Global-stability analysis of transverse modes in laser systems under inhomogeneous pumping

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The stability of a two-level laser system with flat end mirrors, operating in a single longitudinal mode, is considered in the case of inhomogeneous pumping. The pumping is assumed to be of a Gaussian type of one transverse variable and spatially slowly varying. This situation is commonly encountered, for instance, in broad-area semiconductor lasers or slab CO_2 lasers. We study the transverse structure formation induced by the slow variation of the inhomogeneous pumping near the primary instability. In a linear instability analysis, we show that the absolutely unstable state always sets in through a transversal perturbation near the onset of lasing. The spatiotemporal stability analysis separates local and global instabilities since the latter may have a drastic influence on the subsequent behavior. We determine the threshold and frequency for global transverse unstable modes. These modes are nothing but Hermite modes, which appear as a consequence of the spatial inhomogeneity. These predictions are consistent with the recent near-single-mode operations obtained for slab CO_2 lasers. In the weakly nonlinear study, the amplitude equation is derived and appears as a modified Ginzburg-Landau equation with spatially varying coefficients. Numerical investigations are reported to illustrate a possible sequence of convective and absolute instabilities together with global Hopf bifurcations of transverse nonlinear modes. [S1050-2947(96)03705-5]

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I. INTRODUCTION

Lasers have recently appeared as physical systems whose modeling generates specific problems sharing common grounds with very general open questions such as those related to the appearance of complexity in spatially extended systems [1]. Throughout the past 30 years, laser theory has evolved, starting from simple models such as the rate equation model of the monomode laser, which involves a very limited set of ordinary differential equations (ODE's), to deal now with amplitude and phase equations aiming at the detailed description of the transverse structure of beams emitted by lasers with large cross sections. The latter models use partial differential equations (PDE's) such as the complex Landau-Ginzburg or the Swift-Hohenberg equation, originally introduced in completely different fields. Both families of models are presently used to describe spatiotemporal dynamics of lasers, but their respective fields of application are still being discussed now. Let us set the limit cases in which each family of models (i.e., ODE's or PDE's) applies.

Most lasers have a cylindrical geometry with a length L far exceeding the transverse dimensions of the cavity. Therefore the longitudinal coordinate z is often eliminated. Such an approximation is valid when the gain (or loss) per pass in the active medium remains small, i.e., $\alpha L \ll 1$ (where α denotes the gain per unit length). The mean-field approximation in laser theory assumes that the longitudinal variations of the electromagnetic field are negligible. When this is valid, laser dynamics is purely a transverse problem, leading to sets of partial differential equations involving the two transverse coordinates x and y and time t. The interaction of the electromagnetic field with the active medium is described by the reduction of the Schrödinger equation to two energy levels connected by a laser transition. More refined models are sometimes used to represent specific lasers such as the Raman laser, but the two-level Schrödinger equation appears as the most commonly used (and efficient) model.

Pumping and relaxation mechanisms, which are necessary for laser action, are dissipative terms phenomenologically introduced in the equations. The set of quantum plus pumping and relaxation equations is the so-called Bloch model of the two-level system. Together with the field evolution given by Maxwell equations, they form the Maxwell-Bloch model of the laser.

It is then classical to solve Maxwell equations with boundary conditions given by the mirrors that limit the laser cavity. For an empty lossless cavity, Laguerre-Gauss or Hermite-Gauss modes are obtained for cylindrical or rectangular cavities, respectively. The electromagnetic field is then projected onto the basis set provided by the eigenmodes of the empty cavity. Laser dynamics, i.e., the evolution of the field and matter variables, is obtained by solving these Maxwell-Bloch equations. The complexity of this task is greatly reduced when the variables are projected on a suitable basis, which is here provided by the solutions of Maxwell equations (i.e., Hermite-Gauss or Laguerre-Gauss modes, depending on the symmetry of the laser cavity geometry). The problem is then reduced to a set of ODE's whose dimension increases with the number of modes involved. This approach proves extremely efficient in understanding the dynamics with a limited number of active modes, typically up to 3 [2]. Moreover, D'Alessandro and Oppo showed that the Laguerre-Gauss set of modes still provides a sensible basis when a large number of modes (typically up to 5) are involved [3].

In lasers the number of potentially active modes is approximately equal to the Fresnel number $N_F = a^2/\lambda L$, an optical equivalent of the aspect ratio in Rayleigh-Bénard experiments [4], where λ denotes the optical wavelength and a the radius of the laser. This number relates an evaluation of the relative influence of the diffraction losses ($\approx\lambda/a$) to the angular aperture ($\approx a/L$) of the cavity.

There has recently been an increased interest for lasers and more generally optical systems with large cross sections and, accordingly, large Fresnel numbers (N_F up to 100) for which the mode expansion obviously fails [5]. Studies of transverse patterns of these systems from both experimental and numerical points of view revealed a great variety of structures including hexagonal patterns, rolls [6], spirals [7], crystal and quasicrystal structures [8], and vortex dynamics [9]. The description of such transverse dynamics requires a global approach similar to that used in hydrodynamics.

Laser action occurs as some threshold pump power is exceeded. The nonlasing state, which is stable below threshold, destabilizes above it. The transverse structures of laser emission should first be determined in the near-threshold region in which a perturbation expansion for the field-matter interaction may be used. When theoretical results about transverse patterns are compared with experimental findings, boundary conditions and spatial inhomogeneities in the parameters should be carefully discussed.

The mirrors already mentioned above introduce a phase shift depending on the distance to the cavity axis. Their influence has been investigated in details by Lega [10]. Most lasers are axially pumped by either another laser beam or an electric discharge. The transverse dependence of the control parameters may affect drastically the laser dynamics. For instance, Lugiato and Milani [11] showed that all instabilities of the laser Lorenz equations disappear when a field Gaussian profile is assumed for the laser field. It appears then that the influence upon the transverse variable dependence of the pump profile plays a crucial role in selecting the transverse structure. From a theoretical point of view, the universal phenomenon of transverse structure formation near threshold may be described by order parametric equations [12]. The latters depend on symmetry properties of the system, which result from the geometrical configuration of the laser (geometry of the cavity and pumping profile). This situation is shared by both lasers and passive media [13]. Unfortunately, when inhomogeneous pumpings are taken into account for real laser beams, theoretical investigations and the analytical form of the laser solution are usually untractable even for the lowest order [14]. The influence of the transverse dependence of the pump parameter has been investigated in the low-Fresnel-number situation by Chen *et al.* [15], who carried out directly numerical simulations of the Maxwell-Bloch equations of the laser.

In the present paper we study the dynamical evolution of a laser system under inhomogeneous pumping, which is slowly varying in space. Contrary to previous works, we have considered the case of pumping that is homogeneous in one transverse direction (y) and has a slow dependence on the other direction (x). This situation corresponds, for instance, to narrow-gap slab waveguide CO_2 lasers [16] or in stripe geometry semiconductor lasers [17] for which a similar approximation is currently made [18]. We perform a linear spatiotemporal stability analysis near threshold in the simple case of a single longitudinal mode and flat end mirrors. As the pumping profile is slowly varying in an infinite region of space, it is convenient to examine the local and the global instabilities of the system. The term ''local'' refers to the stability of the mean profile of the basic state along local independent spots. This assumption leads one to consider the amplitude of the instability as independent of the transverse coordinate x and one may perform the linear stability analysis by considering local normal modes. The term ''global'' refers to the stability of the basic state over the entire domain. The following question may then be asked: What is the relationship between local and global instability properties?

