Ground state and excited states of a confined condensed Bose gas

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The Bogoliubov approximation is used to study the ground state and low-lying excited states of a dilute gas of N atomic bosons held in an isotropic harmonic potential characterized by frequency ω and oscillator length $d_0=\sqrt{\hbar/m\omega}$. By assumption, the self-consistent condensate has a macroscopic occupation number $N_0 \gg 1$, with $N-N_0 \ll N_0$. A linearized hydrodynamic description yields operator forms of the particle-conservation law and Bernoulli's theorem, expressed in terms of the small density fluctuation operator $\hat{\rho}'$ and velocity potential operator $\hat{\Phi}'$, along with the condensate density n_0 and velocity $\mathbf{v}_0$. For positive scattering length a and large stationary condensate ($N_0 \gg d_0/a$ and $\mathbf{v}_0=\mathbf{0}$), the spherical condensate has a well-defined radius $R_0 \gg d_0$, and the low-lying excited states are irrotational compressional waves localized near the surface. Approximate variational energies E_{0l} of the lowest radial modes $(n=0)$ for successive values of orbital angular momentum l form a rotational band given by $E_{0l} \approx E_{00}+\frac{1}{2}\hbar^2 l(l+1)/mR_0^2$, with radial zero-point energy $E_{00} \propto \hbar\omega(R_0/d_0)^{2/3}=(\hbar^2\, m\omega^4 R_0^2)^{1/3}$. [S1050-2947(96)02006-9]

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The recent experimental demonstration of Bose-Einstein condensation in dilute confined ^{87}Rb [1] has stimulated theoretical research into the physical properties of such a system, based largely on the Bogoliubov approximation [2], originally introduced as a model for bulk superfluid ^{4}He. Although this simple description of liquid He has long been familiar, much of its application to Bose condensed dilute atoms has involved numerical analysis [3–5]. In contrast, Baym and Pethick [6] have provided a more physical description of the confined ground state, emphasizing the relevant dimensionless parameters for ^{87}Rb. The present work introduces a hydrodynamic description of the quantized normal modes, along with a variational analysis of the low-lying excited states for large spherical condensate in an isotropic harmonic confining potential.

The Bogoliubov model is most simply understood by considering the familiar second-quantized field operators that obey boson commutation relations $[\psi(\mathbf{r}),\psi^\dagger(\mathbf{r}')]=\delta(\mathbf{r}-\mathbf{r}')$ (in general, a second-quantized operator will be denoted by a circumflex to distinguish it from its expectation value, although the circumflex will be omitted from the field operator itself). The dynamics follows from the "grand-canonical Hamiltonian" operator

$$\hat{K}\equiv\hat{H}-\mu\hat{N}$$
$$=\int dV\ \psi^\dagger(T+U-\mu)\psi+2\pi a\hbar^2 m^{-1}\int dV\ \psi^\dagger\psi^\dagger\psi\psi, \tag{1}$$

where $\hat{N}=\int dV\ \psi^\dagger\psi$ is the number operator, $\hat{H}$ is the Hamiltonian operator, and μ is the chemical potential [7,8]. Here $T=-\hbar^2\nabla^2/2m$ is the kinetic energy, $U(\mathbf{r})$ is the external confining potential, and the short-range interatomic two-body potential has been approximated by a pseudopotential with an s-wave scattering length a [9]. The presence of Bose condensation implies that the field operator has a macroscopic ensemble average $\langle\psi(\mathbf{r})\rangle\equiv\Psi(\mathbf{r})$, identified as the (temperature-dependent) condensate wave function [10]. For a dilute system at low temperature, most of the particles are in the condensate, and the *deviation* operator $\phi(\mathbf{r})\equiv\psi(\mathbf{r})-\Psi(\mathbf{r})$ is treated as small. By definition, $\langle\phi(\mathbf{r})\rangle$ vanishes, and, in the thermodynamic limit $(N\to\infty)$, these deviation operators obey approximate boson commutation relations $[\phi(\mathbf{r}),\phi^\dagger(\mathbf{r}')]\approx\delta(\mathbf{r}-\mathbf{r}')$ [8].

