QED corrections to $O(\alpha^7 mc^2)$ fine-structure splittings in helium

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$O(\alpha^7 mc^2)$ QED corrections to the fine structure of helium are presented. They are expressed in the form of expectation values of nonrelativistic operators. Self-energy corrections of second order are derived rigorously. Others are obtained phenomenologically. [S1050-2947(96)08205-4]

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I. INTRODUCTION

There has been much effort made over past two decades to test higher-order QED effects in one- or two-body bound states such as hydrogen, positronium, and muonium. A lot of theoretical progress has been made in understanding higher-order QED effects in these systems [1]. Since the theoretical derivation of $O(\alpha^6 mc^2)$ QED and relativistic corrections to fine structure in helium accomplished by Douglas and Kroll [2] and the numerical calculation by Daley [3], there has been little theoretical progress in investigation of higher-order QED effects in helium until recently. One of the reasons is due to being unable to solve the Schrödinger equation for helium accurately. This situation changed after the dramatic development in the search for an extremely accurate nonrelativistic wave function of two electrons, made by Drake [4]. In addition, past and recent experiments [5–7] provide an excellent opportunity to test uncalculated QED and relativistic corrections of one order beyond that of Douglas and Kroll's terms. Although three-body corrections are still beyond reach both experimentally and theoretically, a large number of two-body terms can be tested. In particular, a two-body QED correction of off-leading order has not been tested in any bound-state system due to insufficient experimental accuracy. A typical such term is the vertex or vacuum polarization correction of off-leading order, which is of order $\alpha^7 m^2 c^2/M$ for two distinct particles or $\alpha^7 mc^2$ for positronium. A test of such terms would be of interest since this is the very nature of bound-state systems in view of the fact that a given Feynman diagram contributing to the anomalous magnetic moment of a free electron has only one order. However, these two-body terms are sensitive to experiment of fine-structure splittings in helium. Some of these terms contribute a few kHz or more individually to the fine-structure splittings while the current experimental error is only 3 kHz [7]. The need to calculate these corrections is clear. On the other hand, a modern numerical technique developed by Drake [4] for high precision calculation makes it possible to evaluate all corrections up to the order of interest with high precision and to obtain very accurate theoretical numbers.

In a recent rigorous analysis of QED corrections of off-leading order in bound-state systems [8], a number of fully relativistic formulas in closed form were presented. The crucial idea to derive these formulas is to use the mixed gauge in which the electron-electron interactions are described in Coulomb gauge while radiative interactions are expressed in covariant gauge. Furthermore, the single-particle propagators inside radiative loops are expressed in terms of Feynman propagators and those outside the loops are projected into positive and negative energy operators. That means nonradiative loops may be understood in three-dimensional space while radiative loops are described four dimensionally, which do not have to be understood in three-dimensional space. The difficulty to derive such closed formulas in an explicitly covariant formalism is that the pair and no-pair effects from exchange diagrams are treated on equal footing while only the no-pair part in a diagram causes nonperturbative Coulomb binding on transverse photon exchanged. In a three-dimensional times-order formalism, such effects are treated differently. The no-pair propagators are treated nonperturbatively while the pair propagators are expanded perturbatively. To order $\alpha^6 mc^2$, the only possible QED correction of off-leading order would come from recoil correction to the vertex modification of second order. Douglas and Kroll [2] showed phenomenologically that the correction cancels out between two vertex diagrams in which an additional crossed or uncrossed Coulomb photon is included due to the recoil correction. They concluded their rigorous analysis on this correction by stating that ''the detailed demonstration that all terms which might conceivably contribute to order $\alpha^6 mc^2$ fine structure in fact do not do so has not been completed.'' Although the analysis in Ref. [8] showed that no new nonrelativistic energy correction of order $\alpha^6 mc^2$ is found other than those obtained phenomenologically, it is clear that $O(\alpha^6 mc^2)$ relativistic energy corrections can be obtained using these formulas, together with more formulas involving pairs outside the radiative loops. In particular, the $O(\alpha^6 mc^2)$ self-energy Lamb shift in a two-body system, similar to those in the one-body system obtained by Karplus, Klein, and Schwinger [9] and by Baranger, Bethe, and Feynman [10], may be calculated using these formulas. $O(\alpha^6 mc^2)$ vertex corrections arising from the relativistic momentum region can also be obtained with these formulas. These corrections contribute only to S states in positronium and singlet states in helium in the form of the expectation value of the delta function. However, the main focus in this paper is the QED corrections to the $O(\alpha^7 mc^2)$ fine-structure splittings of helium.

In our previous paper [11], we derived a number of formulas for energy levels in helium and carried out a derivation of nonrelativistic operators contributing to $O(\alpha^7 \ln\alpha mc^2)$ fine-structure splittings in helium. We also obtained $O(\alpha^7 mc^2)$ corrections to the splittings in the nonrel-

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ativistic approximation, arising from exchange diagrams. In this paper, we present our analysis on the $O(\alpha^7 mc^2)$ QED corrections to the triplet splittings, arising from the self-energy and vertex corrections, and vacuum polarization corrections.

II. SELF-ENERGY CORRECTIONS

Self-energy corrections to the $O(\alpha^7 mc^2)$ fine structure of helium arise from a number of diagrams. The most notable one is the self-energy correction of second order, which contributes to the Lamb shift of lowest order. This correction is the most difficult one to calculate to higher order because it contains a correction of lowest order. This correction deserves a rigorous treatment more than any other radiative correction since it is the only one contributing to the Lamb shift of lowest order arising from both the anomalous magnetic moment and the charge density modification. The lowest order of the charge density modification due to vertex corrections is $\alpha^7 mc^2$. Hence, we analyze the self-energy correction of second order first.

For $O(\alpha^7 mc^2)$ fine structure in helium, diagrams responsible for this correction are no-pair and one-pair graphs. The formula to describe the no-pair diagram is presented in Ref. [8] and is given by

$$\Delta E_{S\Sigma_{n=0}^{\infty}C^n} = \frac{-ie^2}{(2\pi)^4}\int \frac{d^4q}{q^2+i\delta} \times \langle \phi_c(\mathbf{p}_1,\mathbf{p}_2)|\,\gamma_1^0\gamma_1^\alpha\gamma_1^0 \frac{1}{E-q_0-H(\mathbf{p}_1-\mathbf{q})-\varepsilon(\mathbf{p}_2)-\mathscr{L}_{2+}(\mathbf{p}_2)I_c\mathscr{L}_{2+}(\mathbf{p}_2)+i\delta}\gamma_{1\alpha}|\,\phi_c(\mathbf{p}_1,\mathbf{p}_2)\rangle. \tag{1}$$

ϕ_c is the eigenfunction in the Coulomb ladder approximation defined in Refs. [12] and [2]. $\mathscr{L}_+$ is defined by

$$\mathscr{L}_+(\mathbf{p}) = \frac{1}{2}\left(1+\frac{H(\mathbf{p})}{\epsilon(\mathbf{p})}\right).$$

The no pair here means no pair on the electron's line where there is no radiative loop since the pairs inside the radiative loop are already included in the above formula through the single-particle Feynman operators. As pointed out by Erickson and Yennie [13], direct expansion of the propagator in a power series in the Coulomb potential does not lead in a straightforward way to an expansion in powers of α. They developed a systematic technique to do expansion in powers of α. Here we extend their method to obtain the self-energy corrections to the $O(\alpha^7 mc^2)$ triplet splittings in helium. Comparing our no-pair self-energy formula for helium with theirs for hydrogen, we find that the mechanical momentum Π is defined by

$$\Pi^\mu = [E - V_1 - \varepsilon(\mathbf{p}_2) - \mathscr{L}_{2+}(\mathbf{p}_2)I_c\mathscr{L}_{2+}(\mathbf{p}_2), \mathbf{p}_1] \tag{2}$$

for helium. For fine structure of order $\alpha^7 mc^2$, the calculation is greatly simplified and is very similar to theirs for the splitting between hydrogenic P states. The simplification comes from two facts. First, no contributions arise from relativistic momentum region. To lowest order, they correspond to hydrogenic terms obtained by Karplus, Klein, and Schwinger [9] and by Baranger, Bethe, and Feynman [10] coming from two potentials. The next-to-leading order is $\alpha^8 mc^2$. To order $\alpha^7 mc^2$, the contributions arise from up to three potentials in nonrelativistic momentum region. Furthermore, only spin-dependent terms contribute to the triplet splittings. A large number of spin-independent terms may be dropped. In particular, all but two corrections give no contribution. These two corrections are relativistic wave function corrections to the magnetic moment term $\Delta E(M)$ and the shift correction $\Delta E(L)$. The magnetic moment term is defined by

$$\Delta E(M) = \frac{\alpha}{4\pi}\int_0^1 dz(2z-2)m\langle \overline{\phi_c}|\frac{M}{m^2}|\phi_c\rangle, \tag{3}$$

where M is the magnetic moment operator and is defined by

$$M = \Pi^2 - \Pi^2 = \boldsymbol{\alpha}_1\cdot[\mathbf{p}_1,\Pi^0] = -\boldsymbol{\alpha}_1\cdot[\mathbf{p}_1,V_1] - \boldsymbol{\alpha}_1\cdot[\mathbf{p}_1,\mathscr{L}_{2+}(\mathbf{p}_2)I_c\mathscr{L}_{2+}(\mathbf{p}_2)]. \tag{4}$$

Note that $\overline{\phi_c}$ is used. Performing integration over the parameter z, we obtain

$$\Delta E(M) = \frac{\alpha}{2\pi}\langle\overline{\phi_c}|\frac{-M}{2m}|\phi_c\rangle = \Delta E_V + \Delta E_c, \tag{5}$$

where

$$\Delta E_V = \frac{\alpha}{4\pi m}\langle\phi_c|\,\gamma_1^0\boldsymbol{\alpha}_1\cdot[\mathbf{p}_1,V_1]|\,\phi_c\rangle = \frac{-Z\alpha}{2\pi^2}\frac{\Delta_2}{2m}\int\frac{d\mathbf{k}_1}{k_1^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\,\boldsymbol{\gamma}_1\cdot\mathbf{k}_1|\,\phi_c(\mathbf{p}_1-\mathbf{k}_1,\mathbf{p}_2)\rangle \tag{6}$$

and

$$\Delta E_c = \frac{\alpha}{4\pi m}\langle\phi_c|\,\gamma_1^0\boldsymbol{\alpha}_1\cdot[\mathbf{p}_1,\mathscr{L}_{2+}(\mathbf{p}_2)I_c\mathscr{L}_{2+}(\mathbf{p}_2)]|\,\phi_c\rangle = \frac{\alpha}{2\pi^2}\frac{\Delta_2}{2m}\int\frac{d\mathbf{k}}{k^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\,\boldsymbol{\gamma}_1\cdot\mathbf{k}|\,\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle. \tag{7}$$

Here $\Delta_2=\alpha/2\pi$ is the anomalous magnetic moment of lowest order. The above two terms can also be obtained phenomenologically. Keeping spin-dependent terms up to order α^7mc^2, we find

$$\langle \boldsymbol{\gamma}_1\cdot\mathbf{k}_1\rangle=-\frac{1}{m}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}_1)+\frac{1}{4m^3}[p_1^2+|\mathbf{p}_1-\mathbf{k}_1|^2]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}_1)-\frac{k_1^2}{2m}. \tag{8}$$

To the lowest order α^5mc^2, the energy correction in coordinate space is

$$\Delta E_V(\alpha^5)=\frac{Z\alpha^5mc^2}{4\pi}\langle\phi_0|2\pi\delta(\mathbf{r}_1)+\frac{1}{r_1^3}\boldsymbol{\sigma}_1\cdot(\mathbf{r}_1\times\mathbf{p}_1)|\phi_0\rangle. \tag{9}$$

To order α^7mc^2, the external potential correction to the fine-structure splittings is given by

$$\Delta E_V(\alpha^7)=\frac{-Z\alpha}{2\pi^2}\frac{\Delta_2}{8m^4}\int\frac{d\mathbf{k}_1}{k_1^2}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|[p_1^2+|\mathbf{p}_1-\mathbf{k}_1|^2]\times i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}_1)|\phi_0(\mathbf{p}_1-\mathbf{k}_1,\mathbf{p}_2)\rangle. \tag{10}$$

