here that for the OPO equations the SW solution (roll pattern), obtained by superposition of two TW states propagating in opposite directions, is also unstable and a pure exact TW state is preferred.

B. Amplitude equations

The nonlinear dynamics of the OPO equations close to threshold can be captured analytically by deriving universal amplitude equations for the system [14]. The derivation of the amplitude equations is based on a multiple-scale expansion [13,14] and is valid when bifurcation to the lasing state near onset is supercritical. We concentrate here on the derivation of the amplitude equations in the negative detuning case ($\Delta < 0$), where bifurcation is always supercritical. In the positive detuning case, derivation of an amplitude equation might be a nontrivial matter because bifurcation could become subcritical; hence for $\Delta > 0$ it appears more convenient to address the stability problem of TW states by direct linear stability analysis of the full OPO equations or by deriving a phase diffusion equation. For $\Delta < 0$, due to the rotational symmetry in the transverse plane, any modes with wave number $|\mathbf{k}| \approx k_C$ can be excited near threshold; however, nonlinear interactions among these modes usually select regular solutions formed by a superposition of a few TWs [14]. Here we focus on the derivation of the amplitude equations considering the competition between two TW states propagating along the x axis with opposite directions. Such equations will be able to establish whether traveling- or standing-wave states are preferred [17]. To this aim, it is convenient to consider the complex conjugate of Eq. (1c) instead of Eq. (1c), and to rewrite Eqs. (1) in the compact form

$$\partial_t \mathbf{v} = \hat{L}\mathbf{v} + \mathbf{N}, \tag{9}$$

where the vector $\mathbf{v} = (B, A_1, A_2^*)^T$ contains the field variables, $\hat{L}$ is the linear operator, and $\mathbf{N}$ is the nonlinear operator of the system,

$$\hat{L} = \begin{pmatrix} -\gamma_0(1+i\Delta_0) + i\gamma_0 a_0 \nabla^2 & 0 & 0 \\ 0 & -\gamma_1(1+i\Delta_1) + i\gamma_1 a_1 \nabla^2 & \gamma_1\mu \\ 0 & \gamma_2\mu & -\gamma_2(1-i\Delta_2) - i\gamma_2 a_2 \nabla^2 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} -\gamma_0 A_1 A_2 \\ \gamma_1 A_2^* B \\ \gamma_2 A_1 B^* \end{pmatrix}.$$

We expand $\mathbf{v}$ as an asymptotic series in a small parameter ε,

$$\mathbf{v} = \varepsilon\mathbf{v}^{(1)} + \varepsilon^2\mathbf{v}^{(2)} + \varepsilon^3\mathbf{v}^{(3)} + \cdots, \tag{10}$$

where $\varepsilon^2 = \mu - 1$ provides a measure of the distance above threshold. The next step is to introduce a multiple-scale expansion for space and time variables. As in the laser case [6], an inspection of the diagram in Fig. 1(a) suggests the following choice of multiple scales for t, x, and y:

$$t = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots, \quad x = X_0 + \varepsilon X, \quad y = \varepsilon^{1/2} Y.$$

Using the chain rule for differentiation, one has the substitution

$$\partial_t = \partial_{T_0} + \varepsilon\partial_{T_1} + \varepsilon^2\partial_{T_2} + \cdots, \tag{11}$$

$$\nabla^2 = \partial_{X_0}^2 + \varepsilon(\partial_Y^2 + 2\partial_{X_0}\partial_X) + \varepsilon^2\partial_X^2.$$

Substituting Eqs. (10) and (11) into Eq. (9) yields a hierarchy of equations for successive corrections of $\mathbf{v}$:

$$(\partial_{T_0} - \hat{L}_0)\mathbf{v}^{(1)} = 0 \quad \text{for } O(\varepsilon),$$

$$(\partial_{T_0} - \hat{L}_0)\mathbf{v}^{(2)} = -\partial_{T_1}\mathbf{v}^{(1)} + \hat{L}_1\mathbf{v}^{(1)} + \mathbf{N}_2 \quad \text{for } O(\varepsilon^2),$$

$$(\partial_{T_0} - \hat{L}_0)\mathbf{v}^{(3)} = -\partial_{T_1}\mathbf{v}^{(2)} - \partial_{T_2}\mathbf{v}^{(1)} + \hat{L}_1\mathbf{v}^{(2)} + \hat{L}_2\mathbf{v}^{(1)} + \mathbf{N}_3 \quad \text{for } O(\varepsilon^3),$$

where $\hat{L}_0$ is the linear operator of the system at threshold (i.e., for $\mu=1$), $\mathbf{N}_2$ and $\mathbf{N}_3$ are the nonlinear terms at $O(\varepsilon^2)$ and $O(\varepsilon^3)$, and

$$\hat{L}_1=\begin{pmatrix} ia_0\gamma_0(\partial_Y^2+2\partial_X\partial_{X_0}) & 0 & 0 \\ 0 & ia_1\gamma_1(\partial_Y^2+2\partial_X\partial_{X_0}) & 0 \\ 0 & 0 & -ia_2\gamma_2(\partial_Y^2+2\partial_X\partial_{X_0}) \end{pmatrix},$$

$$\hat{L}_2=\begin{pmatrix} ia_0\gamma_0\partial_X^2 & 0 & 0 \\ 0 & ia_1\gamma_1\partial_X^2 & \gamma_1 \\ 0 & \gamma_2 & -ia_2\gamma_2\partial_X^2 \end{pmatrix}.$$

At leading order, we find the linearized system studied in Sec. III A, whose solution may be written as a superposition of two TWs propagating along the x axis with opposite directions:

$$\mathbf{v}^{(1)}=\begin{pmatrix} B^{(1)} \\ A_1^{(1)} \\ A_2^{*(1)} \end{pmatrix}=\{A_+(X,Y,T_1,T_2)\exp[i(k_CX_0+\omega_CT_0)] +A_-(X,Y,T_1,T_2)\exp[i(-k_CX_0 +\omega_CT_0)]\}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{12}$$