Recent theoretical works [19–23] concerning fluid dynamical systems have performed a criterion based on local convective instability and absolute convective instability. We summarize here these ideas as follows. Suppose that an initial disturbance is localized in the medium. Then the response of the medium appears like a wave packet propagating with the group velocity of the most unstable mode. If the disturbance is localized at its source and grows exponentially in time, then the instability is called ''absolute instability.'' If, however, the disturbance is convected away from its source, the basic state is said to be ''convectively unstable.'' Recently, this spatiotemporal analysis was applied to the filamentation of two collinear waves in homogeneous plasma [24]. The authors studied the threshold of the transverse modulation of instability, which characterizes conical light emission observed in experiments on phase conjugation in sodium vapor.

Guided by the Ginzburg-Landau model, Chomaz, Huerre, and Redekopp [19] have established a connection between global and local instability properties. In particular they showed that the existence of a finite region of absolute instability is a necessary condition for the onset of time-amplified (self-excited) global oscillations. Here we use this criterion to derive the global mode describing laser instability. The shape of the pumping is assumed to be of the Gaussian type, reaching its maximum at the origin. We recall briefly in Sec. II the equations governing the phenomenon. In Sec. III, after recalling some results from the homogeneous pumping case, we use a two-scale method to show that this instability is largely confined to the neighborhood of the origin. A linear analysis then allows, for inhomogeneous pumping, one to derive a discrete spectrum of critical pumping parameter and to estimate the correspondent global frequencies. Because of the nonlinear character of the phenomenon appearing in experiments (see, for instance, [16]), the evolution equation of the amplitude of the global mode is then obtained (Sec. IV) by using a weakly nonlinear analysis. This evolution is governed by a complex Ginzburg-Landau equation with spatially varying coefficients. Numerical solutions of this equation are summarized in Sec. V. The results confirm the asymptotic estimation of the instability thresholds and the behavior of the system just beyond the thresholds.

II. BASIC EQUATIONS

We consider a unidirectional laser cavity with flat mirrors containing a two-level medium. The electric field $\mathbf{E}(x,y,z,t)$

and the polarization $\mathbf{P}_0(x,y,z,t)$ are assumed to be singly polarized in the same direction and to propagate in the z direction, namely,

$$\mathbf{E}(x,y,z,t)=\mathbf{A}(x,y,z,t)e^{i\omega(z/c-t)},$$
$$\mathbf{P}_0(x,y,z,t)=\mathbf{P}(x,y,z,t)e^{i\omega(z/c-t)}. \tag{1}$$

We denote by D the population difference between the energy levels 1 and 2 of the atomic system. The laser beam has the width d and we denote by L the scale of the pumping profile. The basic equations governing the dynamical behavior of the laser, under the slowly varying envelope and paraxial approximations, are the Maxwell-Bloch equations [25]. These equations may be written for scalar nondimensional variables [4]

$$\frac{\partial A}{\partial z}+\frac{\partial A}{\partial t}-ia\Delta_\perp A=-\sigma A+\sigma P,$$

$$\frac{\partial P}{\partial t}+(1+i\Omega)P=DA, \tag{2}$$

$$\frac{\partial D}{\partial t}+b(D-r)=\frac{1}{2}\,(AP^*+A^*P),$$

where $\Delta_\perp$ denotes the transverse Laplacian $\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2$, r is the function describing the inhomogeneous pumping, a is the inverse Fresnel number, b characterizes the decay rate of population inversion D, σ is the electric-field attenuation, and Ω is the detuning parameter. We assume here that r is a slowly varying function

$$r=r(X),\quad X=\varepsilon x,\quad \varepsilon=d/L. \tag{3}$$

Moreover, we assume that $r(X)$ is an even function with respect to X, which admits a local maximum at $X=0$ (see later) as it is usually the case in real laser systems.

The equilibrium of the system is characterized by $D=r(X)$ and $P=A=0$. These values correspond to an absence of laser emission. In what follows we introduce the variable

$$n=r(X)-D. \tag{4}$$

We assume that the laser system operates under conditions of a single-longitudinal mode and begin with a linear stability analysis, so that we set

$$A=\mu A',\quad P=\mu P',\quad n=\mu n', \tag{5}$$

where μ denotes the order of magnitude of the perturbation. We introduce the primed variables in Eqs. (2). After linearizing around the equilibrium state (and now suppressing the primes), the perturbation equations read

$$\left[\frac{\partial}{\partial t}+(1+i\Omega)\right]\left[\frac{\partial}{\partial t}-ia\Delta_\perp+\sigma\right]A-\sigma r(X)A=0, \tag{6}$$

$$\frac{\partial P}{\partial t}+(1+i\Omega)P=r(X)A,\quad \frac{\partial n}{\partial t}+bn=0. \tag{7}$$

Because the criterion of absolute instability is the link between local and global instabilities, we now recall this criterion by following the development of Chomaz, Huerre, and Redekopp [21].

III. MATHEMATICAL ANALYSIS

A. Homogeneous pumping

It is of interest to first examine the case of a homogeneous pumping. This case is generally considered in the literature (see, e.g., [25–28]). Hence we assume that $r(X)=c^{ste}=r_0$.

The spatiotemporal stability analysis of the nonlasing solution $A=P=n=0$ considers normal modes $\exp i(kx-\nu t)$, with complex wave number $k=k_r+ik_i$ and complex frequency $\nu=\nu_r+i\nu_i$. Hence, from (6), we obtain the corresponding dispersion equation. This equation depends on r_0 and links ν and k:

$$(-i\nu+1+i\Omega)(-i\nu+iak^2+\sigma)-\sigma r_0=0. \tag{8}$$

If we are interested in temporal instability analysis, the local mode is unstable provided that $\nu_i>0$ for any real k, the ''most unstable'' wave number k_c being defined by the equation $\partial\nu_i/\partial k=0$. The most unstable mode moves with the group velocity $(\partial\nu_r/\partial k)_{k=k_c}$. Denoting by r_c the value of r_0 for which a laser solution appears with wave number k_c and frequency ν_c, we have

$$r_c=1,\quad k_c^2=\frac{\Omega}{a},\quad \nu_c=\Omega\quad \text{if }\ \Omega>0,$$
$$r_c=1+\frac{\Omega^2}{(1+\sigma)^2},\quad k_c^2=0,\quad \nu_c=\frac{\sigma\Omega}{1+\sigma}\quad \text{if }\ \Omega<0. \tag{9}$$

Within the unstable domain $r_0>r_c$, we shall distinguish between the convectively unstable region and the absolutely unstable one. The nature of the instability depends on the response of the system to a localized excitation at, say, $t=0$. By evaluating the group velocity $d\nu/dk$ of any unstable mode, we can thus describe the response of the system. The growth rate $\nu_i(k_0)$ corresponding to the zero group velocity is the absolute growth rate. When $\nu_i(k_0)$ is positive, some unstable modes grow in place and the system is said to be absolutely unstable. Alternatively, if $\nu_i(k_0)$ is negative, all unstable modes propagate away from the location of the excitation and the system is said to be convectively unstable.

These ideas manifest themselves in our problem for describing the evolution of some unstable laser modes as follows: For both signs of the detuning parameter Ω and for $r_0>1+\Omega^2/(1+\sigma)^2$, the system is absolutely unstable for the mode $k_0=0$ and $\nu_0=\sigma\Omega/(1+\sigma)$. On the other hand, if we consider oblique waves, the response to a modulated impulse in the y direction must be considered and the dependence of the disturbances is y modulated proportionally to $\exp(ihy)$. The wave number K of the two-dimensional transverse structure is then defined by $K^2=k^2+h^2$.

