Discrete Wigner function and quantum-state tomography

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(Received 2 January 1996)

The theory of discrete Wigner functions and of discrete quantum-state tomography [U. Leonhardt, Phys. Rev. Lett. **74**, 4101 (1995)] is studied in more detail guided by the picture of precession tomography. Odd- and even-dimensional systems (angular momenta and spins, bosons, and fermions) are considered separately. Relations between simple number theory and the quantum mechanics of finite-dimensional systems are pointed out. In particular, the multicomplementarity of the precession states distinguishes prime dimensions from composite ones. [S1050-2947(96)08605-2]

PACS number(s): 03.65.Bz

I. INTRODUCTION

Already in the dawn of quantum mechanics people anticipated that atomic or, more generally, quantum objects would be very different from the things we see as macroscopic observers [1]. The formalism was yet to be discovered but the sheer existence of Planck's quantum of action was proof enough. Observers must interact with the objects they wish to observe and this interaction changes microscopic things significantly. If the overall measurement-induced backaction cannot be reduced much below Planck's constant then microscopic objects or features on the order of $\hbar$ cannot be resolved completely. We cannot see the things *as they are*. They might rather resemble abstract ideas than things we call visible and real. Instead, we see only particular aspects of the physical objects—their quantum shadows (if this excursion into Plato's philosophy [2] is permitted). There is a nice illustration for this principle in terms of the Wigner representation [3]. We express the quantum state of a one-dimensional mechanical system according to Wigner's formula

$$W(q,p)=\pi^{-1}\int_{-\infty}^{+\infty}\exp(2ipx)\langle q-x|\hat{\rho}|q+x\rangle dx, \quad (1)$$

where $\hat{\rho}$ denotes the density matrix. For simplicity we set $\hbar=1$. Here q and p denote position and momentum, respectively, and $|q\rangle$ is an eigenstate of $\hat{q}$. The Wigner function $W(q,p)$ is a quasiprobability distribution for q and p. Any predictable quantity, i.e., any expectation value can be represented as the overlap of the Wigner function $W(q,p)$ with the Wigner representation of an operator $\hat{F}$,

$$\mathrm{Tr}\{\hat{\rho}\hat{F}\}=2\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}W(q,p)W_F(q,p)dq\,dp, \quad (2)$$

where $W_F(q,p)$ is defined as in (1) with $\hat{F}$ replacing $\hat{\rho}$. This means that predictable quantities are filtered projections, i.e., shadows, of the quantum state represented by the Wigner function $W(q,p)$. These projections are particularly simple for the so-called quadratures

$$\hat{q}_\theta=\hat{q}\cos\theta+\hat{p}\sin\theta. \quad (3)$$

The probability distribution for observing a particular value q_θ is simply the shadow of the Wigner function projected onto a line in phase space,

$$\mathrm{pr}_\theta(q_\theta)=\int_{-\infty}^{\pi\infty}W(q\cos\theta-p\sin\theta,q\sin\theta+p\cos\theta)dp. \quad (4)$$

Although we are restrained to see only particular aspects of a physical object in a single observation, nothing prevents us in principle from changing actively the point of view in a series of distinct measurements. The more different the single observations are, or, in other words, the more complementary aspects we observe, the more information we gain about all features of a quantum object. A complete set of observations is sufficient for knowing the quantum state perfectly. Again, this is nicely illustrated in terms of the Wigner representation. If we are able to change the viewpoint in observing the quadratures q_θ, i.e., if we can vary the phase θ, then we can measure the probability distribution $\mathrm{pr}_\theta(q_\theta)$ for all quadrature variables. This is sufficient to infer the quantum state [4–7] represented by the Wigner function $W(q,p)$ [4],

$$W(q,p)=-\frac{1\mathrm{P}}{2\pi^2}\int_0^\pi\int_{-\infty}^{+\infty}\frac{\mathrm{pr}_\theta(q_\theta)dq_\theta d\theta}{(q\cos\theta+p\sin\theta-q_\theta)^2}. \quad (5)$$

[Here P denotes Cauchy's principle value. The integral transformation (5) is known as the inverse Radon transformation.] Changing the point of view in observations to infer the quantum state is the very idea of quantum-state tomography [4]. On the other hand, there are schemes [6,8] for measuring so-called propensities [9], i.e., probability distributions pr (q_0,p_0) defined as

$$\mathrm{pr}(q_0,p_0)=2\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}W(q,p)W_F(q-q_0,p-p_0)dq\,dp. \quad (6)$$

Here $W_F(q,p)$ denotes the Wigner function of a filter state $\hat{F}$. These schemes [6,8] are techniques for measuring position and momentum simultaneously yet not precisely. According to Heisenberg's uncertainty principle (in the version given by Arthurs and Kelly [8]) the resolution of these observations cannot exceed $\hbar$ (which is unity in our units). This means that if we attempt to gain directly the complete picture of a quantum object we are limited in precision.