In order to solve equations at $O(\varepsilon^2)$ and $O(\varepsilon^3)$, solvability conditions expressed by the Fredholm alternative theorem must be satisfied. Specifically, the solvability conditions require that the right-hand side terms $\mathbf{G}_2=\hat{L}_1\mathbf{v}^{(1)}+\mathbf{N}_2-\partial_{T_1}\mathbf{v}^{(1)}$ at $O(\varepsilon^2)$ and $\mathbf{G}_3=\hat{L}_1\mathbf{v}^{(2)}+\hat{L}_2\mathbf{v}^{(1)}+\mathbf{N}_3-\partial_{T_1}\mathbf{v}^{(2)}-\partial_{T_2}\mathbf{v}^{(1)}$ at $O(\varepsilon^3)$ be orthogonal to the two eigenvectors $\mathbf{u}_{1,2}=(0,\gamma_2,\gamma_1)^T\exp[i(\pm k_CX_0+\omega_CT_0)]$ of the adjoint operator of $(\partial_{T_0}-\hat{L}_0)$. Applying the solvability condition at $O(\varepsilon^2)$, we find

$$\partial_{T_1}A_\pm=\frac{i\gamma_1\gamma_2(a_1-a_2)}{\gamma_1+\gamma_2}\,(\partial_Y^2\pm 2ik_C\partial_X)A_\pm \tag{13}$$

and the solution at this order is given by

$$\mathbf{v}^{(2)}=\begin{pmatrix} \alpha+\beta_+\exp(2ik_CX_0)+\beta_-\exp(-2ik_CX_0) \\ \vartheta_+\exp(ik_CX_0+i\omega_CT_0)+\vartheta_-\exp(-ik_CX_0+i\omega_CT_0) \\ 0 \end{pmatrix}, \tag{14}$$

where

$$\alpha=-\frac{|A_+|^2+|A_-|^2}{1+i\Delta_0}, \tag{15a}$$

$$\beta_\pm=-\frac{A_\pm A_\mp^*}{1+i(\Delta_0+4a_0k_C^2)}, \tag{15b}$$

$$\vartheta_\pm=i\,\frac{a_1\gamma_1+a_2\gamma_2}{\gamma_1+\gamma_2}\,(\partial_Y^2\pm 2ik_C\partial_X)A_\pm. \tag{15c}$$

The ε^3-order solvability condition reads explicitly

$$(\gamma_1+\gamma_2)\partial_{T_2}A_\pm=-\gamma_2\partial_{T_1}\vartheta_\pm+ia_1\gamma_1\gamma_2(\partial_Y^2\pm 2ik_C\partial_X)\vartheta_\pm +i(a_1-a_2)\gamma_1\gamma_2\partial_X^2A_\pm+2\gamma_1\gamma_2A_\pm -2\gamma_1\gamma_2\left[\frac{|A_+|^2+|A_-|^2}{1+\Delta_0^2} +\frac{|A_\mp|^2}{1+(\Delta_0+4a_0k_C^2)^2}\right]A_\pm. \tag{16}$$

The total time and space derivatives of the amplitudes are given by $\partial_tA_\pm=\varepsilon\partial_{T_1}A_\pm+\varepsilon^2\partial_{T_2}A_\pm$, $\partial_yA_\pm=\varepsilon^{1/2}\partial_YA_\pm$, and $\partial_xA_\pm=\varepsilon\partial_XA_\pm$. Then, combining Eqs. (13) and (16), using Eq. (15c), and changing variables $\varepsilon A_\pm=S_\pm$, we finally obtain the following *amplitude equations*:

$$\tau(\partial_t S_+ + v_g \partial_x S_+) = \frac{i(a_1 - a_2)}{2} \nabla^2 S_+ - \Lambda(\partial_y^2 + 2ik_C\partial_x)^2 S_+ + \varepsilon^2 S_+ - (\rho_r |S_+|^2 + \sigma |S_-|^2) S_+ , \quad (17a)$$

$$\tau(\partial_t S_- - v_g \partial_x S_-) = \frac{i(a_1 - a_2)}{2} \nabla^2 S_- - \Lambda(\partial_y^2 - 2ik_C\partial_x)^2 S_- + \varepsilon^2 S_- - (\rho_r |S_-|^2 + \sigma |S_+|^2) S_- , \quad (17b)$$

where we have set

$$\tau = \frac{\gamma_1 + \gamma_2}{2\gamma_1\gamma_2},$$

$$\Lambda = \frac{1}{2}\left(\frac{a_1\gamma_1 + a_2\gamma_2}{\gamma_1 + \gamma_2}\right)^2,$$

$$v_g = \frac{2k_C\gamma_1\gamma_2(a_1 - a_2)}{\gamma_1 + \gamma_2},$$

$$\sigma = \frac{1}{1+\Delta_0^2} + \frac{1}{1+(\Delta_0 + 4a_0k_C^2)^2} = \rho_r + \frac{1}{1+(\Delta_0 + 4a_0k_C^2)^2}.$$

These equations are two coupled complex Newell-Whitehead-Segel equations similar to those obtained for laser systems [6], and describe the growth of standing or traveling waves moving in opposite directions with group velocities $\pm v_g$. There are two families of stationary homogeneous solutions: TWs ($S_+ = 0$, $S_- \neq 0$ or vice versa) and SWs ($|S_+| = |S_-|$), the last solution corresponding to roll patterns (stripes). Using the same technique as in Ref. [4], it may easily be shown by linearizing the amplitude equations around the SW solution that this solution is always unstable. This is closely related to the fact that $\sigma > \rho_r$ in Eqs. (17); SWs are in fact favored for $\sigma < \rho_r$ [14].