It appears that there exists an absolute growth rate defined by $\partial\nu/\partial k=0$ that occurs in our problem for $k=0$, $h=h_c=\sqrt{\Omega/a}$, $\nu_c=\Omega$, and $r_0=r_c=1$. This property is similar to the one considered in fluid mechanics problems by Monkewitz [23]. We conclude that the absolute growth rate of disturbances in the y direction is larger than that in the x direction.

Finally, it results that, in the homogeneous case, the absolute instability threshold is reached at $r_0=r_c=1+\alpha^2$, with $h=h_c$, where

$$\alpha=0, \quad h_c^2=\frac{\Omega}{a} \quad \text{if } \Omega>0,$$
$$\alpha=\frac{\Omega}{1+\sigma}, \quad h_c^2=0 \quad \text{if } \Omega<0. \tag{10}$$

We have introduced these values of α and h_c to be able to simultaneously study the two cases $\Omega>0$ and $\Omega<0$. Note, moreover, that ν_c, h_c, and r_c satisfy the relation (8).

B. Inhomogeneous pumping

Now let us return to Eq. (6). We make the following assumptions concerning $r(X)$:

$$r(X)=r_0(1-\delta g(X)), \quad g(0)=g'(0)=0, \quad g''(0)>0;$$
$$g'(X)\geqslant 0 \quad \text{for } X>0, \quad |g(+\infty)|<+\infty. \tag{11}$$

Equation (6) is linear with slowly varying coefficients and can be classically studied by a multiple-scale technique [see, e.g., [29,30]). Hence we introduce a fast variable η and a slow variable X, related to x by the formulas

$$\eta=\frac{\phi(X)}{\varepsilon}, \quad X=\varepsilon x, \tag{12}$$

where $\phi(X)$ is a phase variable, as yet undetermined but satisfying the condition $\phi(0)=0$. With the variables η and X being considered as independent variables, we look for A in the form

$$A=\overline{A(X,\eta,\varepsilon)}e^{-i\nu t+ih_c y}. \tag{13}$$

After some calculations, Eq. (6) reads

$$L(\overline{A})=\varepsilon ia[-i\nu+1+i\Omega]\left[2\phi'(X)\frac{\partial^2\overline{A}}{\partial\eta\partial X}+\phi''(X)\frac{\partial\overline{A}}{\partial\eta}\right]$$
$$+O(\varepsilon^2), \tag{14}$$

where L is the linear operator

$$L=[-i\nu+1+i\Omega]\left[-i\nu-ia\phi'^2(X)\frac{\partial^2}{\partial\eta^2}+iah_c^2+\sigma\right]$$
$$-\sigma r(X)\times. \tag{15}$$

We now expand $\overline{A(X,\eta,\varepsilon)}$ with respect to ε, in the form

$$\overline{A}=\overline{A_0}(X,\eta)+\varepsilon\overline{A_1}(X,\eta)+\varepsilon^2\overline{A_2}(X,\eta)+o(\varepsilon^2). \tag{16}$$

We assume now that $0\leqslant\delta<1$, so that $0<r(X)\leqslant r_0$. Consequently, the instability is first located in the neighborhood of $X=0$. In particular, the critical values of r_0 and ν are assumed as small perturbations of the two preceding values r_c and ν_c corresponding to the absolute unstable mode, namely,

$$r_0=(r_c+\varepsilon r_1+\cdots), \quad \nu=\nu_c+\varepsilon\nu_1+\cdots\ . \tag{17}$$

According to Sec. III A, we assume that $k_c=0$. By introducing the expansions (16) and (17) in (14) and looking for A_j in the form

$$\overline{A_j}(X,\eta)=A_j(X)e^{i\eta}, \tag{18}$$

we obtain at different orders in ε, by applying a multiple scale technique, the following results.

1. Order 0

Denoting by L_c the operator L restricted to the values ν_c, h_c, and r_c, we obtain the equation

$$L_c(\overline{A_0})=0, \tag{19}$$

which provides the dispersion equation

$$\phi'^2(X)=\frac{(\alpha+i)\sigma\delta g(X)}{a}. \tag{20}$$

It results from (20) that the motion is always composed of a fast oscillation combined with a fast damping: Hence this motion is perceptible only near the y axis.

2. Order 1

By canceling secular terms in $\overline{A_1}$, we obtain for $A_0(X)$ a linear amplitude equation, which can be solved as

$$A_0(X)=\frac{1}{\sqrt{\phi'(X)}}\exp\left\{-\frac{\nu_1-i(\alpha\nu_1+r_1)}{2(1+\alpha^2)}\phi(X)\right\}$$
$$\times\exp\left(\int_{X_0}^{X}G(Y)dY\right), \tag{21}$$

where X_0 is an arbitrary constant and where we have set

$$G(X)=\frac{\nu_1[i(1+i\alpha)+\sigma(i+\alpha)]+r_1\sigma}{2a(1+i\alpha)}\frac{1}{\phi'(X)}. \tag{22}$$

It results, from (21) and (22), that the slow variation of $\overline{A_0}$ contains, as its fast variation, an oscillation combined with a damping. But the expression of $A_0(X)$ shows that this expression becomes singular at the turning points Y_0 [i.e., where $\phi'(Y_0)=0$]. The point $Y_0=0$, after (11) and (20), is the only root. Because of the assumptions (11) about $g(X)$, the formula (21) shows also that, since the derivative $g'(0)$ vanishes, the integral $\int G(Y)dY$ is divergent. Hence the expansion is not uniformly valid and must be supplemented, at the neighborhood of 0, by a local expansion. In order to overcome this singularity, we now set in (6)

$$A=\widetilde{A}(x,\varepsilon)e^{ih_c y-i\nu t} \tag{23}$$

and replace $r(X)$ by its expansion with respect to ε, by using (12). Hence (6) takes the form

$$(-i\nu+1+i\Omega)\left[-i\nu-ia\left(\frac{d^2}{dx^2}-h_c^2\right)+\sigma\right]\tilde{A}$$
$$-\sigma r_0\left[1-\frac{\delta\varepsilon^2}{2}g''(0)x^2\right]\tilde{A}=0. \quad (24)$$

Assuming the form of ν and r_0 given by (17), it results, from the balance between the dominant terms of diffraction and inhomogeneity, that the right transverse scaling variable is $\xi=\varepsilon^{1/2}x$, in terms of which we obtain the inner equation

$$\frac{d^2\tilde{A}}{d\xi^2}+\frac{i\sigma(1-i\alpha)\delta g''(0)}{2a}\xi^2\tilde{A}=-\frac{\nu_1}{a}\left[\sigma\frac{1-i\alpha}{1+i\alpha}+1\right]\tilde{A}+\frac{i\sigma}{a(1+i\alpha)}r_1\tilde{A}+O(\varepsilon). \quad (25)$$

Expanding $\tilde{A}$ as

$$\tilde{A}=\tilde{A}_0+\varepsilon\tilde{A}_1+\cdots, \quad (26)$$

after some calculations, we get, to the lowest-order the Weber equation,

$$\frac{d^2\tilde{A}_0}{dz^2}+\left(p+\frac{1}{2}-\frac{1}{4}z^2\right)\tilde{A}_0(z)=0, \quad (27)$$

where we have set

$$z=\lambda\xi, \quad \lambda=\left(-\frac{2i\sigma(1-i\alpha)\delta g''(0)}{a}\right)^{1/4}, \quad (28)$$

$$p=-\frac{1}{2}\pm\frac{i\nu_1[\sigma(1-i\alpha)+1+i\alpha]+\sigma r_1}{(1+i\alpha)\sqrt{2a\sigma\delta g''(0)(i+\alpha)}}, \quad (29)$$

the $\pm$ in (29) depending on the determination chosen of the square root defining λ^2 in (28).