An expansion of $\hat{K}$ through second order in these small field amplitudes immediately yields $\hat{K}\approx K_0+\hat{K}'$, where

$$K_0=\int dV\ \Psi^*(T+U-\mu)\Psi+2\pi a\hbar^2 m^{-1}\int dV|\Psi|^4, \tag{2a}$$

$$\hat{K}'=\int dV\ \phi^\dagger(T+U-\mu)\phi+2\pi a\hbar^2 m^{-1}\times\int dV(4|\Psi|^2\phi^\dagger\phi+\Psi^2\phi^\dagger\phi^\dagger+\Psi^{*2}\phi\phi). \tag{2b}$$

The first-order contribution vanishes because K_0 is stationary for the condensate wave function that satisfies the nonlinear Hartree (or Gross-Pitaevskii) equation [11,12]

$$(T+U-\mu)\Psi+4\pi a\hbar^2 m^{-1}|\Psi|^2\Psi=0. \tag{3}$$

In addition, the ensemble average of the total number operators $\hat{N}\equiv N_0+\hat{N}'$ determines the temperature-dependent number of particles in the condensate $N_0=\int dV|\Psi|^2$ and in the excited states $N'=\int dV\langle\hat{n}'\rangle=\int dV\langle\phi^\dagger\phi\rangle$. The Bogoliubov approximation assumes that $N'\ll N_0$, thus neglecting terms of third and fourth order in the deviation operators; this assumption fails sufficiently close to the onset temperature T_c, since $N_0(T_c)$ vanishes.

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For many purposes, the particle-density operator $\hat{n}=\psi^\dagger\psi$ and particle current-density operator $\hat{\mathbf{j}}=(\hbar/2mi)[\psi^\dagger\nabla\psi-(\nabla\,\psi^\dagger)\psi]$ are the more relevant physical quantities. Rewrite the condensate wave function as

$$\Psi(\mathbf{r})\equiv e^{iS(\mathbf{r})}\sqrt{n_0(\mathbf{r})}, \tag{4}$$

with real phase S and amplitude $|\Psi_0|=\sqrt{n_0}$. To leading order in the deviation operators ϕ and $\phi^\dagger$, the density and current operators separate into two parts.

(1) The condensate contributions,

$$n_0=|\Psi|^2 \quad \text{and} \quad \mathbf{j}_0=n_0\frac{\hbar}{m}\nabla S, \tag{5}$$

identify n_0 as the condensate density and $\hbar\nabla S/m$ as the macroscopic condensate velocity $\mathbf{v}_0=\nabla\Phi_0$, with velocity potential $\Phi_0=\hbar S/m$. For example, taking S to be the cylindrical polar angle ϕ yields a singly quantized rectilinear vortex line in bulk fluid [12–14].

(2) The fluctuation contributions

$$\hat{\rho}'=\Psi^*\phi+\phi^\dagger\Psi=\sqrt{n_0}(e^{-iS}\phi+e^{iS}\phi^\dagger), \tag{6a}$$

$$\hat{\mathbf{j}}'=\frac{\hbar}{2mi}[\Psi^*\nabla\phi+\phi^\dagger\nabla\Psi-(\nabla\Psi^*)\phi-(\nabla\phi^\dagger)\Psi] \tag{6b}$$

show that $\hat{\rho}'$ and $\hat{\mathbf{j}}'$ are transition operators that change the particle number and thus have vanishing ensemble averages; hence $\hat{\rho}'$ differs from the condensate density operator $\hat{n}\equiv\phi^\dagger\phi$, which has a finite ensemble average for $T<T_c$.

