Lang and Kobayashi phase equation

P. M. Alsing,* V. Kovanis,† and A. Gavrielides‡
Nonlinear Optics Center, Phillips Laboratory, 3350 Aberdeen Avenue SE, Kirtland Air Force Base, Kirtland, New Mexico 87117-5776

T. Erneux§
Université Libre de Bruxelles, Optique Nonlinéaire Théorique, Campus Plaine, C.P. 231, 1050 Bruxelles, Belgium

(Received 6 June 1995; revised manuscript received 20 November 1995)

Lang and Kobayashi equations for a semiconductor laser subject to optical feedback are investigated by using asymptotic methods. Our analysis is based on the values of two key parameters, namely, the small ratio of the photon and carrier lifetimes and the relatively large value of the linewidth enhancement factor. For low feedback levels, we derive a third-order delay-differential equation for the phase of the laser field. We then show analytically and numerically that this equation admits coexisting branches of stable periodic solutions that appear at different and almost constant amplitudes. These amplitudes are proportional to the roots of the Bessel function $J_1(x)$. The bifurcation diagram of the phase equation is in good agreement with the numerical bifurcation diagram of the original Lang and Kobayashi equations. We interpret the onset of the periodic solutions as the emergence of a new set of external cavity modes with a more complicated time dependence. [S1050-2947(96)05005-6]

PACS number(s): 42.55.Px

I. INTRODUCTION

Applications of semiconductor lasers span a broad range of areas from optical communication to optical ranging and sensing. Semiconductor lasers are extremely sensitive to optical feedback, which results from undesired reflections from optical elements and detectors. A small amount of optical feedback is sufficient to produce chaotic instabilities that lead to higher intensity or frequency noise [1]. Models of semiconductor lasers subject to optical feedback are formulated by coupled delay-differential equations that are difficult to explore analytically or even numerically. As a result, little is known on the bifurcation mechanisms leading to these chaotic responses. In 1980, Lang and Kobayashi (LK) [2] formulated a simple model consisting of two ordinary delay-differential equations for the complex electrical field and the carrier number. Computer simulations have shown that the LK equations correctly describe the dominant effects observed experimentally. This includes the occurrence of mode hopping [3,4], low-frequency fluctuations [5–7], the onset of coherence collapse [3,8], and coexisting attractors exhibiting time-periodic intensities [9–11].

In dimensionless form, LK equations [2] for the complex electrical field Y and the excess carrier number Z are given by

$$\frac{dY}{ds}=(1+i\alpha)ZY+\eta\,\exp(-i\Omega\theta)Y(s-\theta), \tag{1.1a}$$

$$T\,\frac{dZ}{ds}=P-Z-(1+2Z)|Y|^2. \tag{1.1b}$$

In Eqs. (1.1a) and (1.1b), time is $s\equiv t/\tau_p$ where τ_p is the photon lifetime. Ω is the dimensionless angular frequency of the solitary laser. $T\equiv\tau_s/\tau_p$ is the ratio of the carrier and photon lifetimes. $\theta\equiv\tau/\tau_p$ is the ratio of the external cavity round-trip time and the photon lifetime. P is the dimensionless pumping current above threshold. $\eta>0$ is the strength of the feedback and our bifurcation parameter. α is the linewidth enhancement factor. Typical values of the parameters are $T=O(10^3)$, $\theta=O(10^3)$, $|P|<1$, $\Omega\theta(\mathrm{mod}2\pi)$ is $O(1)$, $\alpha\sim5$, and $\eta=O(10^{-2})$. Without the delay term, Eqs. (1.1a) and (1.1b) are the usual dimensionless equations considered for the semiconductor laser subject to optical injection or for arrays of coupled lasers. In the Appendix, we consider the LK equation as introduced in [5] and show how to formulate Eqs. (1.1a) and (1.1b).