It is quite natural to extend these ideas to objects other than mechanical ones, and truly quantum systems like spins would be interesting candidates. For this we could transcribe the continuous Wigner formalism for discrete quantum mechanics [10]. A discrete version of quantum-state tomography and a discrete Wigner representation was sketched in a brief communication [11]. It is the intention of the present paper to elaborate this theory in both more detail and intuitive insight. In this way the concept of discrete propensities [12] finds a solid foundation as well. Note that discrete Wigner functions can be also applied to define Wigner functions for number and phase [13] and in the Weyl-Wigner formalism for rotation-angle and angular-momentum variables [14]. Given a quantum system with dimension d, which sets of observables are sufficient to infer the state? There are certainly plenty of possibilities [5], yet we would like to restrict ourselves to observables which are simple projections of a discrete Wigner function like the quadratures in continuous quantum-state tomography [4]. In this way we bridge the idea of quantum-state reconstruction with the very concept of a discrete Wigner representation. In addition, we can find a simple way of realizing experimentally the necessary viewpoint changes in the set of observations.

Discrete Wigner representations have been already introduced by Wootters [15] for prime-dimensional systems and by Cohendet *et al.* [16] for odd-dimensional objects. There seems to be a subtle reason why primes, odd, or even dimensions make a difference, why simple number theory gets involved in the quantum mechanics of finite-dimensional systems. In the present paper we investigate this point in more detail. To define a discrete Wigner function on a d-dimensional state space, Wootters [15] proposed to decompose the space into factor spaces of prime dimension according to the factorization of d [17]. The Wigner function of the system is given on the factor spaces [18]. In this paper we take a different approach: We define a Wigner function on the total d-dimensional state space without factorization because this seems to be more natural. Our Wigner function is identical with the previously studied functions [15,16] for prime and odd dimensions, respectively, but we add a number of additional results to the known properties of discrete Wigner functions. It is probably in the nature of things that discrete Wigner functions for even-dimensional systems are a bit odd, and apart from the two-dimensional case they have never been defined before. Note that we have slightly modified the notation of the first communication [11]. More importantly, instead of discrete quadratures we consider phase-precession measurements to make the idea of discrete quantum-state tomography more transparent and perhaps more practical. Note, however, that this does not affect the principal concept [11]. It means only that the parametrization of the measured quantities is changed.

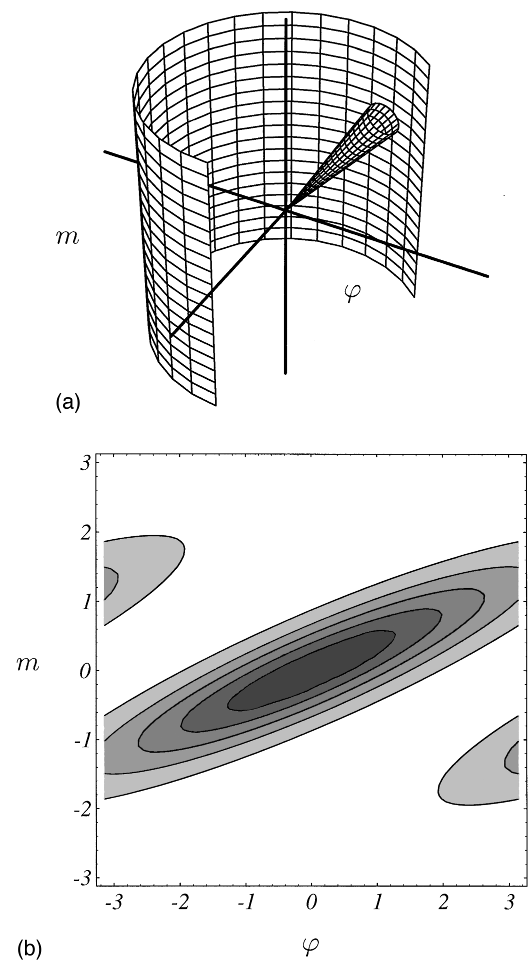

FIG. 1. Spinning top symbolizing a classical angular momentum (a). It is represented in cylinder coordinates, i.e., in terms of the phase φ and the angular-momentum component m. In (b) the cylinder has been rolled out onto the plane in order to show the phase-space density of a statistical ensemble of processing tops.