The Gross-Pitaevskii Eq. (3) determines the spatial form of the condensate wave function Ψ, which then provides static interaction potentials for the low-lying excitations, as seen in the last term of Eq. (2b). A combination of Eqs. (3) and (4) yields a single complex equation, whose real and imaginary parts can be rewritten as

$$n_0^{-1/2}Tn_0^{1/2}+\tfrac{1}{2}mv_0^2+U-\mu+4\pi a\hbar^2m^{-1}n_0=0, \tag{7a}$$

$$\nabla\cdot\mathbf{j}_0=\nabla\cdot(n_0\nabla\Phi_0)=0. \tag{7b}$$

The first relation is simply Bernoulli's theorem for steady irrotational isentropic condensate flow in a compressible inviscid fluid [15], and the second expresses the conservation of the condensate current for steady irrotational flow. In the special case of a bulk uniform stationary condensate with $U=0$, Eqs. (7a) and (2a) show that the chemical potential has the value $\mu=4\pi a\hbar^2n_0/m$ with constant condensate energy density $e_0=2\pi a\hbar^2n_0^2/m$; the thermodynamic relation $p_0=-(\partial E_0/\partial V)_{N_0}$ then gives the condensate pressure $p_0=2\pi a\hbar^2n_0^2/m$.

The next step is to consider the noncondensate, which is described by the boson field operators ϕ and $\phi^\dagger$. In the grand-canonical ensemble, the time dependence of any Heisenberg operator $\hat{\mathcal{O}}$ follows from the general commutator $i\hbar\partial\hat{\mathcal{O}}/\partial t=[\hat{\mathcal{O}},\hat{K}]$. Since $\hat{K}'$ in Eq. (2b) is a quadratic form in the field operators, the relevant commutators are readily evaluated to yield

$$i\hbar\frac{\partial\phi}{\partial t}=\left(T+U-\mu+\frac{8\pi a\hbar^2|\Psi|^2}{m}\right)\phi+\frac{4\pi a\hbar^2\Psi^2}{m}\phi^\dagger, \tag{8a}$$

$$-i\hbar\frac{\partial\phi^\dagger}{\partial t}=\left(T+U-\mu+\frac{8\pi a\hbar^2|\Psi|^2}{m}\right)\phi^\dagger+\frac{4\pi a\hbar^2\Psi^{*2}}{m}\phi. \tag{8b}$$

A straightforward calculation verifies that the fluctuation density $\hat{\rho}'$ and current density $\hat{\mathbf{j}}'$ from Eq. (6) obey the expected operator conservation equation

$$\frac{\partial\hat{\rho}'}{\partial t}+\nabla\cdot\hat{\mathbf{j}}'=0. \tag{9}$$

It is convenient to introduce the transformed deviation operators

$$\tilde{\phi}\equiv e^{-iS}\phi \quad \text{and} \quad \tilde{\phi}^\dagger\equiv e^{iS}\phi^\dagger \tag{10}$$

that obey the transformed equations of motion

$$i\hbar\frac{\partial\tilde{\phi}}{\partial t}=L\tilde{\phi}+\frac{4\pi a\hbar^2n_0}{m}\tilde{\phi}^\dagger$$

and (11)

$$-i\hbar\frac{\partial\tilde{\phi}^\dagger}{\partial t}=L^\dagger\tilde{\phi}^\dagger+\frac{4\pi a\hbar^2n_0}{m}\tilde{\phi}.$$

Here $L\equiv\tilde{T}+U-\mu+8\pi a\hbar^2n_0/m$ is Hermitian, with $\tilde{T}=e^{-iS}Te^{iS}=(\hbar^2/2m)(-i\nabla+\nabla S)^2$. Correspondingly, Eq. (6b) becomes $\hat{\mathbf{j}}'=\hat{\rho}'\mathbf{v}_0+n_0\hat{\mathbf{v}}'$, with $\hat{\mathbf{v}}'=\nabla\hat{\Phi}'$, and

$$\hat{\Phi}'\equiv\frac{\hbar}{2min_0^{1/2}}(\tilde{\phi}-\tilde{\phi}^\dagger) \tag{12}$$

is the linear fluctuation part of the velocity potential operator. As expected from Eqs. (7b) and (9), the difference of the two Eqs. (11) eventually shows that the operators $\hat{\rho}'$ and $\hat{\Phi}'$ obey the conservation law

$$\frac{\partial\hat{\rho}'}{\partial t}+\nabla\cdot(\mathbf{v}_0\hat{\rho}')+\nabla\cdot(n_0\nabla\hat{\Phi}')=0. \tag{13}$$