Upon performing a Fourier transform, we obtain

$$\Delta E_V(\alpha^7)=-\frac{Z\alpha^7mc^2}{8\pi}\langle p_1^2\phi_0|\frac{1}{r_1^3}\boldsymbol{\sigma}_1\cdot(\mathbf{r}_1\times\mathbf{p}_1)|\phi_0\rangle. \tag{11}$$

This single-electron term reduces to

$$\Delta E_V(\alpha^7)=-\frac{Z^6\alpha^7mc^2}{80\pi n^3}\left(1-\frac{13}{12n^2}\right)\mathbf{s}\cdot\mathbf{l} \tag{12}$$

for hydrogen. At first sight, it does not seem to agree with that of Erickson and Yennie [13]. However, one needs to calculate the correction of second order due to wave-function perturbation in order to compare with their result since the correction they obtained is fully relativistic due to relativistic wave function modification to $\Delta E(M)$. Similarly, we find

$$\langle \boldsymbol{\gamma}_1\cdot\mathbf{k}\rangle=-\frac{1}{m}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})+\frac{1}{4m^3}[p_1^2+|\mathbf{p}_1-\mathbf{k}|^2]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})-\frac{k^2}{2m}+\frac{1}{8m^3}[p_2^2+|\mathbf{p}_2+\mathbf{k}|^2]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})+\frac{k^2}{8m^3}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})-\frac{1}{4m^3}\mathbf{p}_2\cdot(\mathbf{p}_2+\mathbf{k})i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})+\frac{1}{4m^3}\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2\times\mathbf{k}). \tag{13}$$

To the lowest order α^5mc^2, the energy correction due to electron-electron Coulomb potential is

$$\Delta E_c(\alpha^5)=-\frac{\alpha^5mc^2}{4\pi}\langle\phi_0|2\pi\delta(\mathbf{r})+\frac{1}{r^3}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1)|\phi_0\rangle. \tag{14}$$

For the fine-structure splittings of order α^7mc^2, we get

$$\Delta E_c(\alpha^7)=\frac{\alpha}{2\pi^2}\frac{\Delta_2}{16m^4}\int\frac{d\mathbf{k}}{k^2}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|2[p_1^2+|\mathbf{p}_1-\mathbf{k}|^2]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})-2\mathbf{p}_2\cdot(\mathbf{p}_2+\mathbf{k})i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})+k^2i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})+[p_2^2+|\mathbf{p}_2+\mathbf{k}|^2]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})+2\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2\times\mathbf{k})|\phi_0(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle. \tag{15}$$

Fourier transforming it yields

$$\Delta E_c(\alpha^7)=\frac{\alpha^7mc^2}{16\pi}\langle\phi_0|\frac{2}{r^3}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1)p_1^2+12\pi\frac{\delta(\mathbf{r})}{r^2}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1)-\frac{1}{r^3}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{p}_2-\frac{3}{r^5}\boldsymbol{\sigma}_1\cdot\{\mathbf{r}\times[\boldsymbol{\sigma}_2\cdot(\mathbf{r}\times\mathbf{p}_2)]\mathbf{p}_1\}|\phi_0\rangle. \tag{16}$$

For the fine-structure splittings of order α^7mc^2, the total contribution of the magnetic moment corrections reads

$$\Delta E(M)=-\frac{Z\alpha^7mc^2}{8\pi}\langle p_1^2\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_1}{r_1^3}\times\mathbf{p}_1\right)|\phi_0\rangle-\frac{Z\alpha^7mc^2}{8\pi}\langle p_2^2\phi_0|\boldsymbol{\sigma}_2\cdot\left(\frac{\mathbf{r}_2}{r_2^3}\times\mathbf{p}_2\right)|\phi_0\rangle+2\frac{\alpha^7mc^2}{16\pi}\left\{\langle p_1^2\phi_0|2\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_1\right)|\phi_0\rangle+\langle\phi_0|12\pi\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^2}\times\mathbf{p}_1\right)-\frac{1}{r^3}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{p}_2-\frac{3}{r^5}\boldsymbol{\sigma}_1\cdot\{\mathbf{r}\times[\boldsymbol{\sigma}_2\cdot(\mathbf{r}\times\mathbf{p}_2)]\mathbf{p}_1\}|\phi_0\rangle\right\}. \tag{17}$$

Now we proceed to the shift correction. This correction is defined as

$$\Delta E(L)=\frac{\alpha}{4\pi}\int_0^1 dz\int_0^\infty dK\int\frac{d^4k}{\pi^2 i}\langle\overline{\phi_c}|I_L|\phi_c\rangle, \tag{18}$$

where

$$I_L=I_{L1}+I_{L2}+I_{L3}+I_{L4}. \tag{19}$$

These shift kernels are given by

$$I_{L1}=8(1-z^2)z^3m^2\int_0^1 d\lambda^2\frac{1}{D_\lambda}\Pi_\nu\frac{1}{D_\lambda}[\Pi^\nu,\not{\Pi}]\frac{1}{D_\lambda^2}||\frac{1}{D_\lambda},$$

$$I_{L2}=-4z(1-z)\Pi_\mu\frac{1}{D}[\Pi^\mu,\not{\Pi}]\frac{1}{D}||\frac{1}{D^2},$$

$$I_{L3}=-4z^2(1-z)\frac{1}{D}\not{\Pi}_\nu\frac{1}{D}[\Pi^\nu,\not{\Pi}]\frac{1}{D}||\frac{1}{D},$$

$$I_{L4}=-8z^3\frac{1}{D}\Pi_\nu\frac{1}{D}[\Pi^\nu,\not{\Pi}]\frac{1}{D^2}. \tag{20}$$

For the operators of lowest order, the correction becomes

$$\Delta E(L)=\frac{\alpha}{\pi}\int_0^1 du\int_0^1 dz P(z,u)\langle\phi_c|\mathbf{\Pi}\cdot\frac{1}{\Delta}[-\Pi^0,\mathbf{\Pi}]|\phi_c\rangle, \tag{21}$$

where

$$P(z,u)=-2(1-z^2)u(1-u)+(1-z)+z(1-z)(1-u)+z^2(1-u)^2 \tag{22}$$

and

$$\Delta=zm^2+u(1-z)H^{NR}. \tag{23}$$

Here

$$H^{NR}=2m\left[\frac{p_1^2}{2m}+\frac{p_2^2}{2m}+V_1+V_2+I_c+W_0\right]. \tag{24}$$

The numerator has a structure of

$$\langle\phi_c|\mathbf{\Pi}\cdot[-\Pi^0,\mathbf{\Pi}]|\phi_c\rangle=\langle\phi_c|\mathbf{p}_1\cdot[V_1+I_c,\mathbf{p}_1]|\phi_c\rangle. \tag{25}$$

Approximating the wave function ϕ_c by nonrelativistic wave function ϕ_0 and following the procedure in Ref. [13] we obtain the shift correction of lowest order as

$$\Delta E(L)=\frac{4\alpha^5mc^2}{3}\left\{\left[\ln(Z\alpha)^{-2}+\frac{11}{24}\right]\langle\phi_0|Z\delta(\mathbf{r}_1)-\delta(\mathbf{r})|\phi_0\rangle+\sum_n(E_n-E_0)\ln\left|\frac{2(E_n-E_0)}{(Z\alpha)^2m}\right||\langle\phi_n|\mathbf{p}_1|\phi_0\rangle|^2\right\}. \tag{26}$$

Including the corresponding terms due to radiative corrections on the other electron line, and combining the magnetic and shift corrections with the binding correction from single transverse photon exchange, we obtain

$$\Delta E(\alpha^5)=\frac{4\alpha^5mc^2}{3}\left\{\left[\ln\frac{1}{(Z\alpha)^{-2}}-\beta(nLS,Z)+\frac{5}{6}\right]\langle\phi_0|Z\delta(\mathbf{r}_1)+Z\delta(\mathbf{r}_2)-2\delta(\mathbf{r})|\phi_0\rangle+2\ln2\langle\phi_0|\delta(\mathbf{r})|\phi_0\rangle\right\}$$
$$+\frac{\alpha^5mc^2}{4\pi}\langle\phi_0|\frac{Z}{r_1^3}\boldsymbol{\sigma}_1\cdot(\mathbf{r}_1\times\mathbf{p}_1)+\frac{Z}{r_2^3}\boldsymbol{\sigma}_2\cdot(\mathbf{r}_2\times\mathbf{p}_1)-\frac{1}{r^3}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1)|\phi_0\rangle, \tag{27}$$

where $\beta(nLS,Z)$ is the Bethe logarithm [14]. This result agrees with that of Araki [15] and Sucher [12]. The procedure used here may be considered to be equivalent to an extension of the method of Baranger, Bethe, and Feynman [10] for a more rigorous calculation discussed by Sucher in his thesis [12].

The shift correction to the fine-structure splittings of order α^7mc^2 due to relativistic wave-function corrections is found to be

$$\Delta E(L)=-2Z\alpha^7mc^2\left[\ln(Z\alpha)^{-2}-\beta'(nLS,Z)+\frac{11}{24}\right]\langle\phi_0|\delta(\mathbf{r}_1)\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_1}{r_1^2}\times\mathbf{p}_1\right)|\phi_0\rangle$$
$$+4\alpha^7mc^2\left(\frac{11}{24}-\ln2\right)\langle\phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^2}\times\mathbf{p}_1\right)|\phi_0\rangle, \tag{28}$$

where $\beta'(nLS,Z)$ is defined as

$$\beta'(nLS,Z)=\frac{\langle\phi_0|\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\mathbf{p}_1\cdot\ln[2(H_0-W_0)/(Z\alpha)^2m][V_1,\mathbf{p}_1]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1|\phi_0\rangle}{\langle\phi_0|\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\mathbf{p}_1\cdot[V_1,\mathbf{p}_1]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1|\phi_0\rangle} \tag{29}$$

similar to the Bethe logarithm [14]. The first line in the above correction is single-electron type and reproduces the hydrogenic result of Erickson and Yennie [13] when reducing to a hydrogenic system. Here, we dropped the $\ln\alpha$ term of the two-electron type since it cancels a similar term from the no-pair single transverse photon exchange [11]. Furthermore, we assume the

nonperturbative Coulomb binding between the two electrons cancels out. In fact, the two-electron binding due to self-energy corrections is

$$\frac{\alpha^5}{3\pi}\left[\left\langle \boldsymbol{\sigma}\cdot\mathbf{p}_1\mathbf{p}_1\ln\frac{(Z\alpha)^2m}{H_0-W_0}[I_c,\mathbf{p}_1]\boldsymbol{\sigma}\cdot\mathbf{p}_1\right\rangle+\left\langle \boldsymbol{\sigma}\cdot\mathbf{p}_1\mathbf{p}_2\ln\frac{(Z\alpha)^2m}{H_0-W_0}[I_c,\mathbf{p}_2]\boldsymbol{\sigma}\cdot\mathbf{p}_1\right\rangle\right]. \tag{30}$$

The two-electron binding correction due to the no-pair single transverse photon exchanged is given by

$$-\frac{\alpha^5}{3\pi}\left\langle \boldsymbol{\sigma}\cdot\mathbf{p}_1\left\{\left[\ln\frac{(Z\alpha)^2m}{H_0-W_0}I_c,\mathbf{p}_1\right],\mathbf{p}_2\right\}\boldsymbol{\sigma}\cdot\mathbf{p}_1\right\rangle. \tag{31}$$

If $\ln(H_0-W_0)$ is approximated by some average value, then we see a cancellation between the above two corrections. A similar cancellation is obtained between vertex correction and double transverse photon exchanged. The above corrections to the triplet splittings of order α^7mc^2 arising from self-energy modification of second order are obtained rigorously using Erickson and Yennie's technique. Now let us compare the rigorous calculation with a phenomenological one. In phenomenological treatment, the magnetic moment correction is identical to that in the above rigorous calculation. The only difference is in the shift correction. Previously, it was found by French [16] that

$$\ln\lambda=\ln(2A)-\frac{5}{6}, \tag{32}$$

where λ is a fictitious photon mass and A is the lower cutoff. The factor 5/6 arises from 11/24+3/8 where 11/24 and 3/8 come from the shift and magnetic moment corrections, respectively. Therefore, we found that if

$$\ln\lambda=\ln(2A)-\frac{11}{24} \tag{33}$$

is used in the phenomenological treatment, then the phenomenological approximation is accurate both for the Lamb shift of lowest order and for the fine-structure splittings of order α^7mc^2. Thus, the above number may also be a good approximation in the vertex calculation to be analyzed later phenomenologically.