II. SETTING THE SCENE

A. Precession tomography

To get an intuitive picture for discrete Wigner functions and quantum-state tomography, imagine a spin or angular-momentum component as our system. Classically, this would correspond to a spinning top with phase φ and angular momentum m in z direction as depicted in Fig. 1. We may also have a statistical ensemble of tops where the phases and angular momenta are statistically distributed according to a phase-space density $P(\varphi,m)$. Although this picture is of course not exactly correct for quantum tops it puts in a nutshell the central idea of discrete quantum-state tomography, and that is *precession tomography*. Suppose we are restricted to observe either the phase distribution or the angular-momentum distribution of an ensemble of tops. Suppose furthermore that the tops are precessing according to the classical Hamiltonian

$$H=\frac{m^2}{2}. \tag{7}$$

The probability distribution $P(\varphi,m)$ is sheared when time progresses

$$P(\varphi,m;t)=P(\varphi-mt,m), \tag{8}$$

simply because tops with larger angular-momentum components precess faster than tops with smaller ones. The phase distribution is given by

$$\mathrm{pr}(\varphi;t)\equiv\int_{-l}^{+l}P(\varphi-mt,m)dm. \tag{9}$$

Here l denotes the total angular momentum. Consequently, the Fourier-transformed distribution

$$\widetilde{\mathrm{pr}}(\nu;t)=\int_{-\pi}^{+\pi}\mathrm{pr}(\varphi;t)\exp(-i\varphi\nu)d\varphi \tag{10}$$

is related to the Fourier-transformed phase-space density (also called characteristic function)

$$\widetilde{P}(\nu,n)\equiv\int_{-\pi}^{+\pi}\int_{-l}^{+l}P(\varphi,m)\exp[-i\varphi\nu-imn]dm\;d\varphi \tag{11}$$

according to the simple rule

$$\widetilde{\mathrm{pr}}(\nu;t)=\widetilde{P}(\nu,\nu t). \tag{12}$$

Precession shears the phase space of the top so that measuring the phase distribution at the times $n\nu^{-1}$ is sufficient to infer the characteristic function $\widetilde{P}(\nu,n)$ apart from values with $\nu=0$. Fortunately, the angular-momentum distribution (invariant during precession)

$$\mathrm{pr}(m)=\int_{-\pi}^{+\pi}P(\varphi,m)d\varphi \tag{13}$$

closes this gap. In fact

$$\widetilde{\mathrm{pr}}(n)\equiv\int_{-l}^{+l}\mathrm{pr}(m)\exp(-imn)dm=\widetilde{P}(0,n). \tag{14}$$

We see that we can reconstruct the characteristic function from observations of the angular momentum and separately of the phase distribution during precession. So without observing directly the phase-space density $P(\varphi,m)$ we can tomographically infer this distribution, i.e., the classical state of an ensemble of tops. Although this draws an entirely classical picture for spin-precession tomography, the quantum version is just a mere refinement which, however, involves some subtleties.

B. Quantization

Let us state more precisely what we mean by the phase and the angular-momentum component of a quantum top. In contrast to the phase of a harmonic oscillator [19,20], the phase of a spin or angular momentum does not involve much theoretical headaches, and it is easily defined [21,22]. Here we follow mostly Vourdas' approach [22]. Suppose we have a system described by the angular-momentum or spin operators $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$. We consider states with fixed total spin

$$(\hat{J}_x^2+\hat{J}_y^2+\hat{J}_z^2)\hat{\rho}=j(j+1)\hat{\rho}. \tag{15}$$

As well known, the quantum number j can be only integer or half odd. When j is integer we call the system an angular momentum (it is in the Bose sector [22]) and when j is half odd we call it a spin (in the Fermi sector [22]). The dimension of the system

$$d=2j+1 \tag{16}$$

is odd for bosons and even for fermions. To simplify future expressions we introduce the following notations for the eigenstates of the $\hat{J}_z$ component: In the Bose sector we have as usual