The solutions of (27) bounded at the origin are the Weber functions $W_p(z)$. The asymptotic expansions of these functions for large $|z|$ (see, e.g., [31]) are regular (for large $|z|$) only when p is a positive integer. The Weber functions, in this case, are related to Hermite polynomials $H_p(z)$. Since the matching of the inner expansion with the outer one is possible only for bounded $W_p(z)$, p must effectively be a positive integer in (29). Therefore the discrete real eigenvalues of the problem appear as two sequences (r_{1p},ν_{1p}). Finally, we have a family of eigensolutions $\tilde{A}_{0p}$ corresponding to these eigenvalues (r_{1p},ν_{1p}),

$$\tilde{A}_{0p}(\xi)=D_p e^{-(1/4)z^2}H_p(z), \quad (30)$$

where D_p is an arbitrary constant.

Since, from (29), r_{1p} and ν_{1p} are linear functions of $p+\frac{1}{2}$, the smallest value of r_{1p} is always r_{10}. Hence the value r_{10} characterizes the alteration of the instability onset due to the inhomogeneity. In the following we denote by r_{1c} this value of r_1.

Once we have obtained the values of the constants of the inner expansion, it is now necessary to relate this inner expansion to the outer one by an asymptotic matching. The procedure is very classical (see [30]). In the present case, the matching is straightforward. Its role is to relate the value of X_0 (say X_{0p}) in (21) to the amplitude D_p in (30). In fact, the main result of the matching is to show that, for X_{0p} of order 1, the corresponding amplitude D_p of the solution in the inner region is of order $\varepsilon^{p/2}$. Since the first unstable state arises for $p=0$, the amplitude of the solution is of the same order of magnitude in the two regions.

IV. WEAKLY NONLINEAR INSTABILITY ANALYSIS

The most important practical result of the preceding analysis is the identification of the localized nature of the transverse modes [formula (30)]. From the point of view of fundamental laser physics, the inhomogeneity in the pumping is responsible for both localization of modes and frequency selection. Thus the spatially inhomogeneous gain causes the appearance of standing-wave (SW) (here through Hermite modes) instead of traveling-wave (TW) patterns. This kind of qualitative change of solutions may also be induced by inhomogeneous phase gain, as pointed out in a recent study by Staliunas and Weiss [32]. Those authors have shown that curved mirrors (i.e., inhomogeneous phase gain) can cause a transition from TW's to SW's (Hermite modes) in the one-dimensional case.

From an experimental point of view, the Hermite mode behavior is consistent with recent results of experiments in slab waveguide CO and CO_2 lasers [16,33]. The localized nature of the laser beam has also been observed both experimentally and numerically in broad-area semiconductor lasers [34,35]. In fact, homogeneous variations in laser parameters may provide a useful technique to select and control spatially localized modes. Indeed, recently, Li *et al.* [36] were not only able to generate experimentally Hermite modes but they were also able to control them by external light injection, in electrically pumped vertical-cavity surface-emitting semiconductor lasers.

It is well known that saturation effects are widely present in most of these experiments. But so far, in our analytical investigations, only a linear spatiotemporal analysis has been performed and the dynamical behavior just beyond the primary threshold of the lasing solution still remains to be investigated. It is the aim of the present section to consider the nonlinear behavior of a linearly unstable state.

A. General analysis

In the previous sections, the quantities A, P, and n were assumed to be small and of the same order μ [see formulas (5)]. We now assume that the nonlinear terms previously neglected are no longer negligible in the equations. Let us return to Eqs. (2). Denoting by μ_A, μ_P, and μ_n the orders of magnitude, it results from the first of (2) that $\mu_A=\mu_P$ and from the last of (2) that $\mu_n=\mu_A\mu_P$. Hence, denoting now by μ the order of magnitude μ_A, we have $\mu_P=\mu$ and $\mu_n=\mu^2$. Thus we set instead of (5)

$$A=\mu A'', \quad P=\mu P'', \quad n=\mu^2 n''. \quad (31)$$

The equations corresponding to (6) and (7) now read

$$\left[\frac{\partial}{\partial t}+(1+i\Omega)\right]\left[\frac{\partial}{\partial t}-ia\Delta_\perp+\sigma\right]A''-\sigma r(X)A''=-\sigma\mu^2 n''A'', \quad (32)$$

$$\frac{\partial P''}{\partial t}+(1+i\Omega)P''=r(X)A''-\mu^2 n''A'',$$
$$\frac{\partial n''}{\partial t}+bn''=\tfrac{1}{2}(A''P''^*+A''^*P''). \quad (33)$$

We consider a state of the system in the neighborhood of the linear onset previously studied. Hence, taking into account the assumptions (11) about $r(X)$, we assume that

$$r(X)=(r_c+\varepsilon r_1)[1-\delta g(X)], \quad (34)$$

with, since we consider the nonlinear stability, $r_1>r_{1c}$.

By inserting (34) in (32) and (33), we see that the balance of terms in (32) indicates that $\mu^2=\varepsilon$. Considering now this case, ε appears in (32) and (33) only by integer powers, so that we finally expand A'', P'', and n'' in the form

$$A''=A_0+\varepsilon A_1+\cdots, \quad P''=P_0+\varepsilon P_1+\cdots,$$
$$n''=n_0+\varepsilon n_1+\cdots, \quad (35)$$

with A_0 and P_0 in the form

$$A_0=\overline{A_0}(x,\varepsilon,T)e^{ih_c y+\lambda t}, \quad P_0=\overline{P_0}(x,\varepsilon,T)e^{ih_c y+\lambda t},$$
$$T=\varepsilon t, \quad (36)$$

and we note the following points.

(i) At the order 0 with respect to ε, the linear equations concerning A_0 and P_0 remain the same as in the linear system: in particular r_c and ν_c are the eigenvalues of this equation and have always the values given by (9). For the same reasons as in the linear analysis, we consider, moreover, the case $\lambda=-i\nu_c$.

(ii) At the order 1, there appears a singularity located at $x=0$. As already seen in the linear study, the boundary layer remains of thickness $X=O(\varepsilon^{1/2})$. Hence the variable $\xi=\varepsilon^{-1/2}X=\varepsilon^{1/2}x$ remains the inner variable associated with this singularity. In the following subsection, we concentrate on the solution in the neighborhood of the origin. Specifically, we seek a solution of (32) and (33) in the form

$$A''=A_0(\xi,T,t,y)+\varepsilon A_1(\xi,T,t,y)+\cdots,$$
$$P''=P_0(\xi,T,t,y)+\varepsilon P_1(\xi,T,t,y)+\cdots, \quad (37)$$
$$n''=n_0(\xi,T,t,y)+\varepsilon n_1(\xi,T,t,y)+\cdots.$$

B. Inner expansion and amplitude equation

Rewriting (32) and (33) with the help of the variables ξ,T,t, we obtain the system

$$L(A_0)=0, \quad (38)$$

$$\frac{\partial P_0}{\partial t}+(1+i\Omega)P_0=r_c A_0, \quad \frac{\partial n_0}{\partial t}+bn_0=\frac{1}{2}(A_0P_0^*+A_0^*P_0),$$

where

$$L=\left(\frac{\partial}{\partial t}+(1+i\Omega)\right)\left(\frac{\partial}{\partial t}-ia\frac{\partial^2}{\partial y^2}+\sigma\right)-\sigma r_c\times. \quad (39)$$