Now let us check out if there is any other correction contributing to the fine-structure splittings of helium. First, we look at the magnetic moment correction $\Delta E(M)$. One possible correction arises from shifting the self-energy loop variables q_0 and $\mathbf{q}$ integrations for I_M. A typical numerator structure is

$$[\mathbf{p}_1,\cdot[\mathbf{p}_1,M]]=[\boldsymbol{\alpha}_1\cdot\mathbf{p}_1,[\mathbf{p}_1,\cdot[\mathbf{p}_1,V_1+I_c]]]. \tag{34}$$

Although the numerator is spin dependent, the matrix element $\Delta E(M-p)$ becomes spin independent after being sandwiched between wave functions. Another potential correction comes from $\Delta E(M-M)$, which has the numerator structure

$$\begin{aligned}&[\boldsymbol{\alpha}_1\cdot\mathbf{p}_1,V_1+I_c][\boldsymbol{\alpha}_1\cdot\mathbf{p}_1,V_1+I_c]\\&\quad=[\mathbf{p}_1,V_1+I_c]\cdot[\mathbf{p}_1,V_1+I_c],\end{aligned} \tag{35}$$

which becomes spin independent. The terms in $\Delta E(a)$ have the numerator structure

$$\begin{aligned}[\Pi_\nu,\mathbf{\Pi}][\Pi^\nu,\mathbf{\Pi}]=&-2[\mathbf{p}_1,V_1+\mathscr{S}_{2+}(\mathbf{p}_2)I_c\mathscr{S}_{2+}(\mathbf{p}_2)][\mathbf{p}_1,V_1\\&+\mathscr{S}_{2+}(\mathbf{p}_2)I_c\mathscr{S}_{2+}(\mathbf{p}_2)],\end{aligned} \tag{36}$$

which is also spin independent to lowest order. Similarly, the numerator structures in ΔE_b, ΔE_d, and ΔE_f all become spin independent. A possible contribution comes from the shift correction $\Delta E(L)$ defined by Erickson and Yennie for hydrogen [13]. To lowest order, the shift correction corresponds to nonperturbative Coulomb binding appearing in the form of the $\ln\alpha$ term and the Bethe logarithm. One of the higher-order terms is due to a nonrelativistic Hamiltonian modification to the lowest-order shift correction. It is given by

$$\begin{aligned}\Delta E(L-H)=&\frac{2\alpha}{\pi m}\int_0^1 u\,du\int_0^1 dz\,\overline{P(z,u)}\langle\phi_0|[\mathbf{p}_1,V_1]\frac{1}{\Delta}\\&\times[-\Pi^0_{\rm red},\mathbf{p}_1]|\phi_0\rangle,\end{aligned} \tag{37}$$

where

$$\begin{aligned}\overline{P(z,u)}=&-2(1+z-z^2)u(1-u)+(2-z)-(1-z)^2(1-u)\\&-z(1-z)(1-u)^2\end{aligned}$$

and the reduced zero component of the four-dimensional mechanical momentum is

$$\Pi^0_{\rm red}=2m+W_0-V_1-E(\mathbf{p}_2)-V_2-I_c \tag{38}$$

to lowest order. Since

$$\begin{aligned}&\langle\phi_0|[\mathbf{p}_1,V_1]\cdot[V_1,\mathbf{p}_1]|\phi_0\rangle=Z^2\alpha^2\langle\phi_0|\frac{1}{r_1^4}(1-\delta_{l_10})|\phi_0\rangle,\\&\langle\phi_0|[\mathbf{p}_1,I_c]\cdot[I_c,\mathbf{p}_1]|\phi_0\rangle=\alpha^2\langle\phi_0|\frac{1}{r^4}|\phi_0\rangle,\\&\langle\phi_0|[\mathbf{p}_1,V_1]\cdot[I_c,\mathbf{p}_1]|\phi_0\rangle=-Z\alpha^2\langle\phi_0|\frac{\mathbf{r}_1}{r_1^3}\cdot\frac{\mathbf{r}}{r^3}|\phi_0\rangle,\end{aligned} \tag{39}$$

we obtain

$$\begin{aligned}\Delta E(L-H)=&\frac{2\alpha^7mc^2}{3\pi}\ln(Z\alpha)^{-2}\langle\phi_0|\frac{Z^2}{r_1^4}(1-\delta_{l_10})\\&+\frac{Z^2}{r_2^4}(1-\delta_{l_20})-2Z\frac{\mathbf{r}_1}{r_1^3}\cdot\frac{\mathbf{r}}{r^3}\\&-2Z\frac{\mathbf{r}_2}{r_2^3}\cdot\frac{\mathbf{r}}{r^3}+\frac{2}{r^4}|\phi_0\rangle\end{aligned} \tag{40}$$

for $\ln\alpha$ terms. These corrections correspond to two potentials inside the self-energy loop, which cannot be described phenomenologically.

Another group of infrared logarithmic corrections to the $O(\alpha^7 mc^2)$ triplet energy levels comes from $\Delta E(L-p)$ defined by Erickson and Yennie. They arise from numerators containing one potential and four momenta. A typical numerator structure is

$$\langle \phi_0 | \mathbf{p}_1 \cdot ([\mathbf{p}_1, \cdot [\mathbf{p}_1, V_1]]) \mathbf{p}_1 | \phi_0 \rangle = -12\alpha^6 m^5 \langle \phi_0 | Z\delta(\mathbf{r}_1) \frac{1}{r_1^2} | \phi_0 \rangle \tag{41}$$

for $\Delta E(L2-p)$, which then becomes

$$\Delta E(L2-p) = 2\alpha^7 mc^2 \ln(Z\alpha)^{-2} \langle \phi_0 | Z\delta(\mathbf{r}_1) \frac{1}{r_1^2} + Z\delta(\mathbf{r}_2) \frac{1}{r_2^2} - c\,\delta(\mathbf{r}) \frac{1}{r^2} | \phi_0 \rangle, \tag{42}$$

where c is some constant to be determined. Similarly,

$$\Delta E(L1-p) = -\frac{4}{5}\alpha^7 mc^2 \ln(Z\alpha)^{-2} \langle \phi_0 | Z\delta(\mathbf{r}_1) \frac{1}{r_1^2} + Z\delta(\mathbf{r}_2) \frac{1}{r_2^2} - c'\,\delta(\mathbf{r}) \frac{1}{r^2} | \phi_0 \rangle, \tag{43}$$

where c' is some constant. In total, they become

$$\Delta E(L-p) = \frac{6}{5}\alpha^7 mc^2 \ln(Z\alpha)^{-2} \langle \phi_0 | Z\delta(\mathbf{r}_1) \frac{1}{r_1^2} + Z\delta(\mathbf{r}_2) \frac{1}{r_2^2} - (c+c')\,\delta(\mathbf{r}) \frac{1}{r^2} | \phi_0 \rangle. \tag{44}$$

Therefore, no more spin-dependent correction is found. Although the above correction comes from a single potential inside the self-energy loop, it cannot be obtained in a phenomenological treatment. This shows the difference between phenomenological and rigorous calculations even for a single potential diagram. One interesting result is the $O(Z^5\alpha^6 mc^2)$ QED correction to the Lamb shift of helium, which is the leading term of order $\alpha^6 mc^2$. They come from the self-energy correction with two external potentials. Since the nominal order of the two-external-potential self-energy correction is $\alpha^7 mc^2$ as shown above, the $O(\alpha^6 mc^2)$ contribution arises from the relativistic momentum region. Therefore, the nonrelativistic operator is the delta function. For hydrogen, this correction was calculated by Karplus, Klein, and Schwinger [9] and by Baranger, Bethe, and Feynman [10], and was recalculated by Erickson and Yennie [13] in a more systematic approach. For helium, they become

$$\Delta E = 4\pi Z^2 \alpha^6 mc^2 (1 + \tfrac{11}{128} - \tfrac{1}{2}\ln 2) \langle \phi_0 | \delta(\mathbf{r}_1) + \delta(\mathbf{r}_2) | \phi_0 \rangle. \tag{45}$$

We have reproduced this by using our self-energy formula and Erickson and Yennie's technique with the help of computer algebra, which is particularly suitable for the calculation of corrections from the relativistic momentum region. If we assume $1/r$ is proportional to Z, the above term is the total $O(Z^5\alpha^6 mc^2)$ QED correction since all other terms in self-energy and vertex corrections contain no external potential. That means no corrections due to the single external potential, which can only lead to the nonrelativistic approximation. Moreover, all other QED corrections of order $\alpha^6 mc^2$ are of actual order $Z^3\alpha^6 mc^2$. No single potential term leads to no $O(Z^4\alpha^6 mc^2)$ QED correction, which is in contrast to exchange corrections.

Now we proceed to the calculation of other self-energy diagrams. Obviously, self-energy corrections of sixth order contribute to the triplet splittings and are readily obtained as

$$\Delta E_c = -2\frac{\alpha^4 mc^2}{2} \Delta_6 \langle \phi_0 | \boldsymbol{\sigma}_1 \cdot \left(\frac{\mathbf{r}}{r^3} \times \mathbf{p}_1 \right) | \phi_0 \rangle \tag{46}$$

due to electron-electron Coulomb interaction, and

$$\Delta E_V = 2\frac{Z\alpha^4 mc^2}{2} \Delta_6 \langle \phi_0 | \boldsymbol{\sigma}_1 \cdot \left(\frac{\mathbf{r}_1}{r_1^3} \times \mathbf{p}_1 \right) | \phi_0 \rangle \tag{47}$$

from the external potential. Here

$$\Delta_6 = 1.176(\alpha/\pi)^3$$

is the anomalous magnetic moment of sixth order [17]. Double self-energy correction to the interelectron Coulomb potential is given by

$$\begin{aligned} \Delta E_{CR^2} &= \frac{\alpha}{2\pi^2} \int \frac{d\epsilon}{-2\pi i} \frac{d\omega}{-2\pi i} \frac{d\mathbf{k}}{k^2} \langle \phi_c(\mathbf{p}_1, \mathbf{p}_2) | [S_{1+}(\mathbf{p}_1, \epsilon) + S_{2+}(\mathbf{p}_2, \epsilon)] \\ &\quad \times \gamma_1^0 \Lambda_{10}^M(k) \gamma_2^0 \Lambda_{20}^M(k) [S_{1+}(\mathbf{p}_1 - \mathbf{k}, \epsilon - \omega) + S_{2+}(\mathbf{p}_2 + \mathbf{k}, \epsilon - \omega)] | \phi_c(\mathbf{p}_1 - \mathbf{k}, \mathbf{p}_2 + \mathbf{k}) \rangle \\ &= \frac{\alpha}{2\pi^2} \left(\frac{\Delta_2}{2m} \right)^2 \int \frac{d\mathbf{k}}{k^2} \langle \phi_c(\mathbf{p}_1, \mathbf{p}_2) | \boldsymbol{\gamma}_1 \cdot \mathbf{k} \boldsymbol{\gamma}_2 \cdot \mathbf{k} | \phi_c(\mathbf{p}_1 - \mathbf{k}, \mathbf{p}_2 + \mathbf{k}) \rangle, \end{aligned} \tag{48}$$

which is of higher order $\alpha^8 mc^2$. Double self-energy correction to the nucleon-electron Coulomb potential is given by

$$\Delta E_{VR^2}=\left(\frac{Z\alpha}{2\pi^2}\right)^2\int\frac{d\epsilon}{-2\pi i}\frac{d\mathbf{k}_1}{k_1^2}\frac{d\mathbf{k}_2}{k_2^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|$$

$$\times[S_{1+}(\mathbf{p}_1,\epsilon)+S_{2+}(\mathbf{p}_2,\epsilon)]\gamma_1^0\Lambda_{10}^M(\mathbf{k}_1)\gamma_2^0\Lambda_{20}^M(-\mathbf{k}_2)[S_{1+}(\mathbf{p}_1-\mathbf{k}_1,\epsilon)+S_{2+}(\mathbf{p}_2+\mathbf{k},\epsilon)]|\phi_c(\mathbf{p}_1-\mathbf{k}_1),(\mathbf{p}_2+\mathbf{k}_2)\rangle$$

$$=-2\left(\frac{Z\alpha}{2\pi^2}\right)^2\left(\frac{\Delta_2}{2m}\right)^2\int\frac{d\mathbf{k}_1}{k_1^2}\frac{d\mathbf{k}_2}{k_2^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\boldsymbol{\gamma}_1\cdot\mathbf{k}_1\boldsymbol{\gamma}_2\cdot\mathbf{k}_2|\phi_c(\mathbf{p}_1-\mathbf{k}_1,(\mathbf{p}_2+\mathbf{k}_2)\rangle, \tag{49}$$

which is of higher order $\alpha^{10}mc^2$.

