Kinetics of photon correlation functions under the time-dependent quadratic Hamiltonian

Shan-Wen Tsai and A. F. R. de Toledo Piza
Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05389-970 São Paulo, São Paulo, Brasil
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The dynamic decoupling of the correlation function hierarchy under the general quadratic Hamiltonian is exploited to extend the results obtained for coherent pumping and squeezing in Ref. [1]. Closed c-number dynamic equations are given explicitly for three- and four-operator averages, for an arbitrary (pure or mixed) initial state of the field. At this level an independent c-number constant of motion appears, related to two-photon correlation properties of the initial state. Extension to higher correlation functions is straightforward.

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In a recent paper [1] we considered the dynamics associated with the general, time-dependent, quadratic Hamiltonian ($[a,a^\dagger]=1$, Schrödinger picture)

$$H(t)=f_1(t)a^\dagger a+f_2(t)a^\dagger a^\dagger+f_2^*(t)aa+f_3(t)a^\dagger+f_3^*(t)a \tag{1}$$

and used the fact that the class of observables referred to as Gaussian observables (dynamical variables involving at most two operators a, $a^\dagger$) is dynamically closed under this Hamiltonian in order to derive exact, closed c-number Hamiltonian equations of motion for parameters which describe coherent pumping and squeezing of arbitrary (pure or mixed) initial states of the system. The coherent content of the state is determined by the expectation value of the annihilation operator a and driven by the linear f_3 terms of the Hamiltonian, while squeezing effects, driven by the ''two-photon'' f_2 terms, can be absorbed in a time-dependent change of scale implemented by means of a Bogolyubov transformation which mixes a and $a^\dagger$. Note that $f_1(t)$ is real so that $H(t)$ is Hermitian. The procedure adopted in Ref. [1] thus makes use of the special (in fact, Gaussian) character of $H(t)$ to avoid having to obtain a complete solution of the full quantum problem in order to determine the exact dynamics of the restricted class of Gaussian observables. The crucial point is that this dynamics is independent of correlation amplitudes of higher order. This ''self-truncation'' of a Bogolynbov-Born-Green-Kirkwood-Yvon hierarchy can be imediately understood when one takes notice of the noninteracting character of $H(t)$, from a many-body (or field theoretical) point of view. An additional byproduct of this fact is moreover the existence of a c-number constant of motion in the time evolution of the Gaussian observables. It is associated with the eigenvalues of the extended one-body density matrix, reflecting the possible incoherence and/or many-boson correlation properties of the particular quantum state under consideration, at the level of one-boson observables.

The literature available on the dynamics of the quantum linear oscillator with time-dependent parameters is quite rich and extensive [2–8]. This is a soluble quantum problem for which solutions obtained within the context of several distinct formulations can now be found [3,5]. Interest in this problem hinges to a large extent on its relation to a variety of characteristically quantum phenomena of current theoretical and experimental relevance, such as squeezing phenomena [2,8] and quantum-mechanical phase problems [4,5].

Even though a rather complete theoretical picture emerges from such studies, the practical tools for dealing with the kinetics of arbitrary initial states still appear as rather cumbersome in so far as they rely ultimately on the exact solution of the full quantum problem. The main purpose of this paper is therefore to show that results similar to those of Ref. [1] can be obtained *also* for higher correlation functions (i.e., depending on expectation values of three or more operators a, $a^\dagger$) associated with arbitrary initial states. The complete solution of the full quantum problem is again avoided. As a result of the noninteracting character of $H(t)$, one obtains closed sets of c-number differential equations for parameters which completely describe functions pertaining to succesive correlation levels. From the lower two levels the results of [1] are recovered. Additional c-number constants of motion appear within the higher correlation levels. They correspond to invariant, irreducible, many-body correlation structures in the time evolution of an arbitrary, given initial state.

A particular development, which in fact goes back to the work of Lewis and Riesenfeld [6] but has been actively pursued in recent years [7,8] and is of relevance in the present context, concerns the existence and the role played by the so-called *invariant operators*, and their relation to coherent states and squeezing [8]. It is worth stressing that the hierarchical constants of motion appearing in the present approach are c-number objects whose value is in each case determined by specific many-body correlation properties *of the adopted initial state*. While their existence may be ultimately traced to invariant operators of $H(t)$, their irreducibility is (and remains) a property of the quantum state under consideration. Thus, while, e.g., one can express general invariant operators in terms of a small number of *basic* invariants (two, in the present case of a one-dimensional system) [8], no corresponding limitation applies to the number of independent, nontrivial c-number constants of motion present in the time evolution of a fully general initial state.