Setting now

$$A_0=\widetilde{A}_0(\xi,T)e^{ih_c y+\lambda t}, \quad (40)$$

the operator L vanishes for $\lambda=-i\nu_c$. The corresponding values of P_0 and n_0 read

$$P_0=\left(1+\frac{i}{\sigma}(ah_c^2-\nu_c)\right)\widetilde{A}_0(\xi,T)e^{ih_c y-i\nu_c t}, \quad n_0=\frac{|\widetilde{A}_0|^2}{b}. \quad (41)$$

At order 1, restricting our attention to A_1, (32) becomes

$$L(A_1)=-\left(\frac{\partial}{\partial t}+(1+i\Omega)\right)\left(\frac{\partial}{\partial T}-ia\frac{\partial^2}{\partial\xi^2}\right)A_0-\frac{\partial}{\partial T}\left(\frac{\partial}{\partial t}-ia\frac{\partial^2}{\partial y^2}+\sigma\right)A_0+\sigma\left(r_1-\frac{\delta g''(0)}{2}r_c\xi^2\right)A_0-\sigma n_0A_0. \quad (42)$$

The operator L being singular, the operator of the right-hand side must satisfy an orthogonality condition. This condition takes the form

$$-[1+\sigma+i\alpha(1-\sigma)]\frac{\partial\widetilde{A}_0}{\partial T}+ia(1+i\alpha)\frac{\partial^2\widetilde{A}_0}{\partial\xi^2}+\sigma\left(r_1-\frac{\delta g''(0)}{2}r_c\xi^2\right)\widetilde{A}_0-\frac{\sigma}{b}\widetilde{A}_0|\widetilde{A}_0|^2=0. \quad (43)$$

Note that, in this equation, contrary to the linear case, it is not necessary to take into account a variation $\varepsilon\nu_1$. In effect this variation is included in the T dependence of $\widetilde{A}_0$. Moreover, Eq. (43) is not valid if, simultaneously, $\Omega>0$ and $h=0$. In this case, the fourth-order diffusion term must be added to prevent from the unphysical blowing up of the solution [37]. Equation (43) is a modified Ginzburg-Landau equation with complex coefficients. For $\Omega\to\infty$ (i.e., $\alpha\to\infty$), this equation becomes a perturbed nonlinear Schrödinger equation.

C. Analysis of the amplitude equation beyond the instability threshold

We are now interested in solutions of (43) that go to zero for large $|\xi|$. The amplitude of these solutions is determined by the values of r_1 such that $r_1>r_{1c}$. In particular, let us obtain the modified eigenfunction of the linearized equation associated with (43), by setting

$$\widetilde{A}_0(\xi,T)=s[\tilde{\tilde{A}}_{00}(\xi,T')+s^2\tilde{\tilde{A}}_{01}(\xi,T')+\cdots]e^{-i\nu_{10}T},$$
$$T'=s^2T, \quad r_1=r_{1c}+s^2\overline{r}, \quad (44)$$

where ν_{10} is the first eigenvalue corresponding to $p=0$ [see formulas (27)–(29)] and s is a small real parameter, not related to ε, which measures the order of magnitude of the distance of r_1 from criticality. T' is a slow time related to s, which is introduced in order to avoid secular terms and to obtain a solution of (43) uniformly valid in time.

By introducing the perturbation expansion (44) in (43) and replacing the derivative $\partial/\partial T$ by $\partial/\partial T+s^2\partial/\partial T'$, the leading order of the expansion gives

$$\tilde{\tilde{A}}_{00}(\xi,T')=D(T')\tilde{A}_{00}(\xi), \tag{45}$$

where $\tilde{A}_{00}(\xi)$ is the function defined by (30) corresponding to $p=0$. The equation for $\tilde{\tilde{A}}_{01}$ is then

$$\begin{aligned}-[1+\sigma+i\alpha(1-\sigma)]\,\frac{\partial\tilde{\tilde{A}}_{01}}{\partial T}+ia(1+i\alpha)\,\frac{\partial^2\tilde{\tilde{A}}_{01}}{\partial\xi^2}\\ +\sigma\left(r_{1c}-\frac{\delta g''(0)}{2}\,r_c\xi^2\right)\tilde{\tilde{A}}_{01}\\ =[1+\sigma+i\alpha(1-\sigma)]\,\frac{\partial\tilde{\tilde{A}}_{00}}{\partial T'}\\ -\sigma\overline{r}\tilde{\tilde{A}}_{00}+\frac{\sigma}{b}\,\tilde{\tilde{A}}_{00}|\tilde{\tilde{A}}_{00}|^2.\end{aligned} \tag{46}$$

In terms of the expression (45) and in order to obtain bounded solutions, the compatibility condition leads to the Landau equation satisfied by $D(T')$,

$$[1+\sigma+i\alpha(1-\sigma)]\,\frac{dD}{dT'}=\sigma\overline{r}D-\frac{\sigma}{b\sqrt{2}}\,D|D|^2. \tag{47}$$

The perturbation is consistent only if $\overline{r}>0$. The inhomogeneity originates, in the laser system, a change of the solution from a spatially periodic and time periodic mode to a spatially localized and time periodic mode. The new frequency is a slight correction of the main frequency.

V. NUMERICAL SIMULATIONS

We have numerically integrated Eq. (43) by the split-step method where the spatially varying coefficients were incorporated in the nonlinear term. To avoid eventual numerical instabilities, we have replaced the varying coefficient ξ^2 by the function $r_1(\xi)=e^{-\xi^2}-1$, which coincides with $-\xi^2$ for small ξ. We now denote by $r(\xi)$ the corresponding varying coefficient in (43),

$$r(\xi)=r_1+\frac{\delta g''(0)}{2}\,r_1(\xi). \tag{48}$$

This function has no incidence on our analytical analysis since most important is the main part in the vicinity of the origin. In addition, we have set systematically the length of the transverse interval of integration wide enough with respect to the finite width of the transverse global modes.

The results can then matched to the solutions of Eq. (27). This equation is, in fact, a linearized version of (43). The conclusions concern, first, the local and the global instability, and, second, the existence of stationary global transverse modes, together with the nonlinear saturation effects giving rise to Hopf global oscillations.

A. Global instability threshold

From Eq. (29) we first deduce the corresponding values (r_{1p},ν_{1p}). The corresponding global modes are given by (30), where z and ξ are related by (28). All modes are of great interest for the nonlinear problem far from the threshold of the primary bifurcation. However, the first unstable mode is obtained for $p=0$ and in all numerical simulations (the linear and nonlinear problems) we restrict our attention to the fundamental mode $p=0$. In this case the solution of (29) reads

$$\begin{aligned}\nu_{10}&=\pm\frac{1}{2}\,\frac{\sqrt{2(\alpha^2+1)a\sigma\delta g''(0)}}{\sigma+1}\,\mathrm{Re}(\sqrt{\alpha-i}),\\ r_{1c}&=r_{10}=\mp\nu_{10}\left[\frac{\sigma+1}{\sigma}\,\frac{\mathrm{Im}(\sqrt{\alpha-i})}{\mathrm{Re}(\sqrt{\alpha-i})}+\alpha\,\frac{\sigma-1}{\sigma}\right],\end{aligned} \tag{49}$$

while the corresponding eigensolution reads

$$\begin{aligned}\tilde{A}_0(\xi)&=D_0\\ &\times\exp\left\{-\frac{i}{4}\left(\frac{\sigma\delta g''(0)}{a}\right)^{1/2}\frac{\xi^2}{(\sqrt{1+\alpha^2}-\alpha)^{1/2}}\right\}\\ &\times\exp\left\{-\frac{1}{4}\left(\frac{\sigma\delta g''(0)}{a}\right)^{1/2}(\sqrt{1+\alpha^2}-\alpha)^{1/2}\xi^2\right\}.\end{aligned} \tag{50}$$

First of all, we emphasize that, for the inhomogeneous pumping, the analytical calculations of the previous sections need no restrictions on the magnitude of δ and $g''(0)$. The small parameter ε measures the ratio between the width of the beam and the size of the pumping profile. Then it may be of order one, which we left with a modified Ginzburg-Landau equation with varying coefficients and not, necessarily, a perturbed version of it. Thus we set $\delta g''(0)=1$.