A basic solution of these equations called an external cavity mode solution is a periodic solution of the form

$$Y=A_e\,\exp(i\Omega_e s)\quad\text{and}\quad Z=Z_e, \tag{1.2}$$

where A_e, Ω_e, and Z_e are constants. Its linear stability can be analyzed and approximations of Hopf bifurcation points have been determined analytically [12,14]. For very low feedback levels, it is reasonable to assume that the intensity of the laser field is almost a constant and that the phase of the laser field is the main dynamical variable. This approximation has led to the formulation of phase equations [4,15,16] that reproduced the mode hopping phenomena but never showed the oscillatory regimes predicted by the linear stability analysis of (1.2). The main purpose of this paper is to derive a phase equation from the LK equations (1.1a) and (1.1b) that exhibits time-periodic intensity solutions. To this end, we seek an asymptotic solution of Eqs. (1.1a) and (1.1b) based on the large value of T and the relatively large value of α compared to the other time constants. Our analysis benefits from a recent study of a semiconductor laser subject to an injected signal [17] for which we derived a phase equation.

*Electronic address: alsing@arom.plk.af.mil
†Also at Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131. Electronic address: kovanis@xaos.plk.af.mil
‡Electronic address: tom@photon.plk.af.mil
§Electronic address: terneux@ulb.ac.be

In this paper we show that branches of time-periodic intensity solutions emerge from either Hopf bifurcation points or from limit points (saddle-node bifurcation points). As a result, coexistence of stable periodic states is possible as was suspected in earlier studies [9,10]. In [18], the coexistence of periodic states has been analyzed in terms of two successive Hopf bifurcations branches. In this paper, we examine a different mechanism involving isolated branches of solutions. We interpret the onset of the periodic solutions as the emergence of a new set of external cavity modes with a time dependence more complicated than that of simple plane waves. These modes arise when the intracavity field reproduces itself after one round trip, up to a scale factor.

The paper is organized as follows. In Sec. II, we derive a phase equation starting from LK equations (1.1a) and (1.1b). In Sec. III, we study this phase equation and determine branches of periodic solutions. In Sec. IV, we examine the validity of our analytical predictions by determining numerically the bifurcation diagrams of the original LK equations and the approximate LK phase equation. In Sec. V, we discuss our main results. In the Appendix we introduce the LK equations with all the relevant definitions of the parameters.

II. ASYMPTOTIC ANALYSIS

We first analyze the behavior of the single-mode solution (1.2) and its linear stability for T large and α large. We consider the following range of values of the feedback rate:

$$\eta = O(T^{-1}\alpha^{-1}) \to O(T^{-1/2}\alpha^{-1}) \tag{2.1}$$

and find that [14]

$$Z_e = O(\eta) \quad \text{and} \quad A_e = P^{1/2} + O(\eta). \tag{2.2}$$

Furthermore, if we assume

$$\Omega = O(T^{-1/2}) \quad \text{and} \quad \theta = O(T^{1/2}), \tag{2.3}$$

then Ω_e is $O(T^{-1/2})$ and its leading approximation satisfies the implicit equation

$$\eta \simeq -\frac{1}{\alpha}\frac{\Omega_e - \Omega}{\cos(\Omega_e \theta)}. \tag{2.4}$$

Equation (2.4) admits several branches of solutions and their number increases as η increases. A stable single-mode solution may change stability as η surpasses a critical value η_H given by [14]

$$\eta_H \simeq \frac{1+2P}{T\alpha}\frac{1}{\sin(\Omega_e \theta)[\cos(\omega\theta)-1]}, \tag{2.5}$$

where $\cos(\omega\theta)-1 \neq 0$ and ω is the laser relaxation frequency defined by

$$\omega \equiv \sqrt{2P/T}. \tag{2.6}$$

The condition $\eta_H > 0$ requires the inequality $\sin(\Omega_e\theta) < 0$. We also note from our analysis of the Hopf bifurcation conditions that the frequency ω_H at the Hopf point is close to ω for large T. These asymptotic properties of the Hopf bifurcation point were obtained by first formulating the linearized LK equations and then taking the limit large T and large α.