A similar but lengthy calculation shows that the sum of the two Eqs. (11) yields the dynamical equation

$$n_0\left(\frac{\partial\hat{\Phi}'}{\partial t}+\mathbf{v}_0\cdot\nabla\hat{\Phi}'\right)+\frac{4\pi a\hbar^2n_0}{m^2}\hat{\rho}'+\frac{\hbar^2}{4m^2}\nabla\cdot\left(\hat{\rho}'\frac{\nabla n_0}{n_0}\right)-\frac{\hbar^2}{4m^2}\nabla^2\hat{\rho}'=0. \tag{14}$$

To interpret this last result, recall the exact classical form of Bernoulli's equation for irrotational isentropic flow of an inviscid compressible fluid [15],

$$U+\frac{1}{2}mv^2+\frac{e+p}{n}+m\frac{\partial\Phi}{\partial t}=\text{const}, \tag{15}$$

which follows directly from thermodynamics and Newtonian dynamics. Here, e is the energy density and Φ is the velocity potential, with $\mathbf{v}=\nabla\Phi$. Let $e(n)$ and $p(n)$ have the functional form

$$e(n)=n^{1/2}Tn^{1/2}+2\pi a\hbar^2 n^2/m \quad \text{and} \quad p(n)=2\pi a\hbar^2 n^2/m, \tag{16}$$

appropriate for the energy operator in Eq. (1). A direct expansion of Bernoulli's theorem (15) to first order in the fluctuation quantities $\rho'\equiv n-n_0$ and $\Phi'\equiv\Phi-\Phi_0$ reproduces Eq. (14), which thus becomes an operator version of Bernoulli's theorem for a dilute Bose gas in an external potential U with steady macroscopic condensate velocity $\mathbf{v}_0$ and condensate density n_0.

Although Eqs. (13) and (14) look identical to their classical counterparts, it is essential to remember that they are operator relations that involve not only annihilation and creation operators but also wave functions. To separate these two aspects, it is preferable to return to the transformed field operators $\tilde{\phi}$ and $\tilde{\phi}^\dagger$, since only the condensate density n_0 appears in Eq. (11), whereas Eqs. (13) and (14) also involve various spatial derivatives of n_0 (note that $n_0^{-1/2}\hat{\rho}'$ and $n_0^{1/2}\hat{\Phi}'$ are merely linear combinations of $\tilde{\phi}$ and $\tilde{\phi}^\dagger$). Define the linear transformation [8]

$$\begin{aligned}\tilde{\phi}(\mathbf{r},t)&=\sum_j{}' \,[u_j(\mathbf{r})\alpha_j(t)-v_j^*(\mathbf{r})\alpha_j^\dagger(t)],\\ \tilde{\phi}^\dagger(\mathbf{r},t)&=\sum_j{}' \,[u_j^*(\mathbf{r})\alpha_j^\dagger(t)-v_j(\mathbf{r})\alpha_j(t)],\end{aligned} \tag{17}$$

where the primed sum means to omit the condensate mode. Here, $\alpha_j^\dagger$ and α_j are "quasiparticle" creation and annihilation operators that obey boson commutation relations $[\alpha_j,\alpha_k^\dagger]=\delta_{jk}$, and u_j and v_j are wave functions. Substitute Eq. (17) into Eq. (11) and assume that the annihilation operator α_j satisfies the simple time-dependent equation $i\hbar\,\partial\alpha_j/\partial t=E_j\alpha_j$ along with its adjoint $-i\hbar\,\alpha_j^\dagger/\partial t=E_j\alpha_j^\dagger$; it follows immediately that the wave functions satisfy the coupled "Bogoliubov" equations

$$\begin{aligned}Lu_j-(4\pi a\hbar^2 n_0/m)v_j&=E_ju_j,\\ L^*v_j-(4\pi a\hbar^2 n_0/m)u_j&=-E_jv_j.\end{aligned} \tag{18}$$