The simplest way to derive the equations of motion is to switch to the Heisemberg picture through the unitary evolution operator (we use units in which $\hbar=1$)

$$U(t,0)=T\ \exp-i\int_0^t dt'H(t'), \tag{2}$$

where T exp denotes a time-ordered exponential. The state representative is now time independent and will be taken with full generality as a density operator F describing the initial state of the system. We are interested in mean values of normal ordered products of m creation and n annihilation operators (now in the Heisenberg picture),

$$A_{mn}(t)\equiv \mathrm{Tr}[(a_H^\dagger(t))^m(a_H(t))^nF]=A_{nm}^*(t). \quad (3)$$

Note that while the $A_{mn}(t)$ are in general complex numbers, the ''diagonal'' mean values A_{nn} are real. The time dependence of these objects is readily obtained from the Heisemberg equations of motion. It is determined by the first-order equations,

$$\frac{dA_{mn}(t)}{dt}=i\ \mathrm{Tr}\{[H_H(t),(a_H^\dagger(t))^m(a_H(t))^n]F\}. \quad (4)$$

In the above expression, $H_H(t)$ is the Hamiltonian (1) written in terms of the Heisemberg picture operators $a_H(t),a_H^\dagger(t)$. Due to the special form of this Hamiltonian, the commutator which appears in (4) can be expressed in normal ordered form as a sum of terms each of which involves at most $m+n$ creation-annihilation operators, so that the time derivative of $A_{mn}(t)$ appears as a function of $A_{pq}(t)$ with $p+q\leqslant m+n$. Taking into account the real character of A_{nn}, this gives a total number of $(m+n)^2+1$ coupled first-order equations in order to fully determine the kinetics of $A_{mn}(t)$.

This general situation will now be first confronted with that of Ref. [1], where one has two independent pairs of real first-order equations, in addition to a constant of motion, in order to deal with the case $p+q\leqslant 2$. Equation (4) gives, in this case,

$$\frac{dA_{01}}{dt}=-i[f_1A_{01}+2f_2A_{10}+f_3],$$

$$\frac{dA_{02}}{dt}=-2i[f_1A_{02}+f_2(2A_{11}+1)+f_3A_{01}],$$

$$\frac{dA_{11}}{dt}=i[2f_2^*A_{02}-2f_2A_{20}+f_3^*A_{01}-f_3A_{10}]. \quad (5)$$

The first of these equations describes in closed form the time development of the usual field quadratures, proportional to $A_{01}\pm A_{10}$. They lead directly to Eqs. (14) of Ref. [1] when one takes the quadratures themselves as (canonical) variables [9]. Furthermore, if we define the shifted creation-annihilation operators $b,b^\dagger$ as

$$b=a_H-A_{01};\quad b^\dagger=a_H^\dagger-A_{10}, \quad (6)$$

the remaining two equations (5) lead to

$$i\frac{dB_{02}}{dt}=2[f_1(t)B_{02}+f_2(t)(2B_{11}+1)],$$

$$i\frac{dB_{11}}{dt}=2[f_2(t)B_{02}^*-f_2^*(t)B_{02}], \quad (7)$$

where B_{pq} are objects analogous to A_{pq} with the operators $a_H,a_H^\dagger$ replaced by $b,b^\dagger$, see Eq. (6). These equations can now be integrated independently of the field quadratures, and it is easy to verify that they support a constant of motion C_2 given by

$$C_2=(B_{11}+\tfrac{1}{2})^2-|B_{02}|^2\equiv(\nu+\tfrac{1}{2})^2, \quad (8)$$

where ν corresponds to the constant defined in Ref. [1]. Full contact with the results of this reference can be achieved by introducing the real variables

$$Q=\sqrt{B_{11}+\tfrac{1}{2}+\mathrm{Re}\ B_{02}},$$

$$P=\frac{\mathrm{Im}B_{02}}{\sqrt{B_{11}+\tfrac{1}{2}+\mathrm{Re}\ B_{02}}}. \quad (9)$$

As shown in [1], Q corresponds to the mean-square deviation of the quadrature $(A_{10}+A_{01})/\sqrt{2}$. Rewriting Eqs. (7) in terms of Q and P, and taking the constant of motion C_2 into account, one obtains Eqs. (17) of Ref. [1], which are the canonical equations of the c-number effective Hamiltonian

$$h_2=\tfrac{1}{2}[f_1(t)-2\ \mathrm{Re}f_2(t)]P^2+\tfrac{1}{2}[f_1(t)+2\ \mathrm{Re}f_2(t)]Q^2+[f_1(t)-2\ \mathrm{Re}f_2(t)]\frac{(2\nu+1)^2}{8Q^2}+2\ \mathrm{Im}f_2(t)PQ. \quad (10)$$