In this section, we propose an alternative strategy. We first approximate the nonlinear LK equations for large α and then determine the Hopf point from the linearized theory using large T. To facilitate our analysis, we introduce a new basic time defined as

$$S \equiv \omega s \tag{2.7}$$

and seek a solution of the LK equations in terms of the rescaled carrier number density x, electric field y, and the phase Φ defined by means of the following relations:

$$Z = x\omega/\alpha, \tag{2.8}$$

$$Y = \sqrt{P}(1 + y/\alpha)\exp[i(\Phi - \Omega s)]. \tag{2.9}$$

After inserting the expressions (2.7), (2.8), and (2.9) into Eqs. (1.1a) and (1.1b), we neglect all $O(\alpha^{-1})$ small terms and obtain the following problem for x, y, and Φ:

$$\frac{dx}{dS} = -y - \omega\xi x, \tag{2.10a}$$

$$\frac{dy}{dS} = x + \Lambda\ \cos[\Phi(S-\Theta) - \Phi(S)], \tag{2.10b}$$

$$\frac{d\Phi}{dS} = \Delta + x, \tag{2.10c}$$

where the control parameter Λ and the fixed parameter Θ, Δ, and ξ are defined by

$$\Lambda \equiv \alpha\eta/\omega, \tag{2.11}$$

$$\Theta \equiv \theta\omega = O(1), \tag{2.12}$$

$$\Delta \equiv \frac{\Omega\,\theta(\mathrm{mod}2\pi)}{\omega\theta} = O(1), \tag{2.13}$$

$$\xi \equiv \frac{1+2P}{2P} = O(1). \tag{2.14}$$

Eliminating y, we find that x satisfies the equation for a harmonic oscillator driven nonlinearly by the phase Φ. We may further eliminate x and obtain a third-order differential equation for Φ only given by

$$\Phi''' + \omega\xi\Phi'' + \Phi' - \Delta + \Lambda\ \cos[\Phi(S-\Theta) - \Phi(S)] = 0, \tag{2.15}$$

where the prime means differentiation with respect to S. The second term is small because $\omega = O(T^{-1/2})$ is small. If the pumping P is too small, the asymptotic approximation may fail. Mathematically, $\xi = O(1)$ means that $\xi = \xi(P)$ is fixed as we take the limit $T \to \infty$. Numerically, our analysis remains valid if the coefficient $\omega\xi$ remains small. We are currently investigating the double limit $T \to \infty$ and $P \to 0$ and we expect the failure of the phase equation for very low P. Note that $\Phi(S)$ can be unbounded as $S \to \infty$ but $\Phi'(S)$ must be bounded as a consequence of Eq. (2.10c) and the fact that x

is a physically bounded function of time. Equation (2.15) is the main point of this paper. In the subsequent sections we describe both analytical and numerical results that show that Eq. (2.15) captures the coexistence of periodic attractors exhibited by the full LK model, Eq. (1.1).

III. PERIODIC SOLUTIONS

In this section, we analyze Eq. (2.15) in detail. We have verified that the particular solution $\Phi=\Omega_e S/\omega$ satisfies Eq. (2.15), where Ω_e satisfies (2.4) now in terms of $\Lambda=\alpha\eta/\omega$. In addition, the linearized phase equation leads to a Hopf bifurcation point that matches the expression (2.5). The analysis of the condition for a single external mode solution and its linear stability motivates an exploration of Eq. (2.15) for periodic solutions in the regime $\Lambda=O(\omega)$.

We apply a two-time perturbation method [19] and seek a solution of Eq. (2.15) of the form

$$\Phi(S,\zeta,\omega)=\Phi_0(S,\zeta)+\omega\Phi_1(S,\zeta)+\cdots, \tag{3.1}$$

where ζ is a slow time defined by

$$\zeta\equiv\omega S. \tag{3.2}$$

We consider S and ζ as two independent time variables, which implies the chain rule $\Phi'=\Phi_S+\omega\Phi_\zeta$ and the expansion $\Phi(S-\Theta)=\Phi(S-\Theta,\zeta-\omega\Theta)=\Phi(S-\Theta,\zeta)-\omega\Theta\Phi_\zeta(S-\Theta,\zeta)+\cdots$. We also expand the control parameter Λ as

$$\Lambda=\omega(\Lambda_1+\cdots). \tag{3.3}$$

After introducing (3.1), (3.2), and (3.3) into Eq. (2.15), we obtain a sequence of linear problems for $\Phi_0,\Phi_1\ldots$. The two first problems are given by

$$\Phi_{0SSS}+\Phi_{0S}=\Delta \tag{3.4}$$

and

$$\Phi_{1SSS}+\Phi_{1S}=-3\Phi_{0SS\zeta}-\Phi_{0\zeta}-\xi\Phi_{0SS}-\Lambda_1\cos[\Phi_0(S-\Theta,\zeta)-\Phi_0(S,\zeta)]. \tag{3.5}$$