Note that $\tilde{T}=T-\frac{1}{2}i\hbar(\nabla\cdot\mathbf{v}_0+\mathbf{v}_0\cdot\nabla)+\frac{1}{2}mv_0^2$ differs from $\tilde{T}^*$ in the presence of a macroscopic condensate velocity $\mathbf{v}_0$; this situation resembles that of a superconductor in a magnetic field, with $\mathbf{v}_0$ analogous to the vector potential.

It is easy to verify that the eigenvalues E_j are real and that the eigenfunctions obey the normalization $\int dV(u_j^*u_k-v_j^*v_k)=\delta_{jk}$. Furthermore, substitution of Eq. (17) into Eq. (2b) yields the elegant and physical result [8]

$$\hat{K}'=-\sum_j{}' \,E_j\int dV|v_j|^2+\sum_j{}' \,E_j\alpha_j^\dagger\alpha_j, \tag{19}$$

so that the linear transformation (17) diagonalizes the operator $\hat{K}'$. In addition, if u_j and v_j are a solution with energy E_j, then the pair v_j^* and u_j^* are also a solution with energy $-E_j$; since the quasiparticle number operator $\alpha_j^\dagger\alpha_j$ has nonnegative integral eigenvalues, it is necessary to take $E_j\geqslant 0$. Finally, Eq. (18) also has the solution $u_0=v_0\propto n_0^{1/2}$ with $E_0=0$, verifying that the Bose condensation indeed occurs in the lowest self-consistent single-particle mode.

The structure of $\hat{K}'$ in Eq. (19) leads to a very simple description of the equilibrium states of the condensed Bose system. The quasiparticle ground state $|\mathbf{0}\rangle$ satisfies the condition $\alpha_j|\mathbf{0}\rangle=0$ for all $j\neq 0$, and the excited states follow by applying an arbitrary number of quasiparticle creation operators $\alpha_j^\dagger$ to $|\mathbf{0}\rangle$. In addition, the well-known properties of these harmonic-oscillator operators mean that the low-temperature behavior is determined entirely by the eigenvalues and eigenfunctions of the Bogoliubov equations (18). If $\langle\cdots\rangle\equiv \mathrm{Tr}[\cdots\exp(-\beta\hat{K}')]/\mathrm{Tr}[\exp(-\beta\hat{K}')]$ denotes a self-consistent ensemble average at temperature $T=(k_B\beta)^{-1}$, then the only nonzero averages of one- or two-quasiparticle operators are $\langle\alpha_j^\dagger\alpha_k\rangle=\langle\alpha_k\alpha_j^\dagger\rangle-\delta_{jk}=\delta_{jk}f_j$, where $f_j\equiv[\exp(\beta E_j)-1]^{-1}$ is the usual Bose-Einstein distribution function. For example [8], the total number density $n(\mathbf{r})$ has a condensate contribution $n_0(\mathbf{r})=|\Psi(\mathbf{r})|^2$ and a noncondensate contribution

$$n'(\mathbf{r})=\sum_j{}' \,[f_j|u_j(\mathbf{r})|^2+(1+f_j)|v_j(\mathbf{r})|^2], \tag{20}$$

where the condition $N=\int dV\, n_0(\mathbf{r})+\int dV\, n'(\mathbf{r})=N_0(T)+N'(T)$ determines the temperature-dependent condensate fraction $N_0(T)/N$; at $T=0$, the quantity f_j vanishes, and n' then involves only $|v_j|^2$.