IV. STABILITY OF TRAVELING WAVES: SECONDARY INSTABILITIES

Up to now it has been shown that, in the nondegenerate case, the OPO equations admit of exact TW solutions which are preferred to roll patterns just above threshold. The central question is now to determine the region of stable TW states (the Busse balloon) beyond the neutral stability curve. As in laser systems [6], the more global stability analysis of TW solutions can be performed by linearizing the full OPO equations about these exact solutions, and this method is sketched in Sec. IV A. Although this is the most powerful approach as it omits no classes of instability, it involves finding the eigenvalues of a complex 6×6 matrix, and therefore this study can be only numerical. Further analytical and physical insights into the stability problem can be nevertheless obtained by deriving amplitude and phase equations of the system. A detailed discussion of these approaches can be found in Refs. [6, 14]. Amplitude equations were derived in the preceding section in the negative detuning case to study competition between traveling- and standing-wave states. Since SWs are unstable, we can deal with only one amplitude equation, say, that for S_+, setting zero the other amplitude S_- in Eq. (17a). The most stringent limit of the amplitude equation is that it describe the nonlinear behavior of the system near threshold. This constraint can be removed by deriving a Cross-Newell phase equation, which gives an exact description of long-wavelength phase instabilities both near and well above threshold. The derivation of the phase equation will be done in Sec. IV B. Higher-order phase instabilities or amplitude instabilities cannot, however, be predicted by the phase equation, and analysis of the matrix eigenvalues in the linearized problem is needed. Examples of amplitude instabilities for the OPO equations in one transverse dimension will be presented in Sec. IV C.

A. Linear stability analysis of TW states

In this section we will work out the linear stability analysis of the TW solutions of the OPO equations. Because of the complex nature of the field variables, we will consider, together with the original Eqs. (1), their complex conjugate equations, and we will assume $B, A_1, A_2, B^*, A_1^*, A_2^*$ as independent variables. Using standard linear stability methods, we write

$$B = B_S(1+b),$$

$$B^* = B_S^*(1+b^*),$$

$$A_1 = A_{1S}(1+x_1),$$

$$A_1^* = A_{1S}^*(1+x_1^*),$$

$$A_2 = A_{2S}(1+x_2),$$

$$A_2^* = A_{2S}^*(1+x_2^*),$$

where B_S, A_{1S}, A_{2S} correspond to the TW solution given by Eq. (2) ($B_S^*, A_{1S}^*, A_{2S}^*$ are their complex conjugate), and $b, b^*, x_1, \ldots$ are perturbations. The most general solution of the perturbations in the linearized system is a superposition of solutions of the form

$$\begin{pmatrix} b \\ b^* \\ x_1 \\ x_1^* \\ x_2 \\ x_2^* \end{pmatrix} \propto \exp(\lambda t + iQ_x x + iQ_y y),$$

where $\lambda = \lambda(Q_x, Q_y)$ is one of the eigenvalues of a 6×6 matrix, whose expression is given in Appendix A, and $\mathbf{Q} = (Q_x, Q_y)$ is the transverse wave vector of the perturbation. Instability of the TW solution to the growth of transverse modulation with wave vector $\mathbf{Q}$ arises when the real part of at least one matrix eigenvalue becomes positive. The translational invariance of Eqs. (1) ensures the existence of a neutral mode with zero eigenvalue at $\mathbf{Q} = \mathbf{0}$, and this is also directly shown in Appendix A. This permits classification of the pattern forming instabilities in two classes: *phase instabilities*, which are instabilities arising from this neutral stable eigenvalue as the transverse wave vector $\mathbf{Q}$ is allowed to vary, and *amplitude instabilities*, which correspond to modes

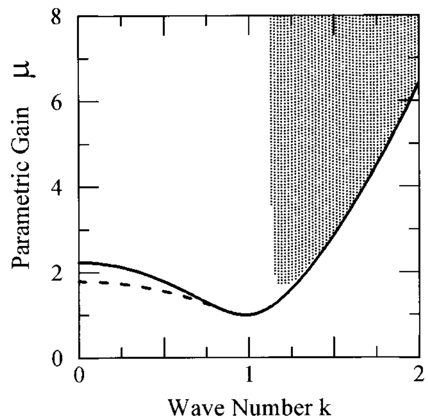

FIG. 2. Instability domain (gray area) of traveling waves to the growth of perturbations with zero transverse wave vector. System parameters are $\gamma_0=0.1$, $\gamma_1=\gamma_2=1$, $\Delta_0=\Delta_1=\Delta_2=-2$, and $v_1=0.4$. In the figure, the solid line is the neutral stability curve, and the dashed line defines the boundary of existence of traveling waves.

that are not neutral at $\mathbf{Q}=\mathbf{0}$. A detailed discussion of these pattern forming instabilities can be found in Ref. [6]. With respect to the laser case, the situation for the OPO equations is further complicated by the fact that there might be unstable eigenvalues at $\mathbf{Q}=\mathbf{0}$, as indicated in Fig. 2. The figure shows, in the plane (k,μ), the domain of TW solutions where Eq. (A1), given in Appendix A, has one eigenvalue with positive real part. This type of instability was previously studied for a degenerate OPO model in the absence of transverse degrees of freedom, and is related to the appearance of a self-pulsing behavior [16]. Because we are mainly interested in the study of transverse pattern instabilities, we will consider parameter values where such instability is absent. Even so, the instability scenario which arises in the numerical study of the matrix eigenvalues is quite complex, due to the presence of amplitude instabilities both longitudinal and transversal to the wave vector of the TW solution. Although a detailed analysis of these instabilities is complicated and goes beyond the purpose of this work, we will present in Sec. IV C a few examples of amplitude instabilities. Universal phase instabilities which arise from long-wavelength perturbations are, however, more dangerous in destabilizing the underlying TW pattern in the case of OPOs with low finesse for the pump field. Stability boundaries of Eckhaus and zigzag phase instabilities are calculated in Sec. IV B.