The solution of the first problem is

$$\Phi_0=A(\zeta)\cos[S+v(\zeta)]+\Delta S+B(\zeta), \tag{3.6}$$

where A, B, and v are unknown functions of ζ. We next introduce (3.6) into the right-hand side (RHS) of Eq. (3.5). Expanding the trigonometric function and using Bessel function identities [20] we find that the right-hand side has the form

$$\begin{aligned}\text{RHS}=&2A'\cos(S+v)-2Av'\sin(S+v)-B'\\&+\xi A\cos(S+v)-\Lambda_1[\cos(\Delta\Theta)J_0(D)\\&+2\sin(\Delta\Theta)J_1(D)\sin(S+v-\Theta/2)+\cdots],\end{aligned} \tag{3.7}$$

where we have omitted the higher-order harmonic functions of S; $J_0(D)$ and $J_1(D)$ denote Bessel functions. D is defined by

$$D\equiv2A\sin(\Theta/2). \tag{3.8}$$

Solvability of the Φ_1 equation requires that Φ_{1S} is bounded with respect to S. This implies that we eliminate in the right-hand side of Eq. (3.5) the terms multiplying $\cos(S+v)$ and $\sin(S+v)$. As a result, we obtain two conditions given by

$$A'=-\frac{\xi}{2}A-\Lambda_1\sin(\Delta\Theta)J_1(D)\sin(\Theta/2) \tag{3.9}$$

and

$$v'=-\frac{\Lambda_1}{A}\sin(\Delta\Theta)J_1(D)\cos(\Theta/2). \tag{3.10}$$

Furthermore, we determine an equation for B by eliminating the constant term in Eq. (3.7). The equation for B is

$$B'=-\Lambda_1\cos(\Delta\Theta)J_0(D). \tag{3.11}$$

Thus B is passively related to A and is obtained by integrating Eq. (3.11). Equation (3.9) is an equation for the amplitude A (the bifurcation equation). Equation (3.10) is an equation for the frequency correction v. If $\sin(\Theta/2)=0$ [or is $O(\omega)$ small], the perturbation analysis needs to be modified. This is a case of resonance that appears when the delay time θ is close to a multiple of the laser relaxation oscillation period $P_0=2\pi/\omega$ (i.e., $P_0\theta\sim2n\pi/\omega$). Similarly, the case $\sin(\Delta\Theta)=0$ or $O(\omega)$ small requires a different perturbation analysis. This case corresponds to the maximum or minimum power mode.

We now analyze the amplitude equation (3.9) in detail. Its steady state represents the amplitude of a periodic solution of the original LK equations. It satisfies the implicit relation

$$\Lambda_1=-\frac{\xi A}{2\sin(\Delta\Theta)J_1(D)\sin(\Theta/2)}. \tag{3.12}$$

By taking the limit $A\to0$ in (3.12), we note that a branch of periodic solutions emerges from a Hopf bifurcation point located at $\Lambda_1=\Lambda_{1H}$ where Λ_{1H} is defined by

$$\Lambda_{1H}\equiv-\frac{\xi}{2\sin(\Delta\Theta)\sin^2(\Theta/2)}. \tag{3.13}$$

$\Lambda_{1H}>0$ implies $\sin(\Delta\Theta)<0$. Equation (3.13) is identical to Eq. (2.5) rewritten in terms of $\Lambda=\alpha\eta/\omega\equiv\omega\Lambda_{1H}$. The Hopf bifurcation is the classical mechanism leading to a periodic state. Expanding (3.12) for small A we find that

$$A\cong\left(\frac{2}{\sin^2(\Theta/2)}\,\frac{\Lambda_1-\Lambda_{1H}}{\Lambda_{1H}}\right)^{1/2} \tag{3.14}$$

as $\Lambda_1-\Lambda_{1H}\to0$. The bifurcation is always supercritical (i.e., defined only for $\Lambda>\Lambda_{1H}$), and according to Hopf theorem, the periodic solution is stable. The behavior of the periodic solution near its bifurcation point has been first examined in [13] using a different method. The expression (3.12) shows that A approaches a constant amplitude as $\Lambda_1\to\infty$. This constant amplitude is denoted by $A=A^*$ and satisfies the condition