Although the actual experimental traps are anisotropic [1], it is simplest to consider an isotropic three-dimensional harmonic potential $U(\mathbf{r})=\frac{1}{2}m\omega^2r^2$, with a characteristic oscillator length $d_0=\sqrt{\hbar/m\omega}$ (the effect of the anisotropy can be treated in perturbation theory). For a stationary spherical condensate in a spherical confining potential $U(r)$, where $\Psi(r)=\sqrt{n_0(r)}$ satisfies Eq. (3), the Bogoliubov equations simplify greatly because the excited states can be characterized by the usual angular-momentum quantum numbers (l,m) associated with the spherical harmonics Y_{lm}, along with a radial quantum number n. Given a solution for $n_0(r)$, standard numerical techniques can determine the eigenvalues E_{nl} and associated $(2l+1)$-fold degenerate radial eigenfunctions $u_{nl}(r)$ and $v_{nl}(r)$ [5]. In order to gain more physical insight, however, it is valuable to consider a special limiting case in which the kinetic energy of the condensate wave function is negligible compared to the confining energy and the repulsive interparticle interaction energy. As discussed in [6] (see also Refs. [3,16]), this condition holds for a harmonic confining potential when the dimensionless parameter N_0a/d_0 is large. As a result, the Hartree equation (3) for the condensate wave function then has the simple solution

$$4\pi a\hbar^2m^{-1}|\Psi(r)|^2=[\mu-U(r)]\,\theta[\mu-U(r)], \tag{21}$$

where $\theta(x)$ denotes the unit positive step function. In this limit, the condensate is spherical with radius R_0 determined by the normalization condition on Ψ; the dimensionless radius $R\equiv R_0/d_0$ has the value $R^5=15N_0a/d_0$ with chemical potential given by $\mu=\frac{1}{2}\hbar\omega R^2$. Although this approximation clearly fails in the immediate vicinity of the condensate surface (see, for example, Fig. 1 of Ref. [3]), its use in the Bogoliubov equations produces only a small error in the limit $N_0 a/d_0 \gg 1$.

A combination of Eqs. (18) and (21) yields the following dimensionless coupled eigenvalue equations:

$$(D_l+V)u_{nl}-V_< v_{nl}=\epsilon_{nl}u_{nl}$$

and (22)

$$-V_< u_{nl}+(D_l+V)v_{nl}=-\epsilon_{nl}v_{nl},$$

where r is the dimensionless radial coordinate measured in units of d_0 and $\epsilon_{nl}=2E_{nl}/\hbar\omega$. Here, $D_l \equiv -r^{-2}(d/dr)r^2(d/dr)+l(l+1)/r^2$ is the kinetic energy, $V=|r^2-R^2|$ is the potential energy, and $V_< =(R^2-r^2)\theta(R-r)$. Apart from the coupling between u and v, which occurs only for $r<R$ through $V_<$, these equations look like those for radial eigenstates with orbital angular momentum l in an isotropic repulsive potential V, which has a central peak of height R^2 at the origin, reaches a minimum near the surface where it vanishes linearly, and rises quadratically for $r \gg R$. Thus the low-lying eigenfunctions (those with $\epsilon \ll R^2$) are expected to be ''surface'' modes localized in the vicinity of the condensate surface.