For the other parameters, we have set $\sigma=1$, $a=0.26$, and $b=1$. This choice of values for the parameters σ and b corresponds to the general case (class C) of laser dynamics, since the number of variables cannot be lowered. However, the transient effects are more important for values of a less than one. The parameter α depends on the sign of the detuning parameter Ω: the two cases $\Omega>0$ and $\Omega<0$ correspond to two qualitatively different behaviors, according to the difference in the pumping threshold. The control parameter r_1 fixes the instability threshold and hence the spatial transverse extent of the region of local absolute instability.

In order to clarify the notion of spatiotemporal instability, we have plotted in Fig. 1 the function $r_1(\xi)$ for $\Omega=-1$ and $r_1=0.5$. The equilibrium of the system (nonlasing solutions) is stable at $\xi=\pm\infty$ and a single region of absolute instability, bordered by symmetric regions of convective instability, develops in the vicinity of the origin. The threshold value corresponding to $\Omega=-1$, from (49), is $r_1=0.36$. In order to check this threshold we have integrated Eq. (43) up to 1500 units in time and measured the $\mathrm{Im}(\nu)$ at $\xi=0$. We found

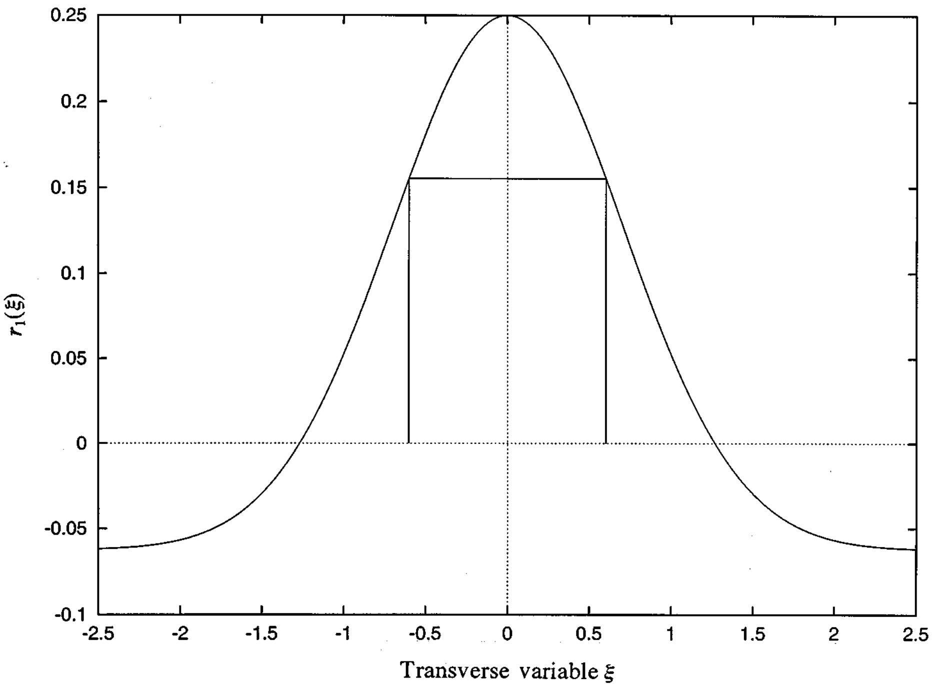


FIG. 1. Local instability characteristic $r_1(\xi)$ in inhomogeneous pumping for $\Omega=-1$ and $r_1=0.5$. The inner region corresponds to a pocket of absolute instability bordered by two convective instability regions. In the outer part $[r_1(\xi)<0]$ the system is linearly stable.

numerically $r_{1c}^{(n)}\approx 0.31$, as shown in Fig. 2. The result is in good agreement with the theoretical one. The relative accuracy is within 15% in the worst case, but we have observed numerically that the accuracy increases with decreasing δ.

B. Nonlinear behavior of the amplitude

The spatiotemporal evolutions of the envelope function $|\widetilde{A}_0|$ are illustrated in Fig. 3 for different values of r_1. The existence of a global mode at the threshold together with the linear attenuation and amplification around the threshold is clear from this figure. Note that the spatiotemporal behavior does not depend on the initial conditions: Indeed we have observed the same behavior with several different initial conditions, including localized, random-phase, and uniform initial conditions.

When $r_1<0$ [hence $r(\xi)<r_c$ for all ξ; Fig. 3(a)], the nonlasing solution is stable everywhere in ξ and the envelope function, after some transient evolution, is asymptotically damped. When r_1 increases up to 0.3 [Fig. 3(b)], the medium is locally unstable in a finite range of values of ξ. Two regions must be distinguished. First, in the vicinity of the origin, an absolute local instability exists already; this region is bordered by a region of convective local instability. In the second region, since the group velocity does not vanish, the traveling-wave packets are amplified and move towards

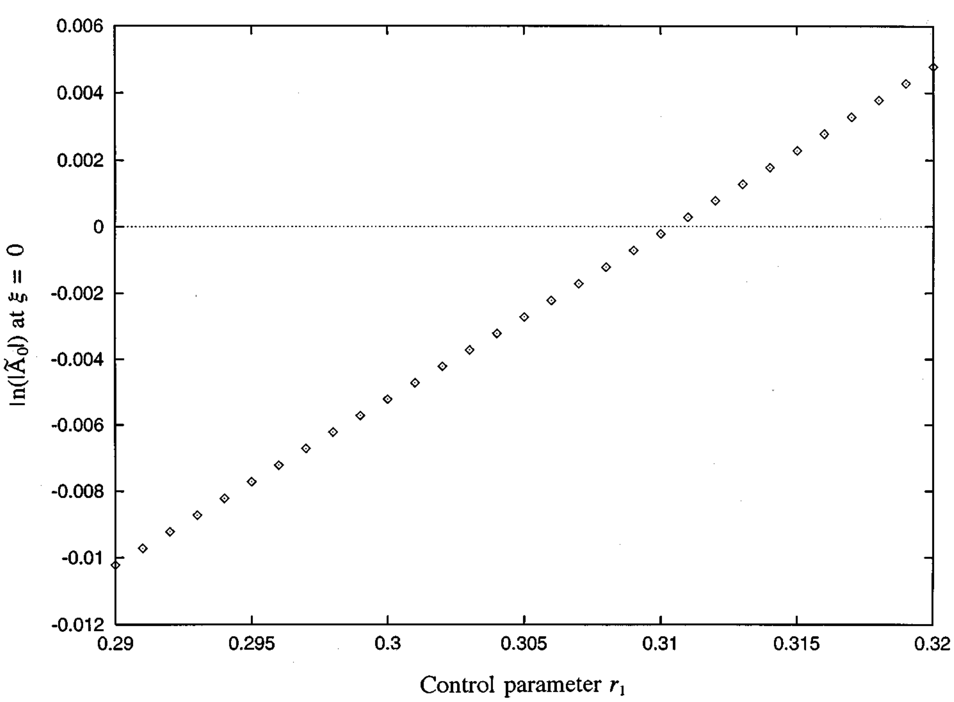


FIG. 2. Global instability thresholds measured by the growth rate $\ln|\widetilde{A}_0|$ at $\xi=0$ for $\Omega=-1$. The numerical threshold is $r_1^{(n)}=0.3105$ while the theoretical one [formula (49)] provides $r_1=0.36$.