B. The Cross-Newell phase equation

The exact boundaries of the phase (Eckhaus and zigzag) instabilities can be obtained analytically by deriving the Cross-Newell phase equation for the TW solutions of the OPO equations. This approach was introduced in hydrodynamics by Cross and Newell for the study of convective patterns [18], and has been recently applied in the optical context to study phase instabilities in the Maxwell-Bloch model of the laser equations [6,8]. The basic idea of the method is to allow the phase ϑ of the field, whose wave vector $\mathbf{k}=\nabla\vartheta$ and frequency $\omega=\partial_t\vartheta$ are constants for the pure TW state, to vary slowly in both time and space. Assuming further that the field amplitude adiabatically follows any phase variation, a diffusion equation for the phase field ϑ can be obtained as a solvability condition in a multiple-scale expansion. The derivation of the phase equation is rather involved, and details of the calculations are reported in Appendix B. Assuming that the wave vector of the TW is oriented along the x axis, the phase diffusion equation reads

$$\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)(\partial_t\vartheta-\omega)=D_{\text{Eck}}\vartheta_{xx}+D_{\text{zig}}\vartheta_{yy}, \tag{18}$$

where the diffusion coefficients are given by

$$D_{\text{Eck}}=-4a_2k^2\frac{d\psi}{dk^2}+\frac{4k^2}{\gamma_2}\frac{d\psi}{dk^2}\frac{d\omega}{dk^2}+\frac{2k^2\tilde{\Delta}(a_1+a_2)}{C}\frac{dC}{dk^2}+\frac{2k^2\tilde{\Delta}}{C}\left(\frac{1}{\gamma_1}-\frac{1}{\gamma_2}\right)\frac{dC}{dk^2}\frac{d\omega}{dk^2}+\tilde{\Delta}(a_1+a_2), \tag{19a}$$

$$D_{\text{zig}}=\tilde{\Delta}(a_1+a_2). \tag{19b}$$

In Eq. (19a), dC/dk^2, $d\psi/dk^2$, and $d\omega/dk^2$ are the derivatives of the TW parameters C, ψ, and ω, respectively, with respect to k^2; they can be easily calculated from Eqs. (4) and (5):

$$\frac{dC}{dk^2}=\frac{\gamma_1a_1+\gamma_2a_2}{\gamma_1+\gamma_2}\frac{C\rho_i-\tilde{\Delta}}{\rho_r(C+1)-\rho_i\tilde{\Delta}}, \tag{20a}$$

$$\frac{d\psi}{dk^2}=\frac{\rho_r}{\tilde{\Delta}-C\rho_i}\frac{dC}{dk^2}, \tag{20b}$$

$$\frac{d\omega}{dk^2}=\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}(a_2-a_1). \tag{20c}$$

The boundaries of the phase instabilities arising from long-wavelength perturbations can be determined analytically from the phase equation (18). In fact, the stability requirement for the TW solution with wave number $\mathbf{k}=(k,0)$ is simply $D_{\text{Eck}}(k)>0, D_{\text{zig}}(k)>0$. The passage of D_{Eck} through zero signals the Eckhaus stability boundary, and D_{zig} passing through zero gives the zigzag stability boundary. Since the wave vector $\mathbf{k}$ has been chosen in the x direction, it follows that the Eckhaus instability corresponds to compression or dilation of the TW state, whereas the zigzag instability corresponds to the bending of the TW. An interesting feature of the diffusion coefficients in the phase equation is that they do not depend on the cavity decay rate of the pump field (γ_0), and therefore the Eckhaus and zigzag stability boundaries are the same for OPOs with high or low finesse for the pump field. In particular, for $\Delta<0$, we note that the zigzag unstable band always lies to the immediate left of the line $k=k_C$ and extends all of the way to the left boundary out to the neutral stability curve. On the contrary, the Eckhaus stability boundary strongly depends on the sign of the pump detuning Δ_0. This is shown in Fig. 3, where the Eckhaus stability domain is reported for three different values of the pump detuning. Note that, as the pump detuning becomes positive [Fig. 3(c)], the Eckhaus stability domain enlarges even below the neutral stability curve. The TW solution with maximum emission corresponds, in the (k,μ) plane, to the line $\mu=1+$

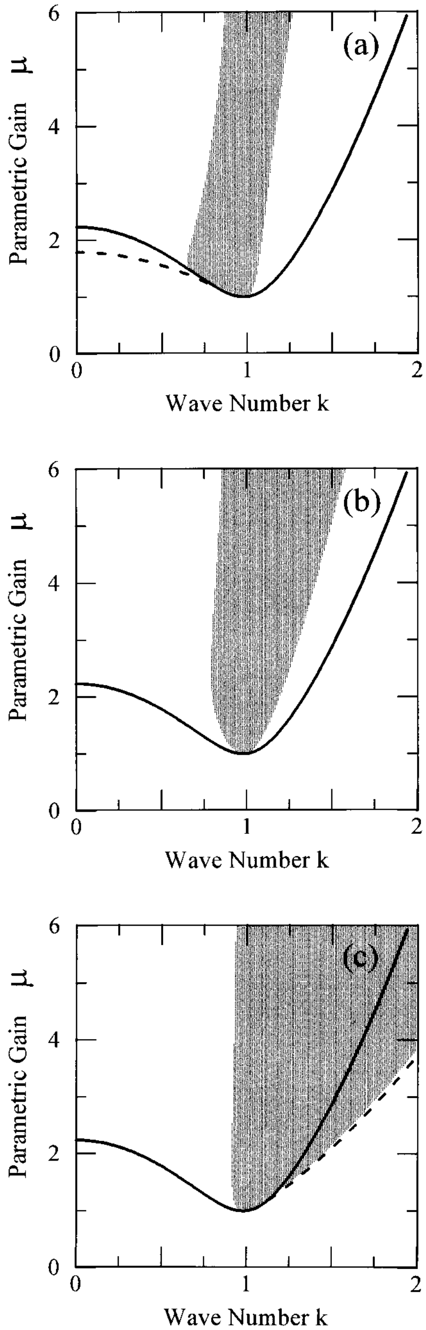

FIG. 3. Eckhaus stability domain (gray area), as computed from the Cross-Newell phase equation, for (a) $\Delta_0=-2$, (b) $\Delta_0=0$, and (c) $\Delta_0=2$; the other parameters are $\Delta_1=\Delta_2=-2$, $\gamma_0=\gamma_1=\gamma_2=1$, and $v_1=0.4$. In the figures, the solid lines are the neutral stability curves, and the dashed lines define the boundaries of existence of traveling waves; in (b) the dashed curve is completely obscured by the solid one.