In principle, these coupled differential equations can be solved numerically, but more physical insight comes from recognizing that they have a variational basis. If $\mathscr{U}$ denotes a two-component vector with elements u and v, Eq. (22) has a simple matrix representation involving $\mathscr{D}_l=D_l\tau_0$, and $\mathscr{V}=V\tau_0-V_<\tau_1$, with τ_0 the 2×2 unit matrix and τ_i the familiar 2×2 Pauli matrices. It follows immediately that the variational quantity

$$\epsilon_{0l} \leq \frac{\int_0^\infty r^2 dr\ \mathscr{U}^\dagger(\mathscr{D}_l+\mathscr{V})\mathscr{U}}{\int_0^\infty r^2 dr\ \mathscr{U}^\dagger \tau_3 \mathscr{U}} \tag{23}$$

provides an upper bound on the lowest eigenvalue ϵ_{0l} for each separate l. As a very simple model, take $u(r)=\cosh\chi f(r)$ and $v(r)=\sinh\chi f(r)$ with $\int_0^\infty r^2 dr |f(r)|^2=1$. Substitution into Eq. (23) gives $\epsilon_{0l} \leq A\cosh 2\chi - B\sinh 2\chi$, where

$$A=\int_0^\infty r^2 dr\ f(r)^*(D_l+V)f(r)$$

and (24)

$$B=\int_0^R r^2 dr\ f(r)^* V_< f(r).$$

Minimization with respect to χ yields the condition $\tanh 2\chi = B/A$, with $\epsilon_{0l} \leq \sqrt{A^2-B^2}$.

If f also depends on a parameter, it can be varied to find the minimum upper bound for ϵ_{0l}. For example, take $f(r) \propto \exp[-\frac{1}{2}\gamma^2(r-R)^2]$; for $R \gg 1$, the potential can be approximated by its linear form $V(r) \approx 2R|r-R|$ near the surface, and it is easy to evaluate the integrals A and B. The minimum of ϵ_{0l} with respect to γ occurs for $\gamma_0 \propto R^{1/3}$, so that the radial state $f(r)$ has a dimensionless width of order $R^{-1/3}$. The corresponding energy eigenvalue is given by the sum of a radial zero-point energy ϵ_{00} and a rotational term $\propto l(l+1)R^{-2}$:

$$\epsilon_{0l} \approx \begin{cases} 1.53R^{2/3}+1.07l(l+1)R^{-2} & \text{for } l(l+1) \ll R^{8/3} \\ 1.63R^{2/3}+l(l+1)R^{-2} & \text{for } l(l+1) \gg R^{8/3}. \end{cases} \tag{25}$$

Evidently, these variational solutions can be used to find the corresponding variational fluctuation operators $\hat{\rho}'$ and $\hat{\Phi}'$, so that an equivalent fully hydrodynamic description is feasible. As noted previously, however, the operator character and the appearance of gradients of n_0 render this latter approach more intricate than one based directly on the field operators and the Bogoliubov equations, whose familiar quantum-mechanical structure also simplifies the boundary conditions.

This variational solution provides an estimate of the noncondensate contribution of these low-lying states. At $T=0$, the spatial integral of Eq. (20) shows that each of the $2l+1$ states with energy eigenvalue E_{0l} makes a contribution $\int dV|v_{0l}|^2$ to N', and the preceding variational solution yields the approximate result

$$\int dV|v_{0l}|^2 \approx \sinh^2\chi_{0l} \approx \begin{cases} 0.035-0.050l(l+1)/R^{8/3} & \text{for } l(l+1) \ll R^{8/3} \\ 0.147[R^{8/3}/l(l+1)]^2 & \text{for } l(l+1) \gg R^{8/3}. \end{cases} \tag{26}$$

Each low-lying mode makes only a small noncondensate contribution that decreases with increasing l.

The fluctuation density operator $\hat{\rho}'(\mathbf{r})$ plays a central role in the response of a physical system to external perturbations. The Heisenberg operators $\tilde{\phi}(\mathbf{r},t)$ and $\tilde{\phi}^\dagger(\mathbf{r},t)$ oscillate harmonically at the frequencies given by the eigenvalues of the Bogoliubov equations, and the noncondensate part of the density-density correlation function becomes simply a correlation function of the transformed deviation operators, given by

$$\begin{aligned}\frac{\langle \hat{\rho}'(\mathbf{r},t)\hat{\rho}'(\mathbf{r}',0)\rangle}{|\Psi(\mathbf{r})\Psi(\mathbf{r}')|} \approx \sum_j{}' \{&(1+f_j)[u_j(\mathbf{r})-v_j(\mathbf{r})] \\ &\times[u_j^*(\mathbf{r}')-v_j^*(\mathbf{r}')]e^{-iE_jt/\hbar} \\ &+f_j[u_j^*(\mathbf{r})-v_j^*(\mathbf{r})][u_j(\mathbf{r}') \\ &-v_j(\mathbf{r}')]e^{iE_jt/\hbar}\}.\end{aligned} \tag{27}$$