The above single-potential and double-potential self-energy corrections are analyzed using Erickson and Yennie's technique. The double-potential correction $\Delta E(L-H)$ covers all possible contributions due to the external potential. However, only the diagram in which two interelectron potentials are inside the radiative loop on the electron line 1 while no pair is on the other electron line is analyzed using Erickson and Yennie's technique. In another diagram, the two Coulomb potentials are inside the radiative loop while a pair appears on the second electron line. The corresponding correction is given by

$$\Delta E_{CCR}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{-ie^2}{(2\pi)^4}\int\frac{d^4q}{q^2+i\delta}\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}$$

$$\times\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\gamma_1^0\gamma_1^\alpha\gamma_1^0\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\frac{1}{E-q_0-H(\mathbf{p}_1-\mathbf{q})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{2+}(\mathbf{p}_2)I_c\mathscr{S}_{2+}(\mathbf{p}_2)+i\delta}$$

$$\times\frac{-1}{E-q_0-H(\mathbf{p}_1-\mathbf{k}-\mathbf{q})-E(\mathbf{p}_2)-E(\mathbf{p}_2+\mathbf{k})-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}$$

$$\times\frac{1}{E-q_0-H(\mathbf{p}_1-\mathbf{k}-\mathbf{k}'-\mathbf{q})-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')I_c\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}\gamma_{1\alpha}$$

$$\times|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle. \tag{50}$$

This correction contributes to the $O(\alpha^6mc^2)$ Lamb shift arising from the relativistic momentum region. To order α^7mc^2, it is spin independent. When one of the two Coulomb potentials is outside the radiative loop, the corresponding correction is given by

$$\Delta E_{CR\times C}=2\left(\frac{\alpha}{2\pi^2}\right)^2\frac{-ie^2}{(2\pi)^4}\int\frac{d^4q}{q^2+i\delta}\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\frac{1}{2m}\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k})\Lambda_{2-}(\mathbf{p}_2+\mathbf{k}')$$

$$\times\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{E-q_0-H(\mathbf{p}_1-\mathbf{k}-\mathbf{q})-E(\mathbf{p}_2)-E(\mathbf{p}_2+\mathbf{k})-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}$$

$$\times\frac{1}{E-q_0-H(\mathbf{p}_1-\mathbf{k}-\mathbf{k}'-\mathbf{q})-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}\gamma_{1\alpha}-\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k}')\gamma_1^0\gamma_1^\alpha\gamma_1^0$$

$$\times\frac{1}{-q_0-H(\mathbf{p}_1-\mathbf{k}-\mathbf{q})-E(\mathbf{p}_1-\mathbf{k})+i\delta}\frac{1}{E-q_0-H(\mathbf{p}_1-\mathbf{k}-\mathbf{k}'-\mathbf{q})-E(\mathbf{p}_1)-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2+\mathbf{k}')+i\delta}$$

$$\times\left[\gamma_{1\alpha}\frac{-1}{2m}+\frac{1}{E-q_0-H(\mathbf{p}_1-\mathbf{k}-\mathbf{k}'-\mathbf{q})-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}\gamma_{1\alpha}\right]|\phi_c(\mathbf{p}_1-\mathbf{k}'_{\mathbf{k}},\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle, \tag{51}$$

where the external potentials and some higher-order corrections are neglected. The factor of 2 comes from a correction due to a similar diagram. Direct application of Erickson and Yennie's method to calculate this correction does not seem to be straightforward. Instead, the correction is treated phenomenologically. In phenomenological treatment, the correction is given by

$$\Delta E_{CR\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\epsilon}{-2\pi i}\frac{d\omega}{-2\pi i}\frac{d\omega'}{-2\pi i}\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|[S_{1+}(\mathbf{p}_1,\epsilon)+S_{2+}(\mathbf{p}_2,\epsilon)][S_1(\mathbf{p}_1-\mathbf{k}',\epsilon-\omega')\gamma_1^0\Lambda_{10}(k)$$
$$+\gamma_1^0\Lambda_{10}(k')S_1(\mathbf{p}_1-\mathbf{k}',\epsilon-\omega')]S_2(\mathbf{p}_2+\mathbf{k},\epsilon-\omega)[S_{1+}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\epsilon-\omega-\omega')$$
$$+S_{2+}(\mathbf{p}_2+\mathbf{k}+\mathbf{k}',\epsilon-\omega-\omega')]|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle$$
$$=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{\Delta_2}{4m^2}\int\frac{d\mathbf{k}}{k^2}\int\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|[\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\boldsymbol{\gamma}_1\cdot\mathbf{k}+\boldsymbol{\gamma}_1\cdot\mathbf{k}'\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')]$$
$$\times\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle, \tag{52}$$

where we have neglected higher-order terms in the last line. Taking the FW transformation, we obtain

$$\Delta E_{CR\times C}=-\left(\frac{\alpha}{2\pi}\right)^2\frac{\Delta_2}{4m^3}\int\frac{d\mathbf{k}}{k^2}\int\frac{d\mathbf{k}'}{k'^2}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|\mathbf{k}\cdot\mathbf{k}'+i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}')|\phi_0(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle, \tag{53}$$

which is spin independent and therefore gives no contribution to the fine-structure splittings as one might expect.

A self-energy correction coupled with a single transverse photon exchanged causes an energy-level shift of nominal order $\alpha^7 mc^2$. In particular, the diagram in which a pair is on the electron line outside the self-energy loop is found to contribute to the fine-structure splittings. The correction is given by

$$\Delta E_{-+}=\frac{\alpha}{2\pi^2}\frac{-ie^2}{(2\pi)^4}\int\frac{d^4q}{q^2+i\delta}\frac{d\omega}{-2\pi i}\frac{d\mathbf{k}}{\omega^2-k^2+i\delta}\frac{1}{2m}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})$$
$$\times\frac{1}{\omega-E(\mathbf{p}_1)-E(\mathbf{p}_1-\mathbf{k})+i\delta}\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{-q_0-H(\mathbf{p}_1-\mathbf{k}-\mathbf{q})-E(\mathbf{p}_1-\mathbf{k})+i\delta}\gamma_{1\alpha}\alpha_2^i$$
$$+\alpha_2^i\frac{1}{E+\omega-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})+i\delta}\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})$$
$$\times\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{E-q_0-H(\mathbf{p}_1-\mathbf{k}-\mathbf{q})-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k})+i\delta}\gamma_{1\alpha}$$
$$+\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{E-\omega-q_0-H(\mathbf{p}_1-\mathbf{k}-\mathbf{q})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{2+}(\mathbf{p}_2)I_c\mathscr{S}_{2+}(\mathbf{p}_2)+i\delta}\gamma_{1\alpha}$$
$$\times\frac{1}{E-\omega-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)+i\delta}\alpha_2^i$$
$$+\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{-q_0-H(\mathbf{p}_1-\mathbf{q})-E(\mathbf{p}_1)+i\delta}\gamma_{1\alpha}\Lambda_{1-}(\mathbf{p}_1)\alpha_1^i\alpha_2^i\frac{1}{-\omega-E(\mathbf{p}_1)-E(\mathbf{p}_1-\mathbf{k})+i\delta}$$
$$+\alpha_2^i\frac{1}{E+\omega-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})+i\delta}$$
$$\times\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{E+\omega-q_0-H(\mathbf{p}_1-\mathbf{q})-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k})+i\delta}\gamma_{1\alpha}\Lambda_{1-}(\mathbf{p}_1)\alpha_1^i$$
$$+\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{E-q_0-H(\mathbf{p}_1-\mathbf{q})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{2+}(\mathbf{p}_2)I_c\mathscr{S}_{2+}(\mathbf{p}_2)+i\delta}\gamma_{1\alpha}\Lambda_{1-}(\mathbf{p}_1)\alpha_1^i$$
$$\times\frac{1}{E-\omega-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)+i\delta}\alpha_2^i|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle. \tag{54}$$

A rigorous calculation of the above correction seems to require an extension of Erickson and Yennie's method. However, since its nominal order is $\alpha^7 mc^2$, a phenomenological treatment is used. The energy correction including zero, one, and two pairs is phenomenologically given by

$$\Delta E_{CR\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\epsilon}{-2\pi i}\frac{d\omega}{-2\pi i}\frac{d\omega'}{-2\pi i}\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{\omega'^2-k'^2+i\delta}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|[S_{1+}(\mathbf{p}_1,\epsilon)+S_{2+}(\mathbf{p}_2,\epsilon)]$$
$$\times[\alpha_1^j S_1(\mathbf{p}_1-\mathbf{k}',\epsilon-\omega')\gamma_1^0\Lambda_{10}(k)S_2(\mathbf{p}_2+\mathbf{k},\epsilon-\omega)\alpha_2^j+\gamma_1^0\Lambda_{10}(k)S_1(\mathbf{p}_1-\mathbf{k},\epsilon-\omega)\alpha_1^j\alpha_2^j S_2(\mathbf{p}_2+\mathbf{k}',\epsilon-\omega')]$$
$$\times[S_{1+}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\epsilon-\omega-\omega')+S_{2+}(\mathbf{p}_2+\mathbf{k}+\mathbf{k}',\epsilon-\omega-\omega')]|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle$$
$$=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}\int\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^j\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k}')$$
$$\times\frac{1}{E-k'-\varepsilon(\mathbf{p}_1-\mathbf{k}')-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2)}\gamma_1^0\Lambda_{10}(k)$$
$$\times\frac{1}{E-k'-\varepsilon(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k})}\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^j$$
$$+\alpha_2^j\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k}')\frac{1}{E-k'-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}')}\gamma_1^0\Lambda_{10}(k)$$
$$\times\frac{1}{E-k'-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k}+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k}+\mathbf{k}')}$$
$$\times\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k})\alpha_1^j+\frac{1}{2m}\alpha_1^j\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\gamma_1^0\Lambda_{10}(k)$$
$$\times\frac{1}{E-k'-\varepsilon(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k})}\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^j$$
$$+\frac{1}{2m}\alpha_2^j\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k}')\frac{1}{E-k'-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}')}\gamma_1^0\Lambda_{10}(k)\Lambda_{1-}$$
$$\times(\mathbf{p}_1-\mathbf{k})\alpha_1^j+\frac{1}{2m}\alpha_1^j\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k}')\frac{1}{E-k'-\varepsilon(\mathbf{p}_1-\mathbf{k}')-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2)}$$
$$\times\gamma_1^0\Lambda_{10}(k)\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j+\frac{1}{2m}\alpha_2^j\Lambda_{2-}(\mathbf{p}_2+\mathbf{k}')\gamma_1^0\Lambda_{10}(k)$$
$$\times\frac{1}{E-k'-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}+\mathbf{k}')}$$
$$\times\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k})\alpha_1^j|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle, \tag{55}$$

where infinite Coulomb binding effects are included. The first two terms come from no-pair diagrams and are of nominal order α^8mc^2. The last two terms arising from one-pair diagrams are of nominal order α^9mc^2 and are dropped. The third and fourth terms also come from one-pair diagrams and contribute to the $O(\alpha^7mc^2)$ fine-structure splittings. Two two-pair terms become zero after integration over the energy variables. Taking the Foldy-Wouthyson transformation and eliminating the transverse indices, we obtain

$$\Delta E_{CR\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{\Delta_2}{8m^3}\int\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|-2i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})+2\mathbf{p}_1\cdot\hat{\mathbf{k}}'i\boldsymbol{\sigma}_1\cdot(\hat{\mathbf{k}}'\times\mathbf{k})$$
$$+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}'|\phi_0(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle. \tag{56}$$

Performing a Fourier transform and doubling it to include the vertex correction due to the other electron, we get

$$\Delta E_{CR\times T}=-2\frac{\alpha^7mc^2}{16\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^4}\times\mathbf{p}_1\right)$$
$$+\boldsymbol{\sigma}_1\cdot\frac{\mathbf{r}}{r^3}\boldsymbol{\sigma}_2\cdot\frac{\mathbf{r}}{r^3}|\phi_0\rangle. \tag{57}$$

This correction involves three photons and comes from a single-pair diagram in which the pair and self-energy loop are on the same electron line. Comparing the phenomenological and rigorous results, we find no difference between phenomenological and rigorous calculations of fine-structure splittings in helium arising from self-energy corrections of second order. This implies that phenomenological calcula-

tions of other self-energy corrections may be accurate to the order of interest since their nominal orders are higher. The lowest-order self-energy correction arises from two sources, the magnetic moment term $\Delta E(M)$ and the shift correction $\Delta E(L)$. $\Delta E(M)$ gives the fine-structure splittings of lowest order while $\Delta E(L)$ only contributes to the Lamb shift of lowest order. The leading order of the shift correction to the fine-structure splittings is $\alpha^7 mc^2$, which can be obtained phenomenologically. The magnetic moment term has the same form for a relativistic wave function in both phenomenological and rigorous calculations. However, the calculation of $O(\alpha^7 mc^2)$ triplet energy levels requires a rigorous treatment as we demonstrated above for $\ln\alpha$ terms.