$$J_1[D(A^*)]=0. \tag{3.15}$$

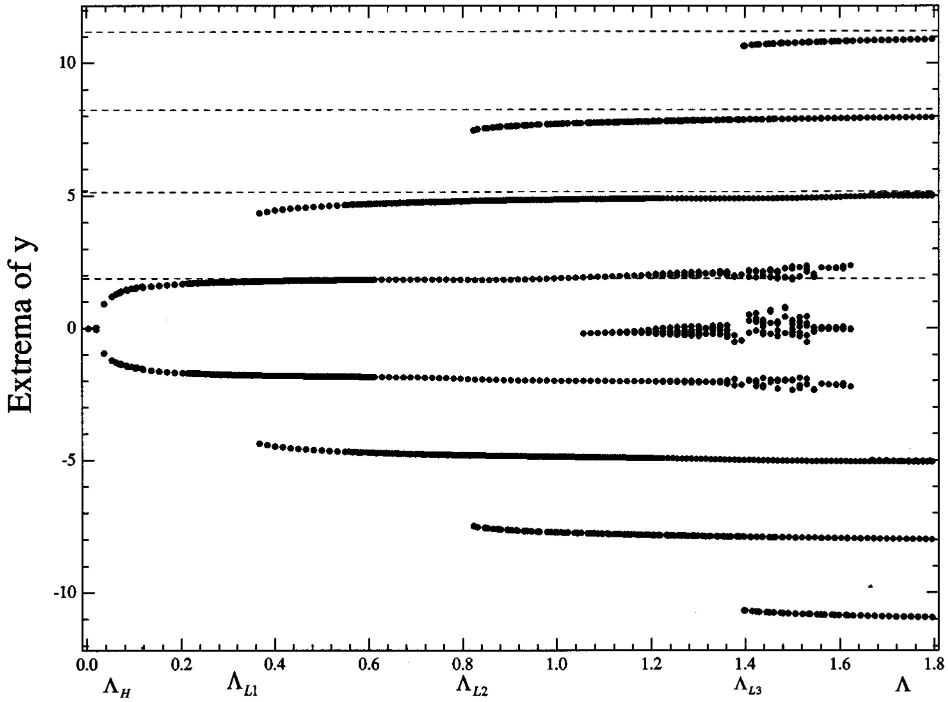

FIG. 1. Bifurcation diagram for the extrema of the electric field variable y for the Lang and Kobayashi phase equation, Eq. (2.15). The values of the parameters are $P=0.75$, $T=2000$, $\Delta=1.5$ and $\Theta=\pi$.

Thus, $A=A^*$ corresponds to the first zero of the Bessel function $J_1(D)$ with $D\neq 0$. But Eq. (3.15) admits additional roots. Equivalently, (3.12) predicts additional, isolated, branches of solutions that appear from limit points. This branching phenomenon is the second mechanism leading to periodic states in the LK problem. From (3.12) and the condition $\Lambda_1'(A)=0$, we find that the amplitude at these limit points satisfies the equation

$$J_1(D)-2AJ_1'(D)\sin(\Theta/2)=2J_1(D)-DJ_0(D)=0. \tag{3.16}$$

The linear stability of the steady state can be analyzed from Eq. (3.9). We eliminate Λ_1 in the expression of the growth rate by using (3.12) and obtain

$$\sigma=-\frac{\xi}{2}\left[1-2A\,\sin(\Theta/2)\,\frac{J_1'(D)}{J_1(D)}\right]=-\xi A\,\sin(\Theta/2)\,\frac{J_2(D)}{J_1(D)}. \tag{3.17}$$

Stability means $\sigma<0$, which is verified for the Hopf bifurcation branch. Note that $\sigma=0$ at the limit points since substituting (3.16) into (3.17) gives $\sigma=0$. From these limit points a stable branch and an unstable branch of the periodic solutions appear as the feedback rate Λ increases.