Consequently, a measurement of the frequency spectrum of density oscillations (for example, by studying the resonant response to small modulations of the trapping potential) would directly characterize the eigenvalues E_j. The density-current and current-current correlation functions involve similar expressions.

Note added in proof. Further analysis, motivated in part by the hydrodynamic description, has yielded much lower (and, hence, better) estimates for the energy eigenvalue of the lowest excited state for each positive $l>0$. The acoustic radial eigenfunctions of a uniform classical gas in a spherical container are spherical Bessel functions $j_l(kr)$, and the variational trial function $f(r)\propto(r/R)^l[1-(r/R)^2]\theta(R-r)$ has the same qualitative behavior. In particular, the resulting Bogoliubov energy is given by $E_{0l}\approx\hbar\omega\sqrt{T_l^2+2T_lV_l}$, where $T_l=(l+\frac{3}{2})(l+\frac{7}{2})/R^2$ and $V_l=3R^2/(2l+9)$ are the expectation values of the kinetic energy and interaction potential energy (21), respectively (this energy is very similar to that for a uniform condensed Bose gas [2,7]). Furthermore, the zero-temperature noncondensate occupation is $N_l'=\sinh^2\chi_{0l}=\frac{1}{2}(T_l+V_l)/\sqrt{T_l^2+2T_lV_l}-\frac{1}{2}$. For $T_l\ll V_l$ (namely large R and small l), the energy is independent of R, and $N_l'\gg 1$. Reference [17] contains a brief summary of this work, and improved variational and numerical studies are in progress. I thank B. V. Svistunov for valuable correspondence.

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[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995).
[2] N. N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947).
[3] M. Edwards and K. Burnett, Phys. Rev. A **51**, 1382 (1995).
[4] P. A. Ruprecht, M. J. Holland, K. Burnett, and M. Edwards, Phys. Rev. A **51**, 4704 (1995).
[5] M. Edwards, P. A. Ruprecht, K. Burnett, and C. W. Clark (unpublished).
[6] G. Baym and C. Pethick, Phys. Rev. Lett. **76**, 6 (1996).
[7] See, for example, A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971), Chaps. 2, 10, and 14.
[8] A. L. Fetter, Ann. Phys. (N.Y.) **70**, 67 (1972).
[9] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1119 (1957); T. D. Lee, K. Huang, and C. N. Yang, *ibid.* **106**, 1135 (1957).
[10] O. Penrose, Philos. Mag. **42**, 1373 (1951); O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956); C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).
[11] V. L. Ginzburg and L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **34**, 1240 (1958) [Sov. Phys. JETP **7**, 858 (1958)].
[12] E. P. Gross, Nuovo Cimento **20**, 451 (1961); J. Math. Phys. **46**, 137 (1963).
[13] L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **40**, 646 (1961) [Sov. Phys. JETP **13**, 451 (1961)].
[14] A. L. Fetter, Phys. Rev. **138**, A429 (1965); **138**, A709 (1965); **140**, A452 (1965).
[15] See, for example, A. L. Fetter and J. D. Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw-Hill, New York, 1980), Sec. 48, and Ref. [48], and Ref. [7], Sec. 55.
[16] D. A. Huse and E. D. Siggia, J. Low Temp. Phys. **46**, 137 (1982).
[17] A. L. Fetter, Czech. J. Phys. (to be published).