III. VERTEX MODIFICATIONS

A rigorous calculation of vertex corrections seems to require an extension of Erickson and Yennie's technique [13] for self-energy corrections to the vertex problem, and to use the no-pair formulas presented in Ref. [8] as well as the pair formulas. However, we treat the vertex corrections phenomenologically based on a expectation that the difference between rigorous and phenomenological results is not great. In fact, we have shown that phenomenological and rigorous results for the self-energy correction of second order are the same for the fine-structure splittings and differ only for the triplet energy corrections. The vertex correction of second order is given by

$$\Delta E_{TR}=\frac{\alpha}{2\pi^2}\int\frac{d\epsilon}{-2\pi i}\frac{d\omega}{-2\pi i}\frac{d\mathbf{k}}{\omega^2-k^2+i\Delta}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|[S_{1+}(\mathbf{p}_1\epsilon)+S_{2+}(\mathbf{p}_2\epsilon)]\gamma_1^0\Lambda_{1i}(k)\alpha_2^i[S_{1+}(\mathbf{p}_1-\mathbf{k}\epsilon-\omega)+S_{2+}(\mathbf{p}_2+\mathbf{k}\epsilon-\omega)]|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle. \tag{58}$$

Integrating over the energy variable ϵ and incorporating the infinite Coulomb binding effects, we find

$$\Delta E_{TR}=\frac{\alpha}{2\pi^2}\int\frac{d\omega}{-2\pi i}\frac{d\mathbf{k}}{\omega^2-k^2+i\delta}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\gamma_1^0\Lambda_{1i}(k)\frac{1}{E-\omega-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)}\alpha_2^i$$
$$+\alpha_2^i\frac{1}{E+\omega-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})}\gamma_1^0\Lambda_{1i}(k)|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle. \tag{59}$$

The correction is divided into two parts. The magnetic moment correction becomes

$$\Delta E_{TR}(M)=\frac{\alpha}{2\pi^2}\frac{\Delta_2}{2m}\int\frac{d\mathbf{k}}{2k}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\boldsymbol{\gamma}_1\cdot\mathbf{k}\frac{1}{E-k-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)}\alpha_2^i$$
$$+\alpha_2^i\frac{1}{E-k-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})}\alpha_1^i\boldsymbol{\gamma}_1\cdot\mathbf{k}|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle \tag{60}$$

after integration over ω. We expand the propagator nonrelativistically up to order $\alpha^7 mc^2$. Taking the FW transformation, we obtain

$$\Delta E_{TR}^M=\frac{\alpha}{2\pi^2}\frac{\Delta_2}{2m}\int\frac{d\mathbf{k}}{-2k^2}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|S_1^i\left[1+\frac{\Delta H_1}{k}+\frac{(\Delta H_1)^2}{k^2}\right]R_2^i+R_2^i\left[1+\frac{\Delta H_2}{k}+\frac{(\Delta H_2)^2}{k^2}\right]S_1^i|\phi_0(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle$$
$$=\Delta E_5+\Delta E_6+\Delta E_7, \tag{61}$$

where

$$S_1^i=\langle\alpha_1^i\boldsymbol{\gamma}_1\cdot\mathbf{k}\rangle, \tag{62}$$

$$R_2^i=\langle\alpha_2^i\rangle, \tag{63}$$

$$\Delta H_1=E-H_{\rm red}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2), \tag{64}$$

and

$$\Delta H_2=E-H_{\rm red}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}). \tag{65}$$

ΔE_5 is of nominal order $\alpha^5 mc^2$ and is given by

$$\Delta E_5=\frac{\alpha}{2\pi^2}\frac{\Delta_2}{2m}\int\frac{d\mathbf{k}}{-2k^2}\langle(\mathbf{p}_1,\mathbf{p}_2)|2S_1^iR_2^i|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle, \tag{66}$$

where

$$S_1^i = -\sigma_1^i \boldsymbol{\sigma}_1\cdot\mathbf{k} + \frac{p_1^2+|\mathbf{p}_1-\mathbf{k}|^2}{8m^2}\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k} + \frac{1}{4m^2}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\sigma_1^i[\mathbf{k}\cdot(\mathbf{p}_1-\mathbf{k}) - i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})] \tag{67}$$

and

$$R_2^i = \frac{1}{2m}(2p_2^i+\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}) - \frac{p_2^i}{8m^3}(4p_2^2+2\mathbf{p}_2\cdot\mathbf{k}+k^2) - \frac{1}{8m^3}(2\mathbf{p}_2\cdot\mathbf{k}+k^2)\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{p}_2 - \frac{1}{16m^3}(4p_2^2+6\mathbf{p}_2\cdot\mathbf{k}+3k^2)\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}$$
$$+\frac{1}{4m^2}[[\boldsymbol{\sigma}_2\cdot\mathbf{p}_2,\sigma_2^i],V_2] \tag{68}$$

accurate up to order α^7mc^2. To lowest order, we find

$$\Delta E_{TR}(\alpha^5) = -2\frac{\alpha^5mc^2}{8\pi}\langle\phi_0|2\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_1\right) + \frac{3}{r^3}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}} - \frac{1}{r^3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + \frac{8\pi}{3}\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle, \tag{69}$$

where the magnetic moment of the other electron is included. For the fine-structure splittings of order α^7mc^2, we have

$$\Delta E_5(\alpha^7) = \Delta E_V + \Delta E. \tag{70}$$

The first term involving an external potential is found to be

$$\Delta E_V = 2\frac{Z\alpha^7mc^2}{8\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\frac{\mathbf{r}}{r^3}\boldsymbol{\sigma}_2\cdot\frac{\mathbf{r}_2}{r_2^3}|\phi\rangle. \tag{71}$$

This comes from a three-photon diagram. The second term coming from the single potential is given by

$$\Delta E = \frac{\alpha}{2\pi^2}\frac{\Delta_2}{16m^4}\int\frac{d\mathbf{k}}{k^2}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|-2\mathbf{p}_1\cdot\mathbf{k}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k} + [2\mathbf{p}_1\cdot\mathbf{p}_2+2k^2+p_2^2+|\mathbf{p}_2+\mathbf{k}|^2-2(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{k}+3p_1^2$$
$$+|\mathbf{p}_1-\mathbf{k}|^2]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}) - [p_1^2+2|\mathbf{p}_1-\mathbf{k}|^2]\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k} - \boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})\boldsymbol{\sigma}_2\cdot(\mathbf{p}_1\times\mathbf{k})|\phi_0(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle. \tag{72}$$

Performing a Fourier transform and including the vertex correction from the second electron, we arrive at

$$\Delta E = 2\frac{\alpha^7mc^2}{32\pi}\langle\phi_0|2\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_1\right)\mathbf{p}_1\cdot\mathbf{p}_2 + 24\pi\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^2}\times\mathbf{p}_1\right) + 2\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_1\right)p_2^2 + 4\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_1\right)p_1^2$$
$$+\frac{6}{r^3}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{p}_1 + \frac{9}{r^3}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}p_1^2 - \frac{1}{r^3}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{p}_1 + \frac{3}{r^5}\boldsymbol{\sigma}_1\cdot\{\mathbf{r}\times[\boldsymbol{\sigma}_2\cdot(\mathbf{r}\times\mathbf{p}_1)]\mathbf{p}_1\}|\phi_0\rangle. \tag{73}$$

ΔE_6 is of nominal order α^6mc^2. Douglas and Kroll showed that the total contribution from ΔE_6 is zero for the fine-structure splittings of order α^6mc^2 although individual terms are not. ΔE_7 is given by

$$\Delta E_7 = \frac{\alpha}{2\pi^2}\frac{\Delta_2}{2m}\int\frac{d\mathbf{k}}{-2k^4}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|S_1^i(\Delta H_1)^2R_2^i + R_2^i(\Delta H_2)^2S_1^i|\phi_0(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle. \tag{74}$$

Further calculation leads to

$$\Delta E_7 = \frac{\alpha}{2\pi^2}\frac{\Delta_2}{16m^3}\int\frac{d\mathbf{k}}{k^4}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|\frac{1}{m}[p_1^2-|\mathbf{p}_1-\mathbf{k}|^2][|\mathbf{p}_2+\mathbf{k}|^2-p_2^2][2i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_2\times\mathbf{k})+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}]$$
$$-4[p_1^2-|\mathbf{p}_1-\mathbf{k}|^2][V_2,i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_2\times\mathbf{k})] - 2\{[p_1^2-|\mathbf{p}_1-\mathbf{k}|^2],[I_c,i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_2\times\mathbf{k})]\}|\phi_0(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle$$
$$=\Delta E+\Delta E_V+\Delta E_c, \tag{75}$$

where ΔE is found to be

$$\Delta E = \left(\frac{\alpha}{2\pi^2}\right)^2\frac{\Delta_2}{16m^4}\int\frac{d\mathbf{k}}{k^4}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|[p_1^2-|\mathbf{p}_1-\mathbf{k}|^2][k^2+2\mathbf{p}_2\cdot\mathbf{k}][2i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_2\times\mathbf{k})+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}]|\phi_0(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle. \tag{76}$$

Fourier transforming it and including the correction from the vertex on the second electron line, we obtain

$$\Delta E=2\frac{\alpha^7 mc^2}{16\pi}\left\{\langle\phi_0|2\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_2\right)\mathbf{p}_1\cdot\mathbf{p}_2+2\frac{\mathbf{r}}{r^3}\cdot\mathbf{p}_2\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{p}_2)-6\boldsymbol{\sigma}_1\cdot\left[\frac{\mathbf{r}}{r^5}\times[\mathbf{r}\cdot(\mathbf{r}\cdot\mathbf{p}_1)\mathbf{p}_2]\mathbf{p}_2\right]\right.$$
$$\left.-12\pi\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^2}\times\mathbf{p}_1\right)|\phi_0\rangle+\langle p_1^2\phi_0|\frac{3}{r^3}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}+i\boldsymbol{\sigma}_1\cdot\frac{\mathbf{r}}{r^3}(2\boldsymbol{\sigma}_2-3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})\cdot\mathbf{p}_2|\phi_0\rangle\right\}. \tag{77}$$

The external potential term reduces to

$$\Delta E_V=-2\frac{Z\alpha^7 mc^2}{8\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_2}{rr_2^3}\times\mathbf{p}_1\right)-\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_2}{r_2^3}\times\frac{\mathbf{r}}{r^3}\right)\mathbf{r}\cdot\mathbf{p}_1-i\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_2}{r_2^3}\times\frac{\mathbf{r}}{r^3}\right)|\phi_0\rangle. \tag{78}$$

The interelectron Coulomb term becomes

$$\Delta E_c=2\frac{\alpha^7 mc^2}{8\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^4}\times\mathbf{p}_1\right)|\phi_0\rangle. \tag{79}$$

For the charge-density correction, the nonperturbative binding cancels the corresponding part in double transverse photon exchange correction. If we use

$$\ln\frac{m}{\lambda}=\ln\frac{m}{2A}+\frac{11}{24},$$

then we find

$$\Delta E_{TR}^C=2\alpha^7 mc^2\left(\frac{11}{24}-\ln 2\right)\langle\phi_0|4\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^2}\times\mathbf{p}_1\right)+5\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}|\phi_0\rangle. \tag{80}$$