IV. NUMERICAL BIFURCATION DIAGRAMS

In all our numerical computations we used fourth-order Runge-Kutta with fixed step size. Figure 1 shows the bifurcation diagram of the LK phase equation (2.15). The figure exhibits the extrema of the electric field y. From Eqs. (2.10a)–(2.10c), we know that y is related to Φ as

$$y=-\Phi''+O(\omega). \tag{4.1}$$

Using Eqs. (3.1) and (3.6), we find that the extrema of y are, in first approximation, given by $\pm A$. A Hopf bifurcation appears at $\Lambda_H=0.023$ and is well approximated by $\Lambda_H=\omega\Lambda_{1H}$ where Λ_{1H} is given by Eq. (3.13). The values of the parameters are given by $P=0.75$, $T=2000$, $\Delta=1.5$, and $\Theta=\pi$. At the Hopf bifurcation point, a branch of periodic solutions appears that then saturates at a constant amplitude as Λ increases. This constant is well predicted by our analysis, which gives $A=A^*\simeq 1.9$. The value of A^* verifies the Bessel function condition Eq. (3.15).

All other branches appear through limit points. Solving Eq. (3.16) numerically and then using Eq. (3.12), we find approximations for the limit points $(\Lambda,A)=(\Lambda_{Ln},A_{Ln})$ listed in Table I. The different Λ_{Ln} are indicated in Fig. 1 and match the numerical estimates obtained from the LK phase equation Eq. (2.15). From each of these limit points a stable branch and an unstable branch of solutions appear. Only the stable branches are shown in Fig. 1. These branches quickly saturate at constant amplitudes. Again, these constants verify, in first approximation, Eq. (3.15). They are shown as dotted lines in Fig. 1 at $A^*\simeq(5.08,8.23,11.38,\ldots)$. For $\Lambda>1.05$, we observe a change of the first branch of periodic solutions that exhibits low-amplitude oscillations. Note that there exists only one single external cavity mode solution for $0<\Lambda<1.5$ and no new Hopf bifurcations points have been observed for $1.5<\Lambda<2.0$.

Figure 2 shows the bifurcation diagram of the original LK

TABLE I. Numerical values of the limit points $(\Lambda,A)=(\Lambda_{Ln},A_{Ln})$ obtained from Eq. (3.16) and Eq. (3.12).

n	Λ_{Ln}	A_{Ln}
1	0.35	4.2
2	0.82	7.4
3	1.39	10.6

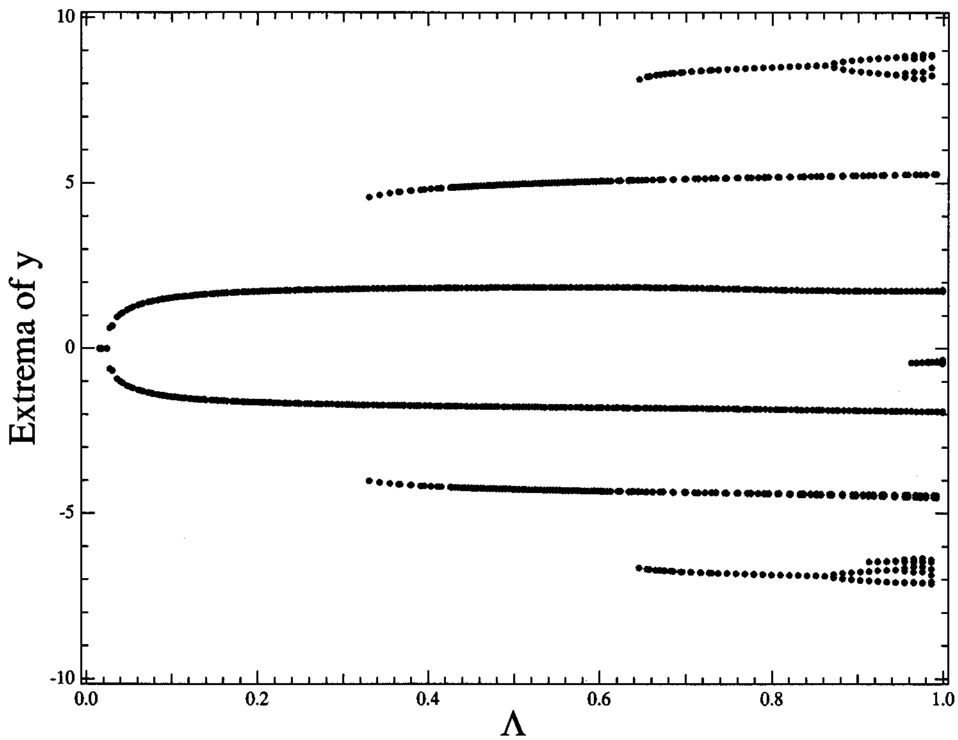

FIG. 2. Bifurcation diagram for the extrema of the electric field variable y for the full Lang and Kobayashi equations, Eq. (1.1a) and Eq. (1.1b). Same values of the parameters as in Fig. 1.