$$\hat{J}_z|m\rangle=m|m\rangle,\quad m\in\{-j,\ldots,j\}, \tag{17}$$

$$\hat{m}\equiv\hat{J}_z, \tag{18}$$

while in the Fermi sector we define

$$\hat{J}_z|m\rangle=(m-\tfrac{1}{2})|m\rangle,\quad m\in\{-j+\tfrac{1}{2},\ldots,j+\tfrac{1}{2}\}, \tag{19}$$

$$\hat{m}\equiv\hat{J}_z+\tfrac{1}{2}. \tag{20}$$

All angular-momentum or spin component eigenstates (in short, *spin states*) are labeled by integer numbers m. According to Vourdas [22] the amplitude $\hat{J}_r$ and the exponential phase operator $\hat{E}$ obey the relations

$$\hat{J}_+\equiv\hat{J}_x+i\hat{J}_y=\hat{J}_r\hat{E},$$
$$\hat{J}_-\equiv\hat{J}_x-i\hat{J}_y=\hat{E}^\dagger\hat{J}_r, \tag{21}$$

with

$$\hat{J}_r\equiv(\hat{J}_+\hat{J}_-)^{1/2}=[\hat{J}-\hat{J}_z^2+\hat{J}_z]^{1/2} \tag{22}$$

and

$$\hat{E}=\sum_m|m+1\rangle\langle m|\equiv\exp(i\hat{\varphi}) \tag{23}$$

with the Pegg-Barnett-type convention [20] that $|j+1\rangle=|-j\rangle$ in the Bose sector and $|j+\tfrac{1}{2}+1\rangle=|-j+\tfrac{1}{2}\rangle$ in the Fermi sector. The phase operator $\hat{\varphi}$ is perfectly Hermitian, and apart from a possible reference phase the phase eigenvalues are discrete and given by the expression

$$\varphi_\mu=\frac{2\pi}{d}\,\mu \tag{24}$$

with integers μ. The *phase states*, i.e., the eigenstates of the phase operator $\hat{\varphi}$ are discrete Fourier transformations of the spin states $|m\rangle$,

$$|\mu\rangle=d^{-1/2}\sum_m\exp\left(-\frac{2\pi i}{d}\,\mu m\right)|m\rangle. \tag{25}$$

To distinguish phase and spin states we denote phases by greek letters (mostly μ and ν) and spins using latin letters (mostly m and n). In case of doubt we put labels φ or m on the state vector. We introduce the rescaled phase operator

$$\hat{\mu} \equiv \frac{d}{2\pi}\,\hat{\varphi} = \sum_{\mu} \mu|\mu\rangle\langle\mu| \tag{26}$$

with integer eigenvalues μ. The spin states are discrete Fourier transformations of the phase states

$$|m\rangle = d^{-1/2}\sum_{\mu} \exp\left(+\frac{2\pi i}{d}\,m\mu\right)|\mu\rangle. \tag{27}$$

Phase states are of course periodic,

$$|\mu+d\rangle = |\mu\rangle, \tag{28}$$

and, if we regard Eq. (27) as a definition for spin states outside the original quantization region, spin states are periodic too,

$$|m+d\rangle = |m\rangle. \tag{29}$$

Note that this relation is just a convenient definition without physical consequences. In this way we have seen that the ''phase space'' for a quantum top is a periodic lattice of integers (μ,m). Finally, we introduce the *modular Kronecker symbol*

$$\delta(k;a) \equiv \begin{cases} 1, & k=0 \pmod{a} \\ 0, & \text{otherwise} \end{cases} \tag{30}$$

$$= \frac{1}{a}\sum_{l=l_0}^{l_0+a-1} \exp\left(\frac{2\pi i}{a}\,lk\right) \tag{31}$$

for integer k. This symbol will simplify future calculations.