$(\gamma_1 a_1+\gamma_2 a_2)(k^2-k_C^2)/[\Delta_0(\gamma_1+\gamma_2)]$; the equation of this curve can be obtained by setting $dC/dk^2=0$, i.e., by looking for the wave number k which maximizes the TW intensity for a given value of the parametric gain. Note that, for $\Delta_0<0$, this line falls entirely in the unstable zigzag band, whereas for $\Delta_0>0$ it lies within the Eckhaus and zigzag stability boundaries.

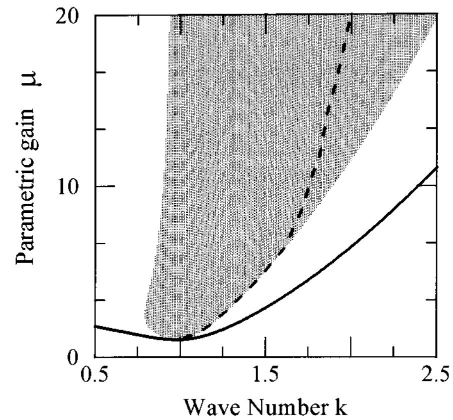

FIG. 4. Results of the stability analysis for traveling waves in one transverse dimension. The solid line is the neutral stability curve, the gray area is the stability domain as computed from the phase equation, and the dashed line is the boundary for emergence of amplitude instabilities as computed from numerical analysis of the matrix eigenvalues. Traveling waves on the right of this curve undergo an amplitude instability. Parameters are $\Delta_1=\Delta_2=-2$, $\Delta_0=0$, $\gamma_1=\gamma_2=1$, $v_1=0.4$, and $\gamma_0=10$.

C. Amplitude instabilities

Although the phase diffusion equation has showed that long-wavelength phase instabilities are independent of the cavity decay rate of the pump field, numerical analysis of the matrix eigenvalues in the linearized problem shows that a more complex instability scenario appears when considering OPOs with a high finesse for the pump field. In this case, in fact, amplitude instabilities at short wavelengths, arising from modes that are not neutral at $\mathbf{Q}=\mathbf{0}$, take place, reducing the Busse balloon to a small region around the line $k=k_C$. For simplicity, we consider here only one transverse dimension, i.e., we consider only perturbations with wave vector parallel to that of the TW state. As an example, Figs. 4 and 5 show the Busse balloon as computed from the phase diffusion equation and from the numerical analysis of the matrix eigenvalues for two different values of the pump decay rate in the cavity. As can be seen, in the low finesse case (Fig. 4), long-wavelength (Eckhaus) phase instabilities are more dangerous, and amplitude instabilities appear only well above

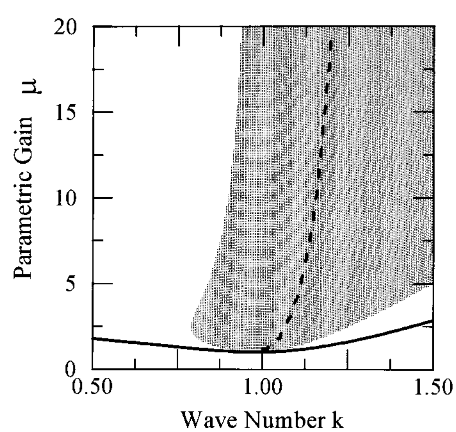

FIG. 5. Same as Fig. 4 with $\gamma_0=1$.

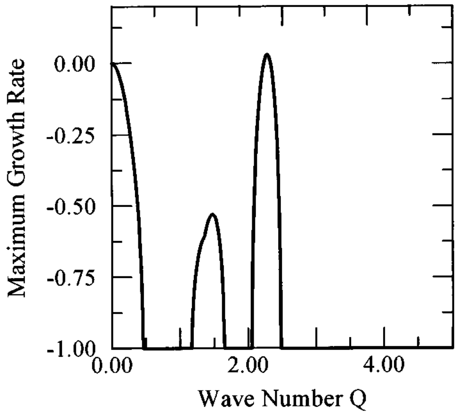

FIG. 6. Largest real part of any eigenvalue in the linearized problem as a function of the perturbing wave number Q for $k=1.125$, $\mu=5$. Parameter values are the same as in Fig. 5.

threshold for wave numbers larger than the critical wave number. On the contrary, in the high finesse case (Fig. 5), the Busse balloon boundary at high wave numbers is strongly pushed toward the line $k=k_C$. The nature of the instability, which reduces the Busse balloon as the pump decay rate decreases, is a short-wavelength amplitude instability longitudinal to the wave vector of the TW state. This is shown in Fig. 6, where the real part of the eigenvalue with maximum growth rate is reported as a function of the perturbing wave number Q at the incipient instability point ($k=1.125$, $\mu=5$) for the case of Fig. 5; similar behaviors were found when analyzing other instability points near the instability boundary.