equations (1.1a) and (1.1b). We represent y as a function of Λ, which are related to the original variables used in Eqs. (1.1a) and (1.1b) by the following expressions:

$$y=\alpha\left(\frac{|Y|}{\sqrt{P}}-1\right),\quad \Lambda=\frac{\alpha\eta}{\omega}. \tag{4.2}$$

The bifurcation diagram is in good agreement with the diagram of the approximate LK equation previously shown in Fig. 1. Successive coexisting branches of periodic solutions appear through Hopf bifurcations or limit point mechanisms. We have verified that these branches numerically overlap the branches shown in Fig. 1 obtained from the phase equation (2.15). We also found that higher-amplitude periodic states predicted from the phase equation are difficult to obtain due to their small basins of attractions. Figure 2 also shows a period-doubling bifurcation for Λ near 1. We did not investigate this bifurcation. As Λ is further increased from 1 the LK equations (1.1a) and (1.1b) exhibit a series of higher-order instabilities leading to chaos. We did not investigate if the phase equation (2.15) captures these higher-order bifurcations. Figure 2 shows the bifurcation diagram for $0<\Lambda<1$ for clarity only.

V. DISCUSSION

We have derived a third-order differential equation for the phase of the laser field that is simpler than the original LK equations and that allows a systematic bifurcation analysis. This equation is equivalent to the system of equations (2.10) that shows that the leading approximation of the LK problem models the laser as a harmonic oscillator driven nonlinearly by the phase. This results from the relatively large value of the linewidth enhancement factor and similar equations have been derived for other semiconductor laser problems [17].

Our asymptotic analysis of the LK equations concentrated on periodic solutions that appear at low feedback rate. We have shown that, in addition to a Hopf bifurcation, periodic solutions branch out from limit points. This allows the coexistence of multiple periodic states. Furthermore, our analysis shows that the amplitude of all periodic states saturates at constant values as the feedback rate increases. In this section, we briefly describe these periodic states as a family of external cavity modes.

The periodic solutions obtained in Sec. III have the form

$$Y_n(S)=\sqrt{P}\,\exp(i\Lambda\overline{B_n}S)\exp[iA_n\cos(S)],$$

where the index n is associated with the solution corresponding to the nth root of $J_1(D)=0$. A_n and B_n are constants. (It is apparent that the frequency of the oscillation of the dominant mode has been shifted.) It can be easily demonstrated that these solutions satisfy the reproducibility relation

$$Y_n(S)=\lambda_n Y_n(S-\Theta),$$

where λ_n is some complex constant. These solutions can be interpreted as a set of approximate eigenfunctions of the external cavity that exhibit a time dependence more complicated than that of simple plane waves.

ACKNOWLEDGMENTS

This research was supported by the U.S. Air Force Office of Scientific Research Grant No. AFOSR-93-1-0084, the National Science Foundation Grant No. DMS-9308009, the Fonds National de la Recherche Scientifique (Belgium), and the InterUniversity Attraction Pole of the Belgian government. P.M.A. would like to thank the National Research Council for support during this work.

APPENDIX

The LK equations are two equations for the slowly varying complex envelope electric field $E(t)$ and the carrier density $N(t)$ given by [5]

$$\frac{dE}{dt}=\frac{1}{2}\,(1+i\alpha)G_N(N-N_{\mathrm{th}})E+\frac{\kappa}{\tau_{\mathrm{in}}}\,e^{-i\omega_0\tau}E(t-\tau), \tag{A1}$$

$$\frac{dN}{dt}=J-\frac{N}{\tau_s}-\left[\frac{1}{\tau_p}+G_N(N-N_{\mathrm{th}})\right]|E|^2. \tag{A2}$$