C. Continuous Wigner representation

Of course, there is no true phase-space density in quantum mechanics and the best compromise we can reach is probably the Wigner function. The quantum characteristic function for a mechanical system described by position $\hat{q}$ and momentum $\hat{p}$ is defined as follows:

$$\widetilde{W}(u,v) \equiv \mathrm{Tr}\{\hat{\rho}\,\exp(-iu\hat{q}-iv\hat{p})\} \tag{32}$$

$$= \mathrm{Tr}\left\{\hat{\rho}\,\exp\left(-i\,\frac{uv}{2}\right)\exp(-iu\hat{q})\times\exp(-iv\hat{p})\right\} \tag{33}$$

$$= \mathrm{Tr}\left\{\hat{\rho}\,\exp\left(+i\,\frac{uv}{2}\right)\exp(-iv\hat{p})\times\exp(-iu\hat{q})\right\}, \tag{34}$$

where the Baker-Hausdorff formula has been used to obtain the last two lines. The Wigner function is the Fourier-transformed characteristic function

$$W(q,p) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\widetilde{W}(u,v)\exp(iuq+ivp)du\,dv \tag{35}$$

and using the expression (33) we derive Wigner's formula (1) for $W(q,p)$. The continuous Wigner function resembles many features of a classical phase-space density, although it might become negative, indicating nonclassical behavior [23]. Let us briefly summarize a few important properties of the continuous Wigner representation for later comparison with the discrete version.

Wigner functions are real and normalized to unity for normalized density operators. The Wigner functions W_1 and W_2 for two arbitrary operators $\hat{F}_1$ and $\hat{F}_2$ obey the overlap relation

$$\mathrm{Tr}\{\hat{F}_1\hat{F}_2\} = 2\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}W_1(q,p)W_2(q,p)dq\,dp, \tag{36}$$

which proves the formula (2) mentioned in the Introduction. The Wigner function has the quantum-mechanically correct marginal distributions

$$\langle q|\hat{\rho}|q\rangle = \int_{-\infty}^{+\infty}W(q,p)dp,$$

$$\langle p|\hat{\rho}|p\rangle = \int_{-\infty}^{+\infty}W(q,p)dq, \tag{37}$$

where $|q\rangle$ and $|p\rangle$ are the eigenstates of $\hat{q}$ and $\hat{p}$, respectively. Linear processes transform Wigner functions as if they were classical probability distributions. In particular, during free evolution described by the Hamiltonian

$$\hat{H} = \tfrac{1}{2}\hat{p}^2 \tag{38}$$

the Wigner function is sheared,

$$W(q,p;t) = W(q-pt,p). \tag{39}$$

Apart from the overlap relation (2) there is another alternative way for calculating expectation values. We consider

$$\mathrm{Tr}\{\hat{\rho}(\alpha\hat{q}+\beta\hat{p})^k\} = i^k\,\frac{\partial^k}{\partial\zeta^k}\,\widetilde{W}(\zeta\alpha,\zeta\beta)\Big|_{\zeta=1} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}W(q,p)(\alpha q+\beta p)^k dq\,dp. \tag{40}$$

In the first line we have used the definition (32) for the characteristic function $\widetilde{W}(u,v)$ while in the second line we have utilized the Fourier relationship (35) between $\widetilde{W}(u,v)$ and the Wigner function. Comparing the powers of α and β we see that

$$\mathrm{Tr}\{\hat{\rho}\mathscr{S}\hat{q}^m\hat{p}^n\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}W(q,p)q^m p^n dq\,dp. \tag{41}$$

The symbol $\mathscr{S}$ means that we should symmetrize all possible products of the m operators $\hat{q}$ and the n operators $\hat{p}$, i.e., we should take the average over all products with the right amount of $\hat{q}$'s and $\hat{p}$'s. Obviously, this ordering (called Weyl ordering) depends on the structure (32) of the continuous characteristic function.

D. Discrete characteristic function

Three major ingredients of our formalism have been presented: the picture of precession tomography, quantized tops, and Wigner functions as *quasi*-phase-space distributions. To find a discrete Wigner representation we must probably begin with a guess. Let us define the discrete characteristic function in a similar way as the continuous one,

$$\widetilde{W}(\nu,n) \equiv \mathrm{Tr}\left\{\hat{\rho}\,\exp\left(-\frac{4\pi i}{d}\,\nu n\right)\exp\left(-\frac{4\pi i}{d}\,\nu\hat{\mu}\right)\times\exp\left(-\frac{4\pi i}{d}\,n\hat{m}\right)\right\} \tag{42}$$

$$\equiv \mathrm{Tr}\left\{\hat{\rho}\,\exp\left(+\frac{4\pi i}{d}\,\nu n\right)\exp\left(-\frac{4\pi i}{d}\,n\hat{m}\right)\times\exp\left(-\frac{4\pi i}{d}\,\nu\hat{\mu}\right)\right\}, \tag{43}$$