V. CONCLUSIONS

It has been demonstrated that the three wave equations describing nonlinear transverse effects in optical parametric oscillators have a family of exact TW solutions when either polarization or frequency degeneracy constraint is removed. These states correspond to an off-axis emission for both signal and idler fields along two symmetric transverse directions due to momentum conservation, and are preferred to SW states previously found in the degenerate case. The tendency to emit TWs instead of SWs has been demonstrated by deriving, with standard weakly nonlinear analysis, two coupled Newell-Whitehead-Segel equations which describe the growth of traveling or standing waves close to threshold. Stability of the TW states above threshold has been investigated by standard linear stability analysis and by deriving a Cross-Newell phase equation, and analytical expressions of Eckhaus and zigzag phase instabilities have been derived from the coefficients of the phase equation. The appearance of amplitude instabilities in OPOs with high finesse for the pump field has also been briefly discussed.

APPENDIX A: LINEAR STABILITY ANALYSIS

The matrix of the equations obtained by linearizing the OPO equations about the TW solution is

$$M=\begin{pmatrix} -\gamma_0(1+i\Delta_0+a_0Q^2) & 0 & \gamma_0(1+i\Delta_0) & 0 & \gamma_0(1+i\Delta_0) & 0 \\ 0 & -\gamma_0(1-i\Delta_0-ia_0Q^2) & 0 & \gamma_0(1-i\Delta_0) & 0 & \gamma_0(1-i\Delta_0) \\ -\dfrac{\gamma_1 C}{1+i\Delta_0} & 0 & -i\omega-\gamma_1\{1+i\Delta_1+ia_1P_+\} & 0 & 0 & \gamma_1(1+i\tilde{\Delta}) \\ 0 & -\dfrac{\gamma_1 C}{1-i\Delta_0} & 0 & i\omega+\gamma_1\{-1+i\Delta_1+ia_1P_-\} & \gamma_1(1-i\tilde{\Delta}) & 0 \\ -\dfrac{\gamma_2 C}{1+i\Delta_0} & 0 & 0 & \gamma_2(1+i\tilde{\Delta}) & i\omega-\gamma_2\{1+i\Delta_2+ia_2P_-\} & 0 \\ 0 & -\dfrac{\gamma_2 C}{1-i\Delta_0} & \gamma_2(1-i\tilde{\Delta}) & 0 & 0 & -i\omega+\gamma_2\{-1+i\Delta_2+ia_2P_+\} \end{pmatrix},$$

where $P_\pm=(Q_x\pm k_x)^2+(Q_y\pm k_y)^2$ and $Q^2=Q_x^2+Q_y^2$. In order to prove that the TW solution corresponding to the lower sign in Eq. (4a) is always unstable, let us restrict our analysis to considering perturbations with zero transverse wave vector, i.e., let us assume $\mathbf{Q}=\mathbf{0}$. In this case, it may be shown that the eigenvalues of the matrix M are the roots of the following sixth-order algebraic equation:

$$\lambda(\lambda^5+c_1\lambda^4+c_2\lambda^3+c_3\lambda^2+c_4\lambda+c_5)=0, \qquad \text{(A1)}$$

where the explicit expressions of the (real) polynomial coefficients are

$$c_1=2(\gamma_0+\gamma_1+\gamma_2),$$

$$c_2=\gamma_0^2+\gamma_1^2+\gamma_2^2+\tilde{\Delta}^2(\gamma_1-\gamma_2)^2+\gamma_0^2\Delta_0^2+2\gamma_1\gamma_2+4\gamma_2\gamma_0 +4\gamma_1\gamma_0+2C\gamma_1\gamma_0+2C\gamma_0\gamma_2,$$

$$\vdots$$

$$c_5=4\gamma_1\gamma_2\gamma_0^2C(\gamma_1+\gamma_2)(1+C-\Delta_0\tilde{\Delta})$$

(we have also evaluated explicitly the expressions of c_3 and c_4, but the formulas are too cumbersome to be given here and are not needed for our purpose). The existence of one zero eigenvalue agrees with the translational invariance of Eqs. (1), and phase instabilities arise from this neutral eigenvalue when the transverse wave vector of the perturbation is taken away from zero. Stability of the TW solutions against the growth of homogeneous perturbations requires negative real part of all other eigenvalues. A *necessary* condition is that all the coefficients $c_1,\ldots,c_5$ be positive. In particular, from the condition $c_5>0$ it follows that the TW solution corresponding to the lower sign in Eq. (4a) is always unstable.

APPENDIX B: DERIVATION OF THE CROSS-NEWELL EQUATION

In this appendix we outline the derivation of the phase equation for the TW solutions of the OPO equations in the nondegenerate case. A detailed description of the procedure can be found in Refs. [6, 8, 14, 18]. The method consists in looking for a solution of the dynamic equations in the form of a TW state whose wave vector $\mathbf{k}=\nabla_x\vartheta$ and frequency $\omega=\partial_t\vartheta$ are allowed to vary slowly in space and time; here ϑ denotes the phase of the TW state. This is done by introducing slow time-space coordinates by setting

$$X=\varepsilon x,\quad Y=\varepsilon y,\quad T_1=\varepsilon t,\quad T_2=\varepsilon^2 t, \qquad \text{(B1)}$$

where the small parameter ε is a bookkeeping parameter which organizes the perturbation expansion. Its inverse specifies the long length scale over which pattern reorientation occurs. Note that, as in the laser case [6], besides the diffusive ε^2 time scaling, another time scale $T_1=\varepsilon t$ is needed to capture slow evolution which occurs at the group velocity of the wave packet. Because we want the wave vector $\mathbf{k}=\nabla_x\vartheta$ to depend on X, Y, T_1, and T_2, we introduce a scaled phase variable $\Theta(X,Y,T_1,T_2)=\varepsilon\vartheta$, so that

$$\mathbf{k}=\nabla_x\vartheta=\nabla_X\Theta, \qquad \text{(B2)}$$

$$\partial_t\vartheta=\partial_{T_1}\Theta+\varepsilon\partial_{T_2}\Theta=\omega+\varepsilon\partial_{T_2}\Theta, \qquad \text{(B3)}$$