In these equations, τ_p, τ_{in}, τ, and τ_s are the photon lifetime, the round-trip time in the laser cavity, the round trip time in the external cavity, and the carrier lifetime, respectively. κ^2 is the power reflected from the external cavity relative to the power reflected from the laser mirror. α is the linewidth enhancement factor and J is the pumping term. The solitary laser is assumed to oscillate in a single longitudinal mode with angular frequency ω_0, N_{th} is the threshold carrier density for the solitary laser, and G_N is a constant defined as $G_N\equiv(\partial G/\partial N)_{\mathrm{th}}$, where $G(N)$ is the gain per unit time:

$$G(N)=G(N_{\mathrm{th}})+G_N(N-N_{\mathrm{th}})=\frac{1}{\tau_p}+G_N(N-N_{\mathrm{th}}). \tag{A3}$$

We now introduce the following dimensionless variables:

$$Y\equiv\sqrt{\frac{\tau_sG_N}{2}}\,E, \tag{A4}$$

$$Z\equiv\left(\frac{\tau_pG_N}{2}\right)(N-N_{\mathrm{th}}), \tag{A5}$$

$$s\equiv\frac{t}{\tau_p}. \tag{A6}$$

In terms of (7.4), (7.5), and (7.6) Eqs. (7.1) and (7.2) can be rewritten as Eqs. (1.1a) and (1.1b), where

$$P\equiv\frac{\tau_pG_NN_{\mathrm{th}}}{2}\left(\frac{J}{J_{\mathrm{th}}}-1\right), \tag{A7}$$

$$J_{\mathrm{th}}\equiv N_{\mathrm{th}}/\tau_s\,, \tag{A8}$$

$$\eta\equiv\kappa\tau_p/\tau_{\mathrm{in}}\,, \tag{A9}$$

$$\theta\equiv\tau/\tau_p\,, \tag{A10}$$

$$\Omega\equiv\omega_0\tau_p\,. \tag{A11}$$

[1] G. R. Gray, A. T. Ryan, G. P. Agrawal, and E. C. Gage, Proc. SPIE **2039**, 45 (1993).
[2] R. Lang and K. Kobayashi, IEEE J. Quantum Electron. **16**, 347 (1980).
[3] N. Schunk and K. Petermann, IEEE J. Quantum Electron. **24**, 1242 (1988).
[4] J. Mork and B. Tromborg, IEEE Photon. Technol. Lett. **2**, 21 (1990).
[5] J. Mork, B. Tromborg, and P. L. Christiansen, IEEE J. Quantum Electron. **24**, 123 (1988).
[6] T. Sano, Phys. Rev. A **50**, 2719 (1994).
[7] G. H. M. van Tartwijk, A. M. Levine, and D. Lenstra, IEEE J. Quantum Electron. **1**, 466 (1995).
[8] H. Olesen, J. H. Osmundsen, and B. Tromborg, IEEE J. Quantum Electron. **22**, 762 (1986).
[9] J. Mork, J. Mark, and B. Tromborg, Phys. Rev. Lett. **65**, 1999 (1990).
[10] J. Mork, B. Tromborg, and J. Mark, IEEE J. Quantum Electron. **28**, 93 (1992).
[11] C. Masoller, Phys. Rev. A **50**, 2569 (1994).
[12] A. Ritter and H. Haug, J. Opt. Soc. Am. B **10**, 130 (1993).
[13] A. Ritter and H. Haug, J. Opt. Soc. Am. B **10**, 145 (1993).
[14] T. Erneux, G. van Tartwijk, D. Lenstra, and A. L. Levine, Proc. SPIE **2399**, 170 (1995); (unpublished).
[15] J. Mork, M. Semkow, and B. Tromborg, Electron. Lett. **26**, 609 (1990).
[16] D. Lenstra, Opt. Commun. **81**, 209 (1991).
[17] T. Erneux, V. Kovanis, A. Gavrieliedes, and P. M. Alsing, Phys. Rev. A (to be published).
[18] A. Ritter and H. Haug, IEEE J. Quantum. Electron. **29**, 1064 (1993).
[19] J. Kervokian and J. D. Cole, *Perturbation Methods in Applied Mathematics*, Applied Mathematical Sciences Vol. 34 (Springer-Verlag, Berlin, 1981).
[20] *Handbook of Mathematical Functions*, 9th ed., edited by M. Abramowitz and I. Stegun (Dover, New York, 1972), p. 361.