using Vourdas' identity, Ref. [22], Eq. (35), for the last line. Note, however, that the discrete characteristic function does not equal $\mathrm{Tr}\{\hat{\rho}\,\exp[-4\pi i(\nu\hat{\mu}+n\hat{m})/d]\}$ anymore, since the commutation relation between phase and spin is different from the canonic relation between position and momentum, and the Baker-Hausdorff formula cannot be applied. (Canonical commutation relations imply a continuous spectrum which violates, of course, the discrete nature of angular momenta and spins.) Consequently, the Weyl ordering of observables is sacrificed in our discrete Wigner representation. Nevertheless, most other important properties of continuous Wigner functions are maintained yet sometimes in a subtly modified form. To find explicit expressions for the discrete characteristic function we note that

$$\exp\left(-\frac{4\pi i}{d}\,n\hat{m}\right)|\mu\rangle = |\mu+2n\rangle_{\varphi}. \tag{44}$$

and

$$\exp\left(-\frac{4\pi i}{d}\,\nu\hat{\mu}\right)|m\rangle = |m-2\nu\rangle_{m}, \tag{45}$$

which is easily verified using the relations (25) and (27) between spin and phase states. In this way we obtain directly from the definition (42), (43) of the discrete characteristic function the explicit formulas

$$\widetilde{W}(\nu,n) = \sum_{\mu} \exp\left[-\frac{4\pi i}{d}\,\nu(\mu+n)\right]\langle\mu|\hat{\rho}|\mu+2n\rangle_{\varphi} \tag{46}$$

$$= \sum_{m} \exp\left[-\frac{4\pi i}{d}\,n(m-\nu)\right]\langle m|\hat{\rho}|m-2\nu\rangle_{m}. \tag{47}$$

Now we are prepared to study discrete Wigner functions and quantum-state tomography.

III. BOSE SECTOR

A. Discrete Wigner function

We have everything on hand to define directly the discrete Wigner function for *odd-dimensional* systems (for genuine angular momenta), i.e., in the Bose sector. Even-dimensional quantum objects require more thought and are considered in the next section. The Wigner function for bosons is the Fourier-transformed characteristic function

$$W(\mu,m) \equiv \frac{1}{d^2}\sum_{\nu n} \exp\left[\frac{4\pi i}{d}\,(\nu\mu+nm)\right]\widetilde{W}(\nu,n). \tag{48}$$

Using the explicit expressions (46) and (47) for the discrete characteristic function we obtain familiar looking formulas for the Wigner function,

$$W(\mu,m) = \frac{1}{d}\sum_{\nu} \exp\left(+\frac{4\pi i}{d}\,m\nu\right)\langle\mu-\nu|\hat{\rho}|\mu+\nu\rangle \tag{49}$$

$$= \frac{1}{d}\sum_{n} \exp\left(-\frac{4\pi i}{d}\,\mu n\right)\langle m-n|\hat{\rho}|m+n\rangle. \tag{50}$$

Bear in mind the periodicity relations (28) and (29) for the phase states $|\mu\rangle$ and spin states $|m\rangle$, respectively, to obtain meaningful results.

B. Examples

1. Line states

How do the discrete Wigner functions look? Let us consider the simplest examples [24] first: *line states*, i.e., phase or spin states. What would we expect? A phase state should have a well-defined phase, of course, and a completely random angular-momentum component. The Wigner function of a phase state should be a line in the discrete lattice of (μ,m) values; see Fig. 2(a). On the other hand, for a spin state we expect a well-defined angular-momentum component and a random phase as depicted in Fig. 2(b). That this is indeed correct is easily verified using the explicit expressions (49) and (50). For a phase state $|\mu\rangle$ we get

$$W(\mu,m) = \frac{1}{d}\sum_{\nu} \exp\left(\frac{4\pi i}{d}\,m\nu\right)\langle\mu-\nu|\mu_0\rangle\langle\mu_0|\mu+\nu\rangle = d^{-1}\delta(\mu-\mu_0;d), \tag{51}$$