where ∇_x (∇_X) means the gradient with respect to fast (slow) space variables, and $\omega=\partial_{T_1}\Theta$ is the wave frequency. Let us now expand the field variables B, A_1, and A_2 in powers of ε:

$$B=B^{(0)}(\vartheta,X,T)+\varepsilon B^{(1)}(\vartheta,X,T)+\cdots,$$

$$A_1=A_1^{(0)}(\vartheta,X,T)+\varepsilon A_1^{(1)}(\vartheta,X,T)+\cdots, \qquad \text{(B4)}$$

$$A_2=A_2^{(0)}(\vartheta,X,T)+\varepsilon A_2^{(1)}(\vartheta,X,T)+\cdots,$$

where $X(T)$ is a global notation for X and Y (T_1 and T_2). Note that, in expansion (B4), we treat the phase variable ϑ as an independent variable, and derivatives of any function $f=f(\vartheta,X,T)$ act according to the rule

$$\partial_t f=\omega\partial_\vartheta f+\varepsilon\partial_\vartheta f\partial_{T_2}\Theta+\varepsilon\partial_{T_1}f+\varepsilon^2\partial_{T_2}f, \qquad \text{(B5)}$$

$$\nabla_x^2 f=\varepsilon^2\nabla_X^2 f+2\varepsilon\mathbf{k}\cdot\nabla_X(\partial_\vartheta f)+\varepsilon\partial_\vartheta f\nabla_X\cdot\mathbf{k}+k^2\partial_\vartheta^2 f. \qquad \text{(B6)}$$

We now substitute Eqs. (B4)–(B6) into Eqs. (1) and collect terms at each order in ε. At $O(\varepsilon^0)$ we have

$$\frac{\omega}{\gamma_0}\partial_\vartheta B^{(0)}=-(1+i\Delta_0)B^{(0)}-A_1^{(0)}A_2^{(0)}+ia_0k^2\partial_\vartheta^2B^{(0)},$$

$$\frac{\omega}{\gamma_1}\partial_\vartheta A_1^{(0)}=-(1+i\Delta_1)A_1^{(0)}+\mu A_2^{(0)*}+ia_1k^2\partial_\vartheta^2A_1^{(0)},$$

$$\frac{\omega}{\gamma_2}\partial_\vartheta A_2^{(0)}=-(1+i\Delta_2)A_2^{(0)}+\mu A_1^{(0)*}+ia_2k^2\partial_\vartheta^2A_2^{(0)},$$

whose solution is the exact TW solution

$$B^{(0)}=-\frac{C\exp(i\psi)}{1+i\Delta_0}\equiv\overline{B^{(0)}}, \qquad \text{(B7)}$$

$$A_1^{(0)}=\sqrt{C}\exp(i\vartheta)\equiv\overline{A_1^{(0)}}\exp(i\vartheta), \qquad \text{(B8)}$$

$$A_2^{(0)}=\sqrt{C}\exp(i\psi)\exp(-i\vartheta)\equiv\overline{A_2^{(0)}}\exp(-i\vartheta), \qquad \text{(B9)}$$

where C, ψ, and ω are given by Eqs. (4) and (5) in the text. At $O(\varepsilon)$, we get

$$-\frac{\omega}{\gamma_0}\partial_\vartheta B^{(1)}-(1+i\Delta_0)B^{(1)}+ia_0k^2\partial_\vartheta^2B^{(1)}-A_1^{(0)}A_2^{(1)} -A_2^{(0)}A_1^{(1)}=\frac{1}{\gamma_0}\partial_{T_1}B^{(0)}, \qquad \text{(B10)}$$

$$-\frac{\omega}{\gamma_1}\,\partial_\vartheta A_1^{(1)}-(1+i\Delta_1)A_1^{(1)}+ia_1k^2\partial_\vartheta^2A_1^{(1)}+\mu A_2^{(1)*}$$
$$+A_2^{(0)*}B^{(1)}+A_2^{(1)*}B^{(0)}$$
$$=-ia_1[2\mathbf{k}\cdot\nabla_X(\partial_\vartheta A_1^{(0)})+\partial_\vartheta A_1^{(0)}\nabla_X\cdot\mathbf{k}]$$
$$+\frac{1}{\gamma_1}\,\partial_\vartheta A_1^{(0)}\partial_{T_2}\Theta+\frac{1}{\gamma_1}\,\partial_{T_1}A_1^{(0)}, \tag{B11}$$

$$-\frac{\omega}{\gamma_2}\,\partial_\vartheta A_2^{(1)}-(1+i\Delta_2)A_2^{(1)}+ia_2k^2\partial_\vartheta^2A_2^{(1)}+\mu A_1^{(1)*}$$
$$+A_1^{(0)*}B^{(1)}+A_1^{(1)*}B^{(0)}$$
$$=-ia_2[2\mathbf{k}\cdot\nabla_X(\partial_\vartheta A_2^{(0)})+\partial_\vartheta A_2^{(0)}\nabla_X\cdot\mathbf{k}]$$
$$+\frac{1}{\gamma_2}\,\partial_\vartheta A_2^{(0)}\partial_{T_2}\Theta+\frac{1}{\gamma_2}\,\partial_{T_1}A_2^{(0)}. \tag{B12}$$