Similar to self-energy corrections, the vertex modification of sixth order also contributes to the fine-structure splittings to the order of interest. The correction is given by

$$\Delta E_{TR}^6=-\alpha^4 mc^2\Delta_6\langle\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_1\right)-\frac{1}{2r^3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})+\frac{4\pi}{3}\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle. \tag{81}$$

In addition, the double vertex correction is given by

$$\Delta E_{TR^2}=\frac{\alpha}{2\pi^2}\int\frac{d\omega}{-2\pi i}\frac{d\mathbf{k}}{\omega^2-k^2+i\delta}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\gamma_1^0\Lambda_{1i}(k)$$
$$\times\frac{1}{E-\omega-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)+i\delta}\gamma_2^0\Lambda_{2i}(-k)+\gamma_2^0\Lambda_{2i}(-k)$$
$$\times\frac{1}{E+\omega-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})+i\delta}\gamma_1^0\Lambda_{1i}(k)|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle. \tag{82}$$

Only the magnetic moment contributes to the fine-structure splittings of order $\alpha^7 mc^2$. After integration over the energy variable ω, the magnetic moment correction then becomes

$$\Delta E_{TR^2}=-\frac{\alpha}{2\pi^2}\left(\frac{\Delta}{2m}\right)^2\frac{d\mathbf{k}}{2k}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\boldsymbol{\gamma}_1\cdot\mathbf{k}\frac{1}{E-k-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)}\alpha_2^i\boldsymbol{\gamma}_2\cdot\mathbf{k}$$
$$+\alpha_2^i\boldsymbol{\gamma}_2\cdot\mathbf{k}\frac{1}{E-k-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})}\alpha_1^i\boldsymbol{\gamma}_1\cdot\mathbf{k}|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle, \tag{83}$$

where $\Delta=\Delta_2+\Delta_4+\Delta_6$. Only the double vertex diagrams, when one vertex is of second order and the other of fourth order, contribute and the correction is

$$\Delta E_{TR^2}=\frac{\alpha^5 mc^2}{4\pi}\Delta_4\langle\phi_0|\frac{1}{r^3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})-\frac{8\pi}{3}\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle, \tag{84}$$

where $\Delta_4=-0.328(\alpha/\pi)^2$ is the magnetic moment of fourth order. Although the recoil correction of the double second-order vertex contributes individually, the total is zero.

The vertex modification coupled with an additional Coulomb photon exchanged between the two electrons contributes to the fine-structure splittings of order $\alpha^7 mc^2$. The correction is

$$\Delta E_{TR\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\epsilon}{-2\pi i}\frac{d\omega}{-2\pi i}\frac{d\omega'}{-2\pi i}\frac{d\mathbf{k}}{\omega^2-k^2+i\delta}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|[S_{1+}(\mathbf{p}_1,\epsilon)+S_{2+}(\mathbf{p}_2,\epsilon)]$$
$$\times[S_1(\mathbf{p}_1-\mathbf{k}',\epsilon-\omega')\gamma_1^0\Lambda_{1i}(k)\alpha_2^i S_2(\mathbf{p}_2+\mathbf{k},\epsilon-\omega)+\gamma_1^0\Lambda_{1i}(k)S_1(\mathbf{p}_1-\mathbf{k},\epsilon-\omega)S_2(\mathbf{p}_2+\mathbf{k}',\epsilon-\omega')\alpha_2^i]$$
$$\times[S_{1+}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\epsilon-\omega-\omega')+S_{2+}(\mathbf{p}_2+\mathbf{k}+\mathbf{k}',\epsilon-\omega-\omega')]|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle. \tag{85}$$

Integrating over ϵ, we get

$$\Delta E_{TR\times C}=2\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\omega}{-2\pi i}\frac{d\mathbf{k}}{\omega^2-k^2+i\delta}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\Bigg[\alpha_2^i\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k})\frac{1}{E+\omega-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k})+i\delta}$$
$$-\frac{1}{-\omega-E(\mathbf{p}_2)-E(\mathbf{p}_2+\mathbf{k})+i\delta}\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\Bigg]\Bigg[\frac{1}{E+\omega-\varepsilon(\mathbf{p}_1-\mathbf{k}')-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k}')$$
$$-\frac{1}{-\omega-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+i\delta}\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\Bigg]\gamma_1^0\Lambda_{1i}(k)|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k})\rangle, \tag{86}$$

where the factor 2 corresponds to the diagrams in which the exchanged Coulomb photon is after the vertex. One of the four terms above corresponds to a no-pair diagram and is already included in the previous vertex corrections. Another term corresponding to a two-pair diagram becomes zero after integration over ω. The other two terms are due to one-pair diagrams and contribute to the fine-structure splittings of order $\alpha^7 mc^2$ only through the magnetic moment. After integration over ω, they reduce to

$$\Delta E_{TR\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{\Delta_2}{4m^2}\int\frac{d\mathbf{k}}{k}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\Bigg[\alpha_2^i\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k})$$
$$\times\frac{1}{E-k-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k})-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})}\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')$$
$$+\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\frac{1}{E-k-\varepsilon(\mathbf{p}_1-\mathbf{k}')-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')}$$
$$\times\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k}')\Bigg]\alpha_1^i\boldsymbol{\gamma}_1\cdot\mathbf{k}|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k})\rangle, \tag{87}$$

where we have included the infinite Coulomb effects. Since the correction is of nominal order $\alpha^7 mc^2$, the propagators can be approximated by the photon momentum k. Then, the correction reduces to

$$\Delta E_{TR\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{-\Delta_2}{4m^2}\int\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_2^i[\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\mathscr{S}_{2+}(\mathbf{p}_2+\mathbf{k})$$
$$+\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k}')\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})]\alpha_1^i\boldsymbol{\gamma}_1\cdot\mathbf{k}|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k})\rangle. \tag{88}$$

The first term is of order $\alpha^8 mc^2$ and is therefore dropped. Taking the FW transformation leads to

$$\Delta E_{TR\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{\Delta_2}{8m^3}\int\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}'|\phi_0(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle. \tag{89}$$

Performing Fourier transform, we arrive at

$$\Delta E_{TR\times C}=-2\frac{\alpha^7 mc^2}{16\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\frac{\mathbf{r}}{r^3}\boldsymbol{\sigma}_2\cdot\frac{\mathbf{r}}{r^3}|\phi_0\rangle, \tag{90}$$

where the factor 2 comes from the diagrams where the vertex is on the second electron line.

One more correction comes from vertex modification coupled with another transverse photon exchanged. The correction is given by

$$\Delta E_{TR\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\epsilon}{-2\pi i}\frac{d\omega}{-2\pi i}\frac{d\omega'}{-2\pi i}\frac{d\mathbf{k}}{\omega^2-k^2+i\delta}\frac{d\mathbf{k}'}{\omega'^2-k'^2+i\delta}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|[S_{1+}(\mathbf{p}_1,\epsilon)+S_{2+}(\mathbf{p}_2,\epsilon)]$$
$$\times[\alpha_1^j S_1(\mathbf{p}_1-\mathbf{k}')\gamma_1^0\Lambda_{1i}(k)\alpha_2^i S_2(\mathbf{p}_2+\mathbf{k})\alpha_2^j+\gamma_1^0\Lambda_{1i}(k)S_1(\mathbf{p}_1-\mathbf{k})\alpha_1^j\alpha_2^j S_2(\mathbf{p}_2+\mathbf{k}')\alpha_2^i]$$
$$\times[S_{1+}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\epsilon-\omega-\omega')+S_{2+}(\mathbf{p}_2+\mathbf{k}+\mathbf{k}',\epsilon-\omega-\omega')]|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle. \quad (91)$$

The no-pair term is of order $\alpha^8 mc^2$ and is therefore neglected. The two-pair correction is found to be

$$\Delta E^{--}_{TR\times T}=2\left(\frac{\alpha}{2\pi^2}\right)^2\frac{\Delta_2}{8m^3}\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|-\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j\frac{E-k'-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')}{E-k-k'-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')}\alpha_1^j\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i$$
$$+\alpha_1^j\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\frac{E-k'-\varepsilon(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-\varepsilon(\mathbf{p}_2)}{E-k-k'-\varepsilon(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-\varepsilon(\mathbf{p}_2)}\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle. \quad (92)$$

To order $\alpha^7 mc^2$, the above two terms cancel each other. The one-pair correction is given by

$\Delta E_{TR\times T}$

$$=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{2m}\int\frac{d\omega}{-2\pi i}\frac{d\omega'}{-2\pi i}\frac{d\mathbf{k}}{\omega^2-k^2+i\delta}\frac{d\mathbf{k}'}{\omega'^2-k'^2+i\delta}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j$$
$$\times\frac{1}{E+\omega+\omega'-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}\alpha_1^j\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k}')$$
$$\times\frac{1}{E+\omega-\varepsilon(\mathbf{p}_1-\mathbf{k}')-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}$$
$$\times\gamma_1^0\Lambda_{1i}(k)+\alpha_1^j\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k}')\frac{1}{E-\omega'-\varepsilon(\mathbf{p}_1-\mathbf{k}')-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2)+i\delta}$$
$$\times\gamma_1^0\Lambda_{1i}(k)\frac{1}{E-\omega-\omega'-\varepsilon(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2)+i\delta}$$
$$\times\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j+\alpha_1^j\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k}')\frac{1}{E-\omega'-\varepsilon(\mathbf{p}_1-\mathbf{k}')-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2)+i\delta}$$
$$\times\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j\frac{1}{E+\omega-\varepsilon(\mathbf{p}_1-\mathbf{k}')-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}$$
$$\times\gamma_1^0\Lambda_{1i}(k)+\alpha_2^j\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^i$$
$$\times\frac{1}{E+\omega+\omega'-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}\gamma_1^0\Lambda_{1i}(k)$$
$$\times\frac{1}{E+\omega'-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k}+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k})\alpha_1^j$$
$$+\gamma_1^0\Lambda_{1i}(k)\frac{1}{E-\omega-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)+i\delta}\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k})\alpha_1^j$$
$$\times\frac{1}{E-\omega-\omega'-\varepsilon(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2)+i\delta}\alpha_2^j\Lambda_{2-}(\mathbf{p}_2+\mathbf{k}')\alpha_2^i$$
$$+\gamma_1^0\Lambda_{1i}(k)\frac{1}{E-\omega-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2)-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)+i\delta}\alpha_2^j\Lambda_{2-}(\mathbf{p}_2+\mathbf{k}')\alpha_2^i$$
$$\times\frac{1}{E+\omega'-\varepsilon(\mathbf{p}_1-\mathbf{k})-\varepsilon(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k}+\mathbf{k}')I_c\mathscr{S}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+i\delta}$$
$$\times\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k})\alpha_1^j|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle, \quad (93)$$

where the infinite Coulomb binding is included. Integrating over the energy variables ω and ω' and keeping only the magnetic moment terms of lowest order, we obtain

$$\Delta E_{TR\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{\Delta_2}{8m^2}\int\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^j\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\boldsymbol{\gamma}_1\cdot\mathbf{k}\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j+\alpha_1^i\boldsymbol{\gamma}_1\cdot\mathbf{k}\mathscr{S}_{1+}(\mathbf{p}_1-\mathbf{k})$$
$$\times\alpha_1^j\alpha_2^j\Lambda_{2-}(\mathbf{p}_2+\mathbf{k}')\alpha_2^i|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle. \tag{94}$$

Further reduction leads to

$$\Delta E_{TR\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{-\Delta_2}{4m^3}\int\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k})-\mathbf{p}_1\cdot\hat{\mathbf{k}}'i\boldsymbol{\sigma}_1\cdot(\hat{\mathbf{k}}'\times\mathbf{k})$$
$$+\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}-\mathbf{p}_1\cdot\hat{\mathbf{k}}'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}'\boldsymbol{\sigma}_2\cdot\mathbf{k}|\phi_0(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle. \tag{95}$$

Fourier transforming it and including the correction from the vertex on the second electron line, we obtain

$$\Delta E_{TR\times T}=-2\frac{\alpha^7mc^2}{32\pi}\langle\phi_0|2\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^4}\times\mathbf{p}_1\right)$$
$$-\frac{5}{r^4}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}|\phi_0\rangle. \tag{96}$$

No more vertex correction is found phenomenologically. To conclude our analysis on vertex corrections, we note that we would not be surprised to see more terms coming out from a rigorous calculation. Differing from self-energy corrections, there are recoil linear terms in the vertex corrections. We have shown previously that these linear terms cancel out to order α^6mc^2 [8]. It is likely that they may show up to order α^7mc^2 and might be different from phenomenological terms. Therefore, a rigorous evaluation of the vertex corrections of second order is the most needed one in a more accurate analysis of QED effects on the $O(\alpha^7mc^2)$ fine structure of helium.