while for a spin state $|m\rangle$ we obtain

$$W(\mu,m) = d^{-1}\delta(m-m_0;d). \tag{52}$$

2. Broken-line states

There is an interesting family of states if the dimension d is composite, i.e.,

$$d = ab, \tag{53}$$

with the integers a and b. Let us call them *broken-line states* defined as

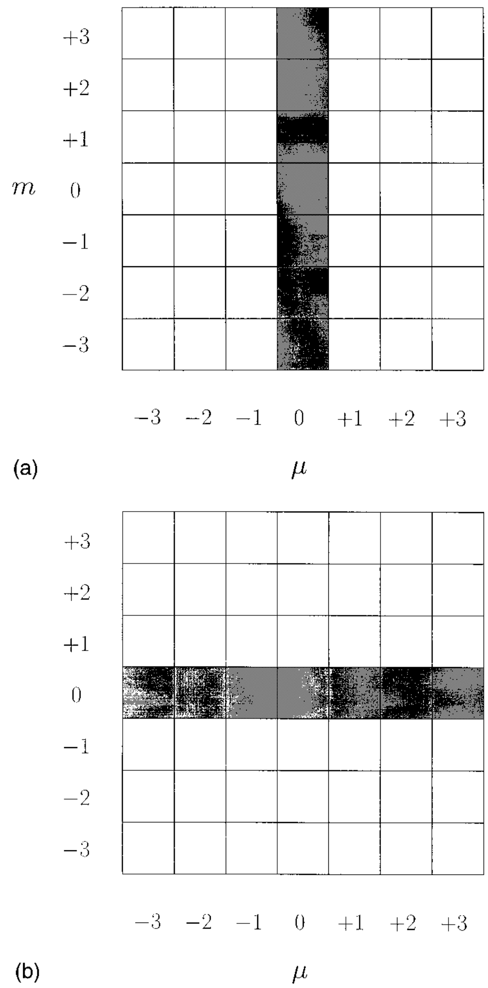

FIG. 2. Example of discrete Wigner functions (51), (52) for line states in odd dimensions. Here the dimension $d=7$. In (a) a phase state $|0\rangle_\varphi$ and in (b) a spin state $|0\rangle_m$ is depicted.

$$|\mu_0,m_0\rangle_{ab}\equiv b^{-1/2}\sum_{m'}\exp\left(-\frac{2\pi i}{b}m'\mu_0\right)|am'+m_0\rangle, \tag{54}$$

where m' ranges from $-(b-1)/2$ to $+(b-1)/2$. We may express these states also in a different way using the representation (27) of the spin states in terms of phase states,

$$|\mu_0,m_0\rangle_{ab}\propto\sum_{m'}\sum_{\mu}\exp\left[\frac{2\pi i}{d}[\mu(am'+m_0)-am'\mu]\right]|\mu\rangle$$
$$\propto\sum_{\mu}\exp\left(\frac{2\pi i}{d}\mu m_0\right)\delta(\mu-\mu_0;b)|\mu\rangle, \tag{55}$$

and utilizing the normalization of $|\mu_0,m_0\rangle_{ab}$,

$$|\mu_0,m_0\rangle_{ab}=a^{-1/2}\sum_{\mu'}\exp\left(+\frac{2\pi i}{a}\mu' m_0\right)|b\mu'+\mu_0\rangle, \tag{56}$$

where μ' ranges from $-(a-1)/2$ to $+(a-1)/2$. The Wigner function for broken-line states is easily calculated using the explicit expression (49)

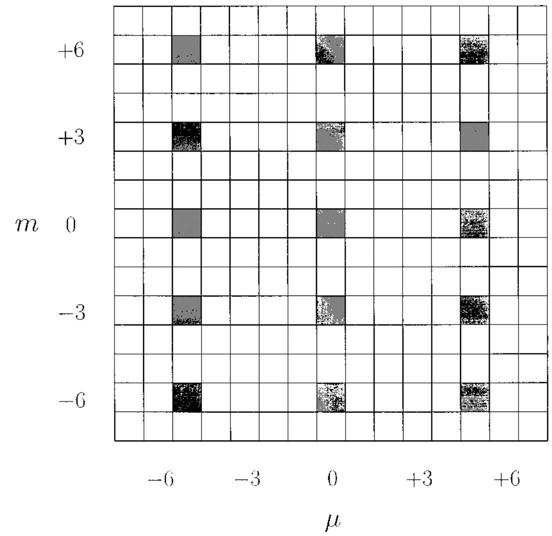

FIG. 3. Example of discrete Wigner functions (60) for broken-line states (54) in odd and composite dimensions. Here the dimension $d=15$. The state is $\frac{1}{3}(|-5\rangle_\varphi+|0\rangle_\varphi+|+5\rangle_\varphi)$ in phase representation. Three broken lines at the phases $\{-5,0,+5\}$ are clearly visible. They correspond to five lines at the spins $\{-6,-3,0,+3,+6\}$ since the state can be expressed as $\frac{1}{5}(|-6\rangle_m+|-3\rangle_m+|0\rangle_m+|+3\rangle_m+|+6\rangle_m)$ in spin representation.