Extending the use of Eq. (4) to higher correlation functions is, of course, straightforward. The kinetics of mean values of three operators (i.e., $p+q=3$) requires eventually the consideration of a closed set of two complex (four real) equations involving B_{03} and B_{12}, which can be written in the form

$$i\frac{dB_{03}}{dt}=3[f_1(t)B_{03}+2f_2(t)B_{12}],$$

$$i\frac{dB_{12}^*}{dt}=-[f_1(t)B_{12}^*-4f_2^*(t)B_{12}+2f_2(t)B_{03}^*]. \quad (11)$$

These equations follow from Eq. (4) using the definitions (6). Their canonical character can be made explicit by intoducing the (complex) variables

$$z_2\equiv\frac{B_{03}}{\sqrt{3}};\quad w_2\equiv B_{12}^* \quad (12)$$

in terms of which Eqs. (11) can be obtained in the complex canonical form [10]

$$i\frac{dz_2}{dt}=\frac{\partial h_3}{\partial z_2^*};\quad i\frac{dw_2}{dt}=\frac{\partial h_3}{\partial w_2^*},$$

where the effective c-number Hamiltonian h_3 is

$$h_3=f_1(t)(3|z_2|^2-|w_2|^2)+3\sqrt{3}[f_2(t)w_2^*z_2^*+f_2^*(t)w_2z_2]-2[f_2^*(t)w_2^{*2}+f_2(t)w_2^2]. \quad (13)$$

The kinetics of four operator averages (i.e., $p+q=4$) requires two complex equations and one real additional equation (or five additional real equations) which read

$$i\frac{dB_{04}}{dt}=4[f_1(t)B_{04}+2f_2(t)B_{13}+3f_2(t)B_{02}],$$

$$i\frac{dB_{13}^*}{dt}=-2f_1(t)B_{13}^*+2f_2(t)B_{04}^*-6f_2(t)^*(B_{22}+B_{11}),$$

$$i\frac{dB_{22}}{dt}=4f_2(t)B_{13}^*+2f_2(t)B_{02}^*-4f_2^*(t)B_{13}-2f_2^*(t)B_{02}^*. \tag{14}$$

Since they involve B_{02} and B_{11}, these equations are not closed, but must be supplemented by Eqs. (7). The full set of equations can, however, be simplified through the introduction of variables which combine two- and four-operator averages:

$$w_4\equiv B_{13}^*+\frac{3}{2}B_{02}^*;\quad z_4\equiv\frac{B_{04}}{2};\quad u_4\equiv\left(\frac{B_{22}}{2}+B_{11}+\frac{1}{4}\right).$$

Note that z_4 and w_4 are complex, but u_4 is real. In terms of these variables, and using Eqs. (7), Eqs. (14) lead to the closed set of equations,

$$i\frac{dz_4}{dt}=4[f_1(t)z_4+f_2(t)w_4^*],$$

$$i\frac{dw_4}{dt}=-2f_1(t)w_4+4f_2(t)z_4^*-12f_2^*u_4,$$

$$i\frac{du_4}{dt}=2[f_2(t)w_4-f_2^*(t)w_4^*]. \tag{15}$$

It is now easy to identify in these equations an additional constant of motion involving up to four operator averages. It is given by the expression [cf. Eq. (8)]

$$C_4=3u_4^2-|w_4|^2+|z_4|^2. \tag{16}$$

This constant corresponds to a two-photon correlation property which is invariant under the quadratic Hamiltonian (1) for arbitrary $f_i(t)$. As was the case with Eqs. (7), it can be used to eliminate one real variable reducing Eqs. (15) to a set of four real equations. The question whether they can be obtained (after a suitable change of variables) as the canonical equations of some effective c-number Hamiltonian is, however, in this case considerably more involved and not particularly relevant for our purposes. We therefore do not consider this question here.

In order to illustrate the use of the present approach we evaluate the time evolution of some typical two-photon correlation functions for the example of the parametric oscillator treated in Ref. [1]. Expressing the correlation functions in terms of the variables appearing in the differential equations is a straightforward algebraic exercise. In the examples below we take in Eq. (1) $f_1=$ const $=1$ as the unit of energy and use $f_2(t)=0.3\exp(-0.7it)$ and $f_3(t)=0.5\exp(-0.7it)$. We choose this particular frequency as it generates periodic solutions. Note that the coherent pumping terms involving $f_3(t)$ do not appear in Eqs. (10), (11), and (15). These equations, together with the first equation in Eq. (5), are integrated numerically with standard methods and with initial conditions appropriate to the adopted initial state of the field. As an example of a mixed initial state we use a thermal distribution of photons described by the density operator