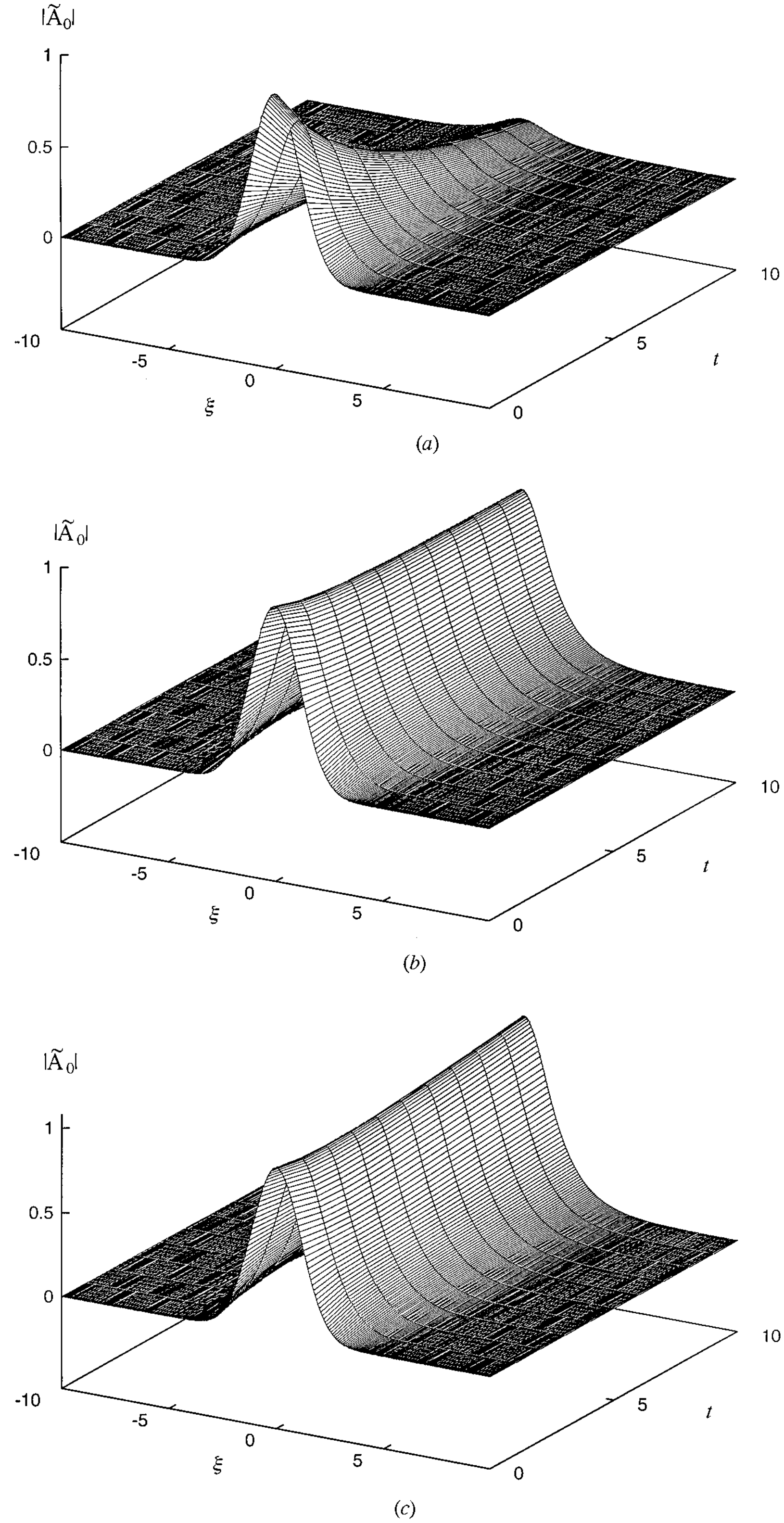


FIG. 3. Spatiotemporal evolution of the modulus $|\widetilde{A}_0|$ of the amplitude for $\Omega=-1$ and different values of r_1: (a) $r_1=-0.1$ (below the threshold), the system is stable; (b) $r_1=r_{10}$ (at the threshold), global stationary mode; and (c) $r_1=0.35$ (beyond the threshold), amplification of the global mode above the global instability.

$\xi=\mp\infty$. Note that the local instability is already very developed without giving rise to global oscillations, as may be seen from Fig. 4. In this figure, the envelope function is still attracted by the equilibrium state. Moreover, the existence of a pocket of local absolute instability around the origin is not sufficient for global oscillations, in agreement with the fact that the value of the control parameter is below the global Hopf bifurcation.

When increasing further r_1 beyond the threshold [Fig. 3(c)], the spatiotemporal envelope function globally increases with time: This means that the system is globally unstable. In fact, at the threshold, in the vicinity of the origin, $\mathrm{Im}(\nu)=0$ and at the same time the group velocity vanishes, giving rise to amplifications in place. On the contrary, in the convective instability, all unstable modes propagate so that the power is carried away. The Hopf bifurcation is confirmed

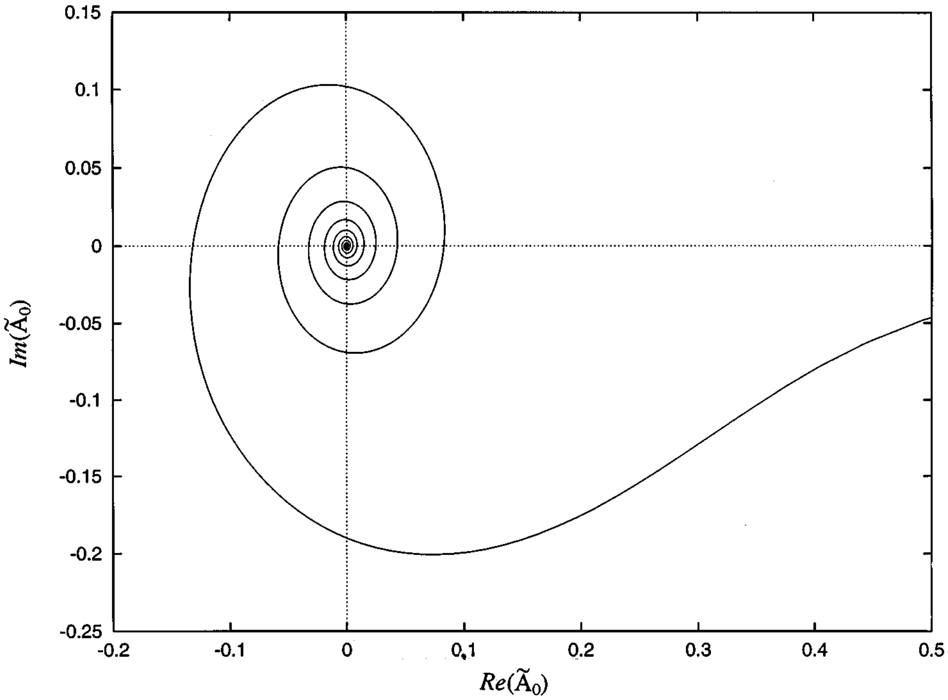


FIG. 4. Asymptotic state just below the threshold of the global instability ($\Omega=-1$ and $r_1=0.3$). Even though the local global instability is very developed, there are no global oscillations.

in Fig. 5. Indeed the growth rate saturates, giving rise to a limit cycle that characterizes the Hopf bifurcation predicted by the Landau equation (47). Figure 6 displays the asymptotic spatiotemporal nonlinear mode and global oscillations of the imaginary part of the amplitude.