IV. VACUUM POLARIZATION

Vacuum polarization corrections of lowest order arise from the nonrelativistic momentum region in terms of momentum of the exchanged photon. The corresponding nonrelativistic operator is a simple delta function and is spin independent. Relativistic wave-function corrections lead to spin-dependent terms of order α^7mc^2, which contribute to the triplet splittings of helium. Vacuum polarization corrections of fourth order are of nominal order α^6mc^2 with the delta function operator. Vacuum polarization corrections of sixth order are of nominal order α^7mc^2 and are expressed in the form of the expectation value of the spin-independent delta function operator, and therefore give no contribution to the triplet splittings. Terms arising from vacuum polarization corrections to two-photon exchange are also of nominal order α^6mc^2. However, they come from the relativistic momentum region. The corresponding operator is also the delta function. Spin-dependent terms are of order α^8mc^2. The vacuum polarization corrections contributing to the $O(\alpha^7mc^2)$ fine-structure splittings of helium come from the following operators:

$$g_{VC}\psi(\mathbf{p}_1,\mathbf{p}_2,\epsilon)=\frac{\alpha}{2\pi^2}\int\frac{d\omega}{-2\pi i}\frac{d\mathbf{k}}{k^2}[-\Pi_c(k^2)]\psi(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k},\epsilon-\omega),$$

$$g_{VV}\psi(\mathbf{p}_1,\mathbf{p}_2,\epsilon)=\frac{-Z\alpha}{2\pi^2}\int\frac{d\mathbf{k}_1}{k_1^2}[-\Pi_c(\mathbf{k}^2)]S_2^{-1}(\mathbf{p}_2,\epsilon)\times\psi(\mathbf{p}_1-\mathbf{k}_1,\mathbf{p}_2,\epsilon),$$

$$g_{VT}\psi(\mathbf{p}_1,\mathbf{p}_2,\epsilon)=\frac{\alpha}{2\pi^2}\int\frac{d\omega}{-2\pi i}\frac{d\mathbf{k}}{\omega^2-k^2+i\delta}[-\Pi_c(k^2)]\times\alpha_1^i\alpha_2^i\psi(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k},\epsilon-\omega), \tag{97}$$

where

$$\Pi_c(k^2)=-\frac{2\alpha}{\pi}\int dzz(1-z)\ln\left[1-\frac{z(1-z)k^2}{m^2}\right] \tag{98}$$

is the vacuum polarization function. For the fine-structure splittings of order $O(\alpha^7mc^2)$, the contribution comes from the nonrelativistic k momentum region and therefore the vacuum polarization function may be expanded nonrelativistically and is given by

$$\Pi_c(k^2)=\frac{\alpha}{\pi}\frac{k^2}{m^2}\left[\frac{1}{15}+\frac{141}{162}\frac{\alpha}{\pi}\right]+\cdots. \tag{99}$$

This is very different from the QED effect due to the vacuum polarization correction to two-photon exchange where the contribution comes solely from the relativistic k region [18]. Due to the interelectron Coulomb potential, the energy correction is

$$\Delta E_{VC}=\frac{\alpha}{2\pi^2}\int\frac{d\epsilon}{-2\pi i}\frac{d\omega}{-2\pi i}\frac{d\mathbf{k}}{k^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|[S_{1+}(\mathbf{p}_1\epsilon)+S_{2+}(\mathbf{p}_2\epsilon)][-\Pi_c(k^2)][S_{1+}(\mathbf{p}_1-\mathbf{k}\epsilon-\omega)+S_{2+}(\mathbf{p}_2+\mathbf{k}\epsilon-\omega)]|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle. \tag{100}$$

After reduction, we find

$$\Delta E_{VC}=\frac{\alpha^2}{30\pi^3 m^2}\int \frac{d\mathbf{k}}{k^2}\langle \phi_c(\mathbf{p}_1,\mathbf{p}_2)|k^2|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle. \tag{101}$$

To lowest order, we get

$$\Delta E_{VC}(\alpha^5)=\frac{4\alpha^5 mc^2}{15}\langle \phi_0|\delta(\mathbf{r})|\phi_0\rangle. \tag{102}$$

For the fine-structure splittings of order $\alpha^7 mc^2$, we obtain

$$\Delta E_{VC}(\alpha^7)=-\frac{2\alpha^7 mc^2}{5}\langle \phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^2}\times\mathbf{p}_1\right)|\phi_0\rangle. \tag{103}$$

The correction due to the external potential is given by

$$\begin{aligned}\Delta E_{VV}=\frac{-Z\alpha}{2\pi^2}\int \frac{d\epsilon}{-2\pi i}\frac{d\mathbf{k}_1}{k_1^2}\langle \phi_c(\mathbf{p}_1,\mathbf{p}_2)|[S_{1+}(\mathbf{p}_1\epsilon)\\ +S_{2+}(\mathbf{p}_2\epsilon)][-\Pi_c(\mathbf{k}_1^2)]S_2^{-1}(\mathbf{p}_2\epsilon)[S_{1+}(\mathbf{p}_1-\mathbf{k}_1\epsilon)\\ +S_{2+}(\mathbf{p}_2\epsilon)]|\phi_c(\mathbf{p}_1-\mathbf{k}_1,\mathbf{p}_2)\rangle,\end{aligned} \tag{104}$$

which may be reduced to

$$\Delta E_{VV}=\frac{-Z\alpha^2}{30\pi^3 m^2}\int d\mathbf{k}_1\langle \phi_c(\mathbf{p}_1,\mathbf{p}_2)||\phi_c(\mathbf{p}_1-\mathbf{k}_1,\mathbf{p}_2)\rangle. \tag{105}$$

To lowest order

$$\Delta E_{VV}(\alpha^5)=2\frac{-4Z\alpha^5 mc^2}{15}\langle \phi_0|\delta(\mathbf{r}_1)|\phi_0\rangle, \tag{106}$$

where the factor 2 corresponds to the vacuum polarization on V_2. For the fine-structure splittings of order $\alpha^7 mc^2$, we obtain

$$\Delta E_{VV}(\alpha^7)=2\frac{Z\alpha^7 mc^2}{5}\langle \phi_0|\delta(\mathbf{r}_1)\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_1}{r_1^2}\times\mathbf{p}_1\right)|\phi_0\rangle. \tag{107}$$

One interesting result could be the corresponding correction to the Lamb shift of hydrogen. Including spin-independent terms, the correction becomes

$$\Delta E_{VV}(\alpha^7)=\frac{Z^6\alpha^7 mc^2}{5}\langle \phi_0|\delta(\mathbf{r}_1)\frac{1}{r_1^2}[\boldsymbol{\sigma}_1\cdot(\mathbf{r}_1\times\mathbf{p}_1)-1]|\phi_0\rangle \tag{108}$$

or

$$\begin{aligned}\Delta E_{VV}(\alpha^7)=\frac{Z^6\alpha^7 mc^2}{45\pi n^3}\left(1-\frac{1}{n^2}\right)[j(j+1)-s(s+1)\\ -l(l+1)-1].\end{aligned} \tag{109}$$

For the $2P_{1/2}$ state, the correction is -0.27 kHz. Although the number is too small, it is a part of the whole picture of the $O(\alpha^7 mc^2)$ Lamb shift of hydrogen.

The correction associated with a transverse photon is given by

$$\begin{aligned}\Delta E_{VT}=\frac{\alpha}{2\pi^2}\int \frac{d\epsilon}{-2\pi i}\frac{d\omega}{-2\pi i}\frac{d\mathbf{k}}{\omega^2-k^2+i\delta}\langle \phi_c(\mathbf{p}_1,\mathbf{p}_2)\\ \times|[S_{1+}(\mathbf{p}_1\boldsymbol{\epsilon})+S_{2+}(\mathbf{p}_2\boldsymbol{\epsilon})][-\Pi_c(k^2)]\alpha_1^i\alpha_2^i\\ \times[S_{1+}(\mathbf{p}_1-\mathbf{k}\boldsymbol{\epsilon}-\omega)+S_{2+}(\mathbf{p}_2+\mathbf{k}\boldsymbol{\epsilon}-\omega)]\\ \times|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\rangle,\end{aligned} \tag{110}$$

which is of nominal order $\alpha^7 mc^2$. Reduction yields

$$\begin{aligned}\Delta E_{VT}(\alpha^7)=-\alpha^7 mc^2\langle \phi_0|\frac{4}{5}\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^2}\times\mathbf{p}_1\right)\\ +\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}|\phi_0\rangle.\end{aligned} \tag{111}$$

Unlike self-energy and vertex corrections, vacuum polarization corrections seem to be better understood. It is likely that the vacuum polarization corrections analyzed in this section may be the full correction to the fine-structure splittings of order $\alpha^7 mc^2$.

V. RESULTS AND CONCLUSIONS

In previous sections, we have derived $O(\alpha^7 mc^2)$ QED corrections to the fine-structure splittings of helium in the nonrelativistic approximation. We expect no contribution from the relativistic momentum region. From the number of potential points of view, the relativistic correction can arise from only two potentials that correspond to a nonrelativistic operator scaled by $1/r^4$. On the other hand, the spin-dependent relativistic contribution must come from a nonrelativistic operator scaled by $1/r^5$, which contradicts the above result. The absence of ultraviolet logarithmic terms also shows no relativistic contribution.

In summary, we have obtained nonrelativistic operators arising from QED corrections to the $O(\alpha^7 mc^2)$ fine-structure splittings of helium. The contributions from self-energy corrections are given by Eqs. (17), (28), (46), (47), and (57). The vertex corrections are given by Eqs. (71), (73), (77), (78), (79), (80), (81), (84), (90), and (96). The total contribution due to vacuum polarization correction is

$$\begin{aligned}\Delta E_{\text{vacuum}}=2\frac{Z\alpha^7 mc^2}{5}\langle \phi_0|\delta(\mathbf{r}_1)\left[\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_1}{r_1^2}\times\mathbf{p}_1\right)-\frac{1}{r_1^2}\right]|\phi_0\rangle\\ -\alpha^7 mc^2\langle \phi_0|\frac{6}{5}\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^2}\times\mathbf{p}_1\right)-\delta(\mathbf{r})\frac{2}{5r^2}\\ +\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}|\phi_0\rangle,\end{aligned} \tag{112}$$

where we have included spin-independent terms. The above correction is the total energy-level shift of order $\alpha^7 mc^2$ due to vacuum polarization. Especially interesting here is that the correction together with the well-known vacuum polarization terms of lowest order provides a bridge between the nonrel-

ativistic variational calculation and the relativistic many-body perturbation calculation. One may calculate triplet energy levels of heliumlike ions using the two totally different methods. The two methods may be checked against each other for moderate Z ions. For low Z atoms or ions, the variational method may be favored while the relativistic method is expected to be superior in high Z systems. However, such a check cannot be made in this paper until numerical calculation using the variational method is finished by Yan and Drake for heliumlike ions.