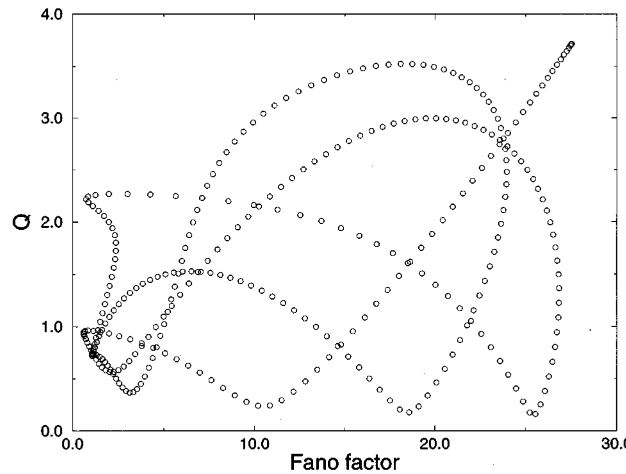

FIG. 1. Strobe view of the trajectory of the dispersion Q against the Fano factor $\mathscr{F}$ for a thermal initial state with $\beta=3$. The time interval between flashes is 0.1 ($\hbar=f_1=1$). See the text for Hamiltonian parameters and other details.

$$F_\beta=\sum_{n=0}^{\infty}|n\rangle(1-e^{-\beta})e^{-\beta n}\langle n|,$$

where the $|n\rangle$'s are the usual eigenvectors of $a^\dagger a$ (Fock states), which gives

$$\nu=\frac{e^{-\beta}}{1-e^{-\beta}},\quad Q(0)=\sqrt{\nu+\tfrac{1}{2}},$$

$$u_4(0)=\frac{2e^{-\beta}-e^{-2\beta}}{(1-e^{-\beta})^3}+\frac{1}{4}$$

and zero for the initial values of the remaining variables. Note that F_β is a Gaussian density in the sense of Ref. [1].

Figure 1 shows a parametric plot, for $\beta=3$, of the time evolution of the variable Q against the so-called Fano factor,

$$\mathscr{F}\equiv\frac{\langle(a^\dagger a)^2\rangle-\langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle},$$

which serves to compare the quadratic number dispersion of the current state with the Poissonian standard. The full periodic orbit is shown in the figure. The motion of the representative point time reverses itself at the extremes of the trajectory, near the lower left (initial state) and upper right corners of the figure. This particular feature is a consequence of the even character of the initial state under time reversal. Moreover, since, unlike $\mathscr{F}$, Q is independent of $f_3(t)$, a change in the frequency of this object will destroy the peri-

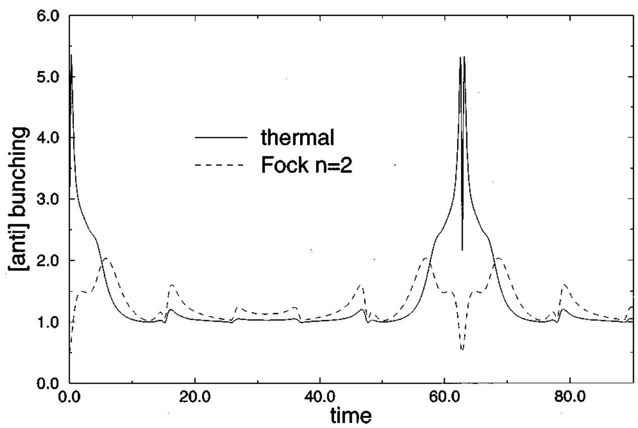

FIG. 2. Time dependence of the bunching and/or antibunching parameter g^2 for a thermal initial state with $\beta=3$ (solid line) and for a Fock initial state with $n=2$ (dashed line). The Hamiltonian parameters are the same as for Fig. 1.

odicity of $\mathscr{F}$ but not of Q. Recall that the value of Q for the Fock vacuum $|n=0\rangle$ is $1/\sqrt{2}\approx 0.7071$.

As a second example, Fig. 2 shows plots against time of the correlation function

$$g^2 \equiv \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2}$$

usually employed to characterize photon bunching and/or antibunching properties. The two curves shown correspond, respectively, to the same initial conditions used for Fig. 1 and to a pure Fock state with $n=2$. The periodicity and time-reflection properties of the solutions, for the adopted Hamiltonian parameters, are clearly displayed here.

We mention finally that the c-number equations of motion at the level of the Gaussian approximation [i.e., at the level of the treatment of Ref. [1] and of Eqs. (5) of the present paper] have recently been considered as a tool for investigating a ''semiquantal'' regime of nonlinear Hamiltonian dynamics [11]. In contrast to the situation considered here, in that wider context the c-number dynamics for the Gaussian parameters has an approximate character which corresponds to a mean-field approximation of the Hartree-Bogolyubov type. Quantum corrections to this approximation for the more general problem have been obtained in the form of collision integrals with memory effects and studied in Refs. [12] and [13]. In this last reference, Hamiltonian aspects of the collisional c-number dynamics are discussed.

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