VI. CONCLUSION

In this paper we have investigated the onset of lasing solutions in the form of transverse structures when the pumping parameter takes the form of an x Gaussian distribution $r=r_0[1-\delta g(\varepsilon x)]$, ε being the ratio of the beam width to the scale of the variation of the pumping. Our choice of the form of a slowly varying function $g(X)$, $X=\varepsilon x$, ensures that the most unstable region is centered around the origin. The influence of this inhomogeneity has been studied by using an asymptotic expansion with respect to ε (the ''outer'' solution). The solution breaks down in a region of magnitude $O(\varepsilon^{1/2})$ around the origin and an inner solution is obtained by a local study. The matching of the outer solution and the inner solution allows one to determine both the onset and frequency of a global laser solution. They are constrained to

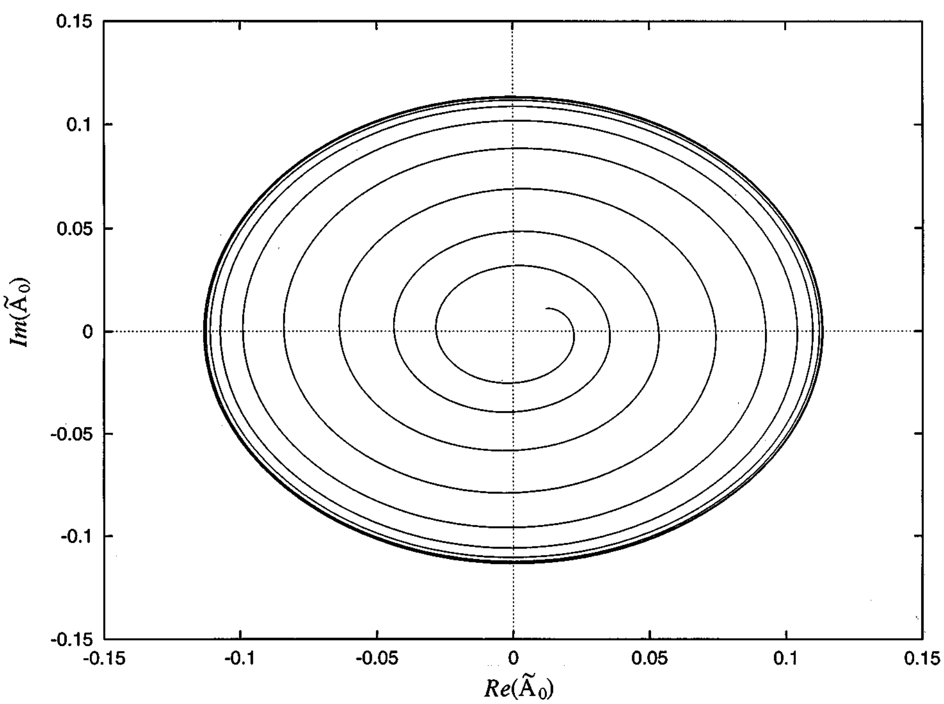
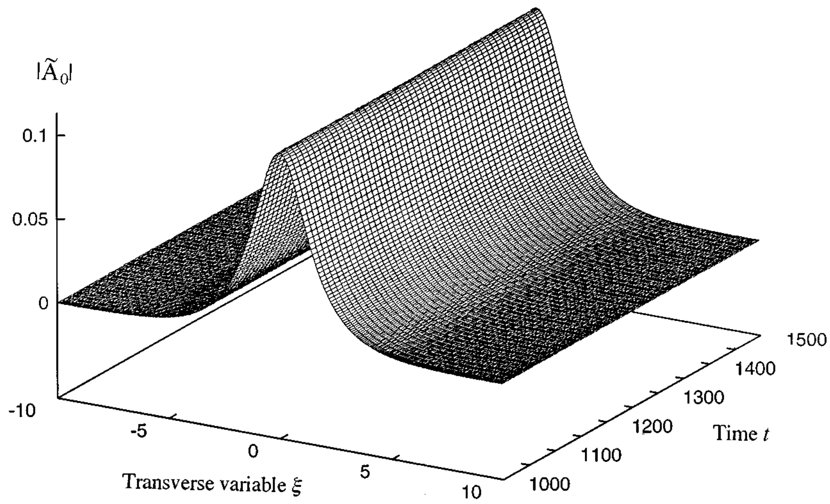
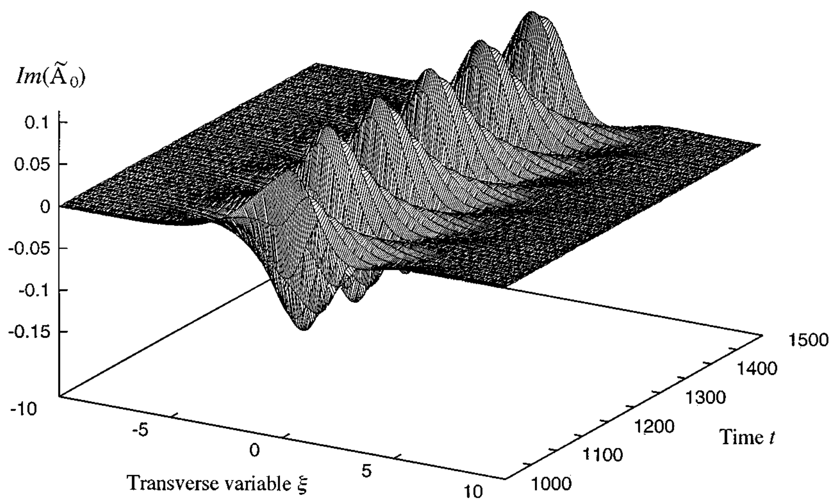


FIG. 5. Global Hopf bifurcation and attractive limit cycle just above the threshold ($\Omega=-1$ and $r_1=0.32$).



(*a*)



(*b*)

FIG. 6. Nonlinear global modes for $\Omega=-1$ and $r_1=0.32$: (a) asymptotic spatiotemporal evolution of the modulus of the amplitude and (b) global oscillations of the imaginary part of the amplitude.

take one of a discrete set of values, in contrast with the case of homogeneous pumping where the threshold and frequency are restricted to a continuous spectrum of values. Such properties are similar to other results in fluid mechanics, concerning the local and global instabilities of dissipative flows (see [22]). In particular, the regions of local absolute instability must reach a critical order of magnitude $X_c=O(\varepsilon^{1/2})$ to induce a global instability. The local amplitude of a laser solution cannot, of course, be determined from a linear analysis.

Considering nonlinear effects, we have shown that the amplitude is described by a modified Ginzburg-Landau equation with a stabilizing cubic nonlinearity. A weakly nonlinear analysis shows that, beyond global onset, the system undergoes a Hopf bifurcation to global self-sustained oscillations.

Numerical simulations solving directly the nonlinear Ginzburg-Landau equation confirm (i) the onset of global instability and (ii) the nonlinear behavior of the amplitude in the neighborhood of the instability onset and beyond it. In particular, the Hopf bifurcation has been qualitatively recognized. Furthermore, we showed that the exponential growth of the unstable transverse modes saturates by the nonlinear terms, giving rise to localized nonlinear modes whose amplitude are governed by the spatially modified Ginzburg-Landau equation.

An extension of the present study with a view to experiment would include curved mirrors for the cavity and a two-dimensional Gaussian profile (works on this is in progress). Further, the main idea is that a spatiotemporal study is necessary for any analysis since the linear stability analysis is no longer sufficient.

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