Regrouping all terms and combining like terms, we express them explicitly in terms of the numbers of photons involved. From the two-photon processes, we have

$$\Delta E_1=-\frac{Z\alpha^7mc^2}{4\pi}\langle p_1^2\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_1}{r_1^3}\times\mathbf{p}_1\right)|\phi_0\rangle, \quad (113)$$

$$\Delta E_2=-2Z\alpha^7mc^2\left[\ln(Z\alpha)^{-2}-\beta'(nSL,Z)+\frac{31}{120}\right]\langle\phi_0|\delta(\mathbf{r}_1)\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_1}{r_1^2}\times\mathbf{p}_1\right)|\phi_0\rangle,$$

$$\Delta E_3=\frac{\alpha^7mc^2}{2\pi}\langle p_1^2\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_1\right)|\phi_0\rangle,$$

$$\Delta E_4=-\frac{\alpha^7mc^2}{8\pi}\langle p_1^2\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_2\right)|\phi_0\rangle,$$

$$\Delta E_5=\alpha^7mc^2\left(\frac{29}{5}-12\ln 2\right)\langle\phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^2}\times\mathbf{p}_1\right)|\phi_0\rangle,$$

$$\Delta E_6=-\frac{\alpha^7mc^2}{8\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_1\right)\mathbf{p}_1\cdot\mathbf{p}_2|\phi_0\rangle,$$

$$\Delta E_7=\frac{\alpha^7mc^2}{4\pi}\langle\phi_0|\frac{\mathbf{r}}{r^3}\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{p}_2)|\phi_0\rangle,$$

$$\Delta E_8=\frac{3\alpha^7mc^2}{4\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\left[\frac{\mathbf{r}}{r^5}\times[\mathbf{r}\cdot(\mathbf{r}\cdot\mathbf{p}_1)\mathbf{p}_2]\mathbf{p}_1\right]|\phi_0\rangle,$$

$$\Delta E_9=-\frac{\alpha^7mc^2}{8\pi}\langle\phi_0|\frac{1}{r^3}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{p}_2|\phi_0\rangle,$$

$$\Delta E_{10}=-\frac{\alpha^7mc^2}{16\pi}\langle\phi_0|\frac{1}{r^3}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{p}_1|\phi_0\rangle,$$

$$\Delta E_{11}=-\frac{3\alpha^7mc^2}{8\pi}\langle\phi_0|\frac{1}{r^5}\boldsymbol{\sigma}_1\cdot\{\mathbf{r}\times[\boldsymbol{\sigma}_2\cdot(\mathbf{r}\times\mathbf{p}_2)]\mathbf{p}_1\}|\phi_0\rangle,$$

$$\Delta E_{12}=\frac{3\alpha^7mc^2}{16\pi}\langle\phi_0|\frac{1}{r^5}\boldsymbol{\sigma}_1\cdot\{\mathbf{r}\times[\boldsymbol{\sigma}_2\cdot(\mathbf{r}\times\mathbf{p}_1)]\mathbf{p}_1\}|\phi_0\rangle,$$

$$\Delta E_{13}=\frac{15\alpha^7mc^2}{16\pi}\langle p_1^2\phi_0|\frac{1}{r^3}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}|\phi_0\rangle,$$

$$\Delta E_{14}=\frac{3\alpha^7mc^2}{8\pi}\langle\phi_0|\frac{1}{r^3}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{p}_1|\phi_0\rangle,$$

$$\Delta E_{15}=\frac{\alpha^7mc^2}{4\pi}\langle p_1^2\phi_0|\frac{i}{r^3}\boldsymbol{\sigma}_1\cdot\mathbf{r}\boldsymbol{\sigma}_2\cdot\mathbf{p}_2|\phi_0\rangle,$$

$$\Delta E_{16}=-\frac{3\alpha^7mc^2}{8\pi}\langle p_1^2\phi_0|\frac{i}{r^3}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\mathbf{r}\cdot\mathbf{p}_2|\phi_0\rangle,$$

$$\Delta E_{17}=9\alpha^7mc^2\langle\phi_0|\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}|\phi_0\rangle.$$

One approximation made here is to replace $\ln(H_0-W_0)$ in two-electron terms by some average value, which leads to a cancellation among the two-electron terms. A precise calculation of the exact Bethe logarithmic type terms needs a more rigorous treatment of vertex corrections of second order. As one observes, a number of operators here are the same as some of Douglas and Kroll, whose expectation values have been calculated numerically by Yan with high precision [19]. ΔE_1, ΔE_5, ΔE_9, ΔE_{11}, and ΔE_{13} have the same operators as E_1, E_9, E_{13}, E_{14}, and E_{10}, respectively. ΔE_3 and ΔE_4 correspond to E_8. ΔE_{15} and ΔE_{16} match E_{12}. ΔE_1 contributes 3.2 and 6.4 kHz to ν_{01} and ν_{12}, respectively. ΔE_{13} contributes -10.7 and 4 kHz. This may be the largest one of the electron-electron type in the above two-photon corrections. $\Delta E_{15}+\Delta E_{16}$ gives 4.6 and -1.9 kHz. ΔE_2 gives possibly the largest contribution of the order of interest. However, the final number depends on the numerical calculation of the Bethe logarithm. Since the above terms come from two-photon diagrams, they are of next-to-leading order. The above two-body terms have never been tested in a two-body system. It would be interesting to see their high-precision test in helium. It could provide one of the most accurate tests in bound-state QED physics. From three-photon diagrams, we obtain

$$\Delta E_{18}=-\frac{Z\alpha^7mc^2}{4\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_2}{rr_2^3}\times\mathbf{p}_1\right)|\phi_0\rangle,$$

$$\Delta E_{19}=\frac{Z\alpha^7mc^2}{4\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_2}{r_2^3}\times\frac{\mathbf{r}}{r^3}\right)\mathbf{r}\cdot\mathbf{p}_1|\phi_0\rangle,$$

$$\Delta E_{20}=\frac{Z\alpha^7mc^2}{4\pi}\langle\phi_0|i\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_2}{r_2^3}\times\frac{\mathbf{r}}{r^3}\right)|\phi_0\rangle,$$

$$\Delta E_{21}=\frac{Z\alpha^7mc^2}{4\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\frac{\mathbf{r}}{r^3}\boldsymbol{\sigma}_2\cdot\frac{\mathbf{r}_2}{r_2^3}|\phi_0\rangle,$$

$$\Delta E_{22}=\frac{\alpha^7mc^2}{16\pi}\langle\phi_0|\boldsymbol{\sigma}_1\cdot\frac{\mathbf{r}}{r^3}\boldsymbol{\sigma}_2\cdot\frac{\mathbf{r}}{r^3}|\phi_0\rangle. \quad (114)$$

Operators ΔE_{19}, ΔE_{21}, and ΔE_{22} correspond to E_2, E_3, and E_5, respectively, of Douglas and Kroll. The correction due to four-photon diagrams is given by

$$\Delta E_{23}=Z\alpha^4 mc^2\Delta_6\langle\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}_1}{r_1^3}\times\mathbf{p}_1\right)|\phi_0\rangle,$$

$$\Delta E_{24}=-2\alpha^4 mc^2\Delta_6\langle\phi_0|\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}_1\right)|\phi_0\rangle,$$

$$\Delta E_{25}=-\alpha^4 mc^2(\Delta_2\Delta_4+\Delta_6)\langle\phi_0|-\frac{1}{2r^3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - 3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})+\frac{4\pi}{3}\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle. \quad (115)$$

All operators arising from four-photon diagrams are Breit operators. The total correction of ΔE_1, ΔE_2 (nonlogarithmic part), ΔE_3, ΔE_4, ΔE_5, ΔE_6, ΔE_9, ΔE_{11}, ΔE_{13}, ΔE_{15}, ΔE_{17}, ΔE_{19}, ΔE_{21}, ΔE_{22}, ΔE_{23}, ΔE_{24}, and ΔE_{25} is $\Delta\nu_{01}=-7.2$ kHz and $\Delta\nu_{12}=4.0$ kHz. Numerical calculation of the other terms is in progress by Yan and Drake. The estimate of those uncalculated numerically (excluding logarithmic terms) is expected to be less than 5 kHz. The Bethe logarithm presented here is spin dependent and different from the normal one, which already has been difficult to calculate accurately for many years. Therefore, the new Bethe logarithm could be the biggest challenge for numerical calculation.

Since the Douglas and Kroll's work in 1974, there has been very active study of QED effects in many-body atomic systems in the framework of relativistic many-body perturbation theory (RMBPT) [20]. It would be interesting to compare our perturbative results with those obtained in RMBPT. First, we need to compare the advantages in our perturbative calculation and the nonperturbative calculation in RMBPT. Our perturbative scheme is developed mainly for low Z atoms or helium in particular. The dominant physics here is nonrelativistic effects. All relativistic and QED corrections are calculated perturbatively to a given order. In this way a very high precision can be obtained to the order of interest due to an extremely high precision variational calculation of nonrelativistic effects. Since all corrections are obtained in a power series of α, theoretical uncertainty can be estimated to a given order. On the other hand, RMBPT takes advantage of the fact that relativistic effects dominate in the binding energy of high Z ions. A relativistic solution of the Dirac type is a good starting point. Therefore, the corresponding calculation is nonperturbative for a given dynamic kernel arising from a Feynman diagram. Although self-energy and vacuum polarization have been calculated in RMBPT, it is not clear how much contribute to order $\alpha^5 mc^2$, $\alpha^6 mc^2$, and $\alpha^7 mc^2$ in the case of low Z atoms such as helium. Note that the terms of order $\alpha^7 mc^2$ are only 1/10 000 of the leading QED terms. Since the QED corrections obtained in RMBPT are not analytical, they have to be expressed in terms of powers of α in order to make a comparison. For example, Mohr calculated numerically the self-energy correction of second order to the hydrogenic Lamb shift [21] expressed in terms of power series of α. Alternatively, one may subtract the lower-order perturbative corrections from the nonperturbative result obtained in RMBPT. To compare with the corrections presented here, Araki and Sucher's terms of order $\alpha^5 mc^2$ and Douglas and Kroll's terms of order $\alpha^6 mc^2$ must be subtracted. The calculations of the triplet $2P$ splittings in heliumlike ions, carried out in a nonperturbative scheme such as RMBPT, are reported in Ref. [22]. However, they acknowledged that the QED corrections were taken from a perturbative calculation [23] that takes only Araki and Sucher's terms of order $\alpha^5 mc^2$ into account. It is important to note the splittings, not the spin average energy level of the triplet P state. Only the splittings can provide a QED test of order $\alpha^7 mc^2$ because the triplet energy levels of lower-order $\alpha^6 mc^2$ have not been calculated yet. Even if the QED calculation in RMBPT could be done for the triplet splittings, there would still be a challenge to compare with our high-order QED corrections, which may only consist of 1/10 000 of the total nonperturbative corrections. As pointed out in Ref. [20] the accuracy of their QED results for $Z<18$ is affected by severe numerical cancellations in their method for calculating electron self-energies. A more realistic comparison could be made in moderate Z ions. The corrections presented here provide an excellent test for high-precision variational calculation, which is superior to the many-body perturbation calculation for low Z atoms because the variational approach has the advantage that the nonrelativistic wave function of helium can be obtained with high precision.

In conclusion, we have presented an analysis on $O(\alpha^7 mc^2)$ QED corrections to the fine-structure splittings of helium. The result is expressed in terms of expectation values of nonrelativistic operators similar in form to those of Douglas and Kroll, and is a correction not only for helium but also for positronium. The final numbers for positronium can be obtained easily by sandwiching the nonrelativistic operators presented here between nonrelativistic wave functions of positronium. However, only the vacuum polarization correction to the hydrogen P levels is close in magnitude to the current experimental errors. Corrections in positronium are of order $\alpha^7 mc^2$ and therefore are far too small to be meaningful. This shows that helium is a better candidate than two-body systems (equivalent corrections in hydrogen and muonium are of order $\alpha^7 m/M$, too small) in a high precision test of the two-body terms analyzed here. This is conceivable since the 2^3P helium levels have a lifetime about 100 times of that of the $2P$ hydrogen levels due to the fact that the triplet P state decays to the triplet S state instead of the ground state. Although many of the corrections are obtained phenomenologically, we expect them to account for a major part of QED effects to the order of interest. Partial justification of phenomenological treatment has been made by rigorous calculation of self-energy corrections of second order. We found that phenomenological treatment leads to the same result as the rigorous one for the self-energy corrections of second order. Full justification, one way or another, requires a rigorous calculation of one-loop vertex corrections. The QED corrections of second order perturbed with the Breit corrections contribute to the fine-structure splittings to the order interest. They were presented in our previous paper [11] in the form of expectation values of nonrelativistic operators of second order. One more piece for a complete picture of the $O(\alpha^7 mc^2)$ fine-structure splittings of helium is corrections from the relativistic momentum region arising from exchange diagrams, which will be presented elsewhere. These corrections are of special interest in view of the fact that they may be the first corrections of off-leading order

arising from the relativistic momentum region ever to be tested in any one-, two-, or three-body system. Note that similar corrections due to self-energy modification in the one-body system are of order $\alpha^8 mc^2$, which are the corrections of order α^2 relative to those of Karplus, Klein, and Schwinger [9] and Baranger, Bethe, and Feynman [10]. The off-leading-order terms in two-body systems are of order $\alpha^7 m/M$. Precise comparison with experiment cannot be made until all theoretical numbers of $O(\alpha^7 mc^2)$ QED effects in helium fine structure are obtained numerically, which is in progress by Yan and Drake.

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