The phase-diffusion equation is derived as a solvability condition of the system of these equations. In fact, the associated homogeneous system [obtained by setting equal to zero the right-hand sides of Eqs. (B10)–(B12)] has a nontrivial solution, which is simply the translation mode $(\partial_\vartheta B^{(0)},\partial_\vartheta A_1^{(0)},\partial_\vartheta A_2^{(0)})^T$. This result follows from the translational invariance of the original OPO equations (1). Hence the solvability condition requires that the right-hand sides of Eqs. (B10)–(B12) not drive this translation mode. To obtain the solvability condition in a simple way, let us look for a solution of Eqs. (B10)–(B12) of the form

$$B^{(1)}=\overline{B^{(1)}}, \tag{B13}$$

$$A_1=\overline{A_1^{(1)}}\exp(i\vartheta), \tag{B14}$$

$$A_2^{(1)}=\overline{A_2^{(1)}}\exp(-i\vartheta), \tag{B15}$$

where $\overline{B^{(1)}}$, $\overline{A_1^{(1)}}$, and $\overline{A_2^{(1)}}$ may depend on X and T, but not on ϑ. Putting Eqs. (B13)–(B15) into Eqs. (B10)–(B12) and eliminating the variables $\overline{B^{(1)}}$ and $\overline{A_2^{(1)}}$ from the linear system of equations so obtained, we have

$$\overline{A_1^{(1)}}+\overline{A_1^{(1)}}^*=g\,\exp(i\psi)/I, \tag{B16}$$

where

$$g=g_1-\left(1-\frac{\beta}{\mu}\right)g_2\exp(-i\psi)+\frac{\beta}{\mu}\,g_2^*\exp(-i\psi),$$

$$g_1=\frac{i}{\gamma_1}\,\overline{A_1^{(0)}}\partial_{T_2}\Theta+a_1[2\mathbf{k}\cdot\nabla_X\overline{A_1^{(0)}}+\overline{A_1^{(0)}}\nabla_X\cdot\mathbf{k}]$$
$$+\frac{1}{\gamma_1}\,\partial_{T_1}\overline{A_1^{(0)}}+\frac{\overline{A_2^{(0)}}^*}{\gamma_0(1+i\Delta_0)}\,\partial_{T_1}\overline{B^{(0)}},$$

$$g_2=-\frac{i}{\gamma_2}\,\overline{A_2^{(0)}}\partial_{T_2}\Theta-a_2[2\mathbf{k}\cdot\nabla_X\overline{A_2^{(0)}}+\overline{A_2^{(0)}}\nabla_X\cdot\mathbf{k}]$$
$$+\frac{1}{\gamma_2}\,\partial_{T_1}\overline{A_2^{(0)}}+\frac{\overline{A_1^{(0)}}^*}{\gamma_0(1+i\Delta_0)}\,\partial_{T_1}\overline{B^{(0)}},$$

$$I=\beta\left(-2+\frac{\beta}{\mu}+\frac{\beta^*}{\mu}\right),$$

$$\beta=(1+i\tilde{\Delta})\exp(i\psi).$$

The solvability condition of Eq. (B16) requires $\mathrm{Im}[g\,\exp(i\psi)/I]=0$, which reads explicitly

$$\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\partial_{T_2}\Theta=-2a_2\mathbf{k}\cdot\nabla_X\psi+\frac{1}{\gamma_2}\,\partial_{T_1}\psi+\tilde{\Delta}\left\{(a_1+a_2)\right.$$
$$\times\left[\frac{\mathbf{k}\cdot\nabla_XC}{C}+\nabla_X\cdot\mathbf{k}\right]$$
$$\left.+\frac{1}{2C}\left(\frac{1}{\gamma_1}-\frac{1}{\gamma_2}\right)\partial_{T_1}C\right\}. \tag{B17}$$

Assuming that the field amplitudes follow adiabatically any change of the phase Θ, the dependence of C and ψ on the slow variables X, Y, and T_1 is implied in Eqs. (4) through $k^2=|\nabla_X\Theta|^2$. Using the derivatives rules

$$\partial_{T_1}C=2\,\frac{dC}{dk^2}\,\mathbf{k}\cdot\,\frac{\partial\mathbf{k}}{\partial T_1}=2\,\frac{dC}{dk^2}\,\frac{d\omega}{dk^2}\,(\mathbf{k}\cdot\nabla_Xk^2),$$

$$\mathbf{k}\cdot\nabla_XC=\frac{dC}{dk^2}\,\mathbf{k}\cdot\nabla_Xk^2$$

(and similar formulas for ψ), after observing that

$$\nabla_X\cdot\mathbf{k}=\Theta_{XX}+\Theta_{YY},$$

$$\mathbf{k}\cdot\nabla_Xk^2=2[k_x^2\Theta_{XX}+2k_xk_y\Theta_{XY}+k_y^2\Theta_{YY}],$$

where $\mathbf{k}=(k_x,k_y)$, Eq. (B17) takes the form

$$\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\partial_{T_2}\Theta=2\left\{-2a_2\,\frac{d\psi}{dk^2}+\frac{2}{\gamma_2}\,\frac{d\psi}{dk^2}\,\frac{d\omega}{dk^2}\right.$$
$$+\tilde{\Delta}\,\frac{(a_1+a_2)}{C}\,\frac{dC}{dk^2}+\frac{\tilde{\Delta}}{C}\left(\frac{1}{\gamma_1}\right.$$
$$\left.\left.-\frac{1}{\gamma_2}\right)\,\frac{dC}{dk^2}\,\frac{d\omega}{dk^2}\right\}(k_x^2\Theta_{XX}+2k_xk_y\Theta_{XY}$$
$$+k_y^2\Theta_{YY})+(a_1+a_2)$$
$$\times\tilde{\Delta}(\Theta_{XX}+\Theta_{YY}). \tag{B18}$$

In order to capture the various phase instabilities more easily, let us choose the x coordinate axis along the $\mathbf{k}$ wave vector of the TW solution, i.e., let us assume $k_y=0$ in Eq. (B18). Using $\partial_t\vartheta=\omega+\varepsilon\partial_{T_2}\Theta$ and the relations $\varepsilon\Theta_{XX}=\vartheta_{xx}$, $\varepsilon\Theta_{YY}=\vartheta_{yy}$, we finally get from Eq. (B18) the Cross-Newell equation

$$\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)(\partial_t\vartheta-\omega)=D_{\mathrm{Eck}}\vartheta_{xx}+D_{\mathrm{zig}}\vartheta_{yy},$$

where the expressions of the phase diffusion coefficients D_{Eck} and D_{zig} are those given in the text [Eqs. (19)].

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