$$W(\mu,m)=\frac{1}{da}\sum_{\nu}\sum_{\mu_1'\mu_2'}\exp\left[\frac{2\pi i}{d}[2\nu m+bm_0\times(\mu_1'-\mu_2')]\right]\times\delta(\mu-\nu-\mu_0-b\mu_1';d)\,\delta(\mu+\nu-\mu_0-b\mu_2';d). \tag{57}$$

The modular Kronecker symbol in this equation implies that

$$2\nu=b(\mu_2'-\mu_1')\quad(\text{mod } d),$$
$$2(\mu-\mu_0)=b(\mu_2'+\mu_1')\quad(\text{mod } d). \tag{58}$$

Consequently,

$$W(\mu,m)=\frac{1}{da}\sum_{\nu'}\exp\left[\frac{4\pi i}{d}(m-m_0)\nu'\right]\delta(\mu-\mu_0;b) \tag{59}$$

and, finally,

$$W(\mu,m)=d^{-1}\delta(m-m_0;a)\,\delta(\mu-\mu_0;b). \tag{60}$$

The Wigner function of broken-line states displays a lattice of broken lines in the discrete phase space; see Fig. 3. The period of this lattice is b in phase and a in spin direction and the parameters μ_0 and m_0 define the displacement of the lattice. Although broken-line states are superpositions of a few line states their Wigner function is non-negative, quite in contrast to Wigner functions of superpositions in continuous quantum mechanics [25]. Note that broken-line states are only possible if the dimension d is composite. In this paper they give a first indication that simple number theory plays a role in the quantum mechanics of finite-dimensional systems.

C. Properties

Let us study the properties of discrete Wigner functions in the Bose sector. Of course, they are real-valued for Hermitian density matrices $\hat{\rho}$ and normalized to unity,

$$\sum_{\mu m} W(\mu,m)=1, \tag{61}$$

as we would expect for a suitable quasiprobability distribution.

3. Marginals

The marginal distribution of the discrete Wigner function should yield the phase and spin histograms, respectively. In fact, we obtain from the explicit formula (49)

$$\sum_{m} W(\mu,m)=\frac{1}{d}\sum_{m\nu} \exp\left(\frac{4\pi i}{d}\, m\nu\right)\langle\mu-\nu|\hat{\rho}|\mu+\nu\rangle$$

$$=\sum_{\nu} \delta(2\nu;d)\langle\mu-\nu|\hat{\rho}|\mu+\nu\rangle \tag{62}$$

and since the dimension d is odd in the Bose sector,

$$\sum_{m} W(\mu,m)=\langle\mu|\hat{\rho}|\mu\rangle. \tag{63}$$

The other explicit expression (50) leads to the spin marginals

$$\sum_{\mu} W(\mu,m)\equiv\langle m|\hat{\rho}|m\rangle. \tag{64}$$

This shows that discrete Wigner functions have the quantum-mechanically correct marginal distributions.

4. Overlap relation

Let $W_1(\mu,m)$ and $W_2(\mu,m)$ be the discrete Wigner functions for two (not necessarily Hermitian) operators $\hat{F}_1$ and $\hat{F}_2$. The overlap of two Wigner functions is given by

$$d\sum_{\mu m} W_1(\mu,m)W_2(\mu,m)$$

$$=\frac{1}{d}\sum_{m\nu}\sum_{\nu_1\nu_2} \exp\left[\frac{4\pi i}{d}\, m(\nu_1+\nu_2)\right]\langle\mu-\nu_1|\hat{F}_1|\mu+\nu_1\rangle$$

$$\times\langle\mu-\nu_2|\hat{F}_2|\mu+\nu_2\rangle. \tag{65}$$

The summation with respect to m produces a modular Kronecker symbol $\delta(2\nu_1+2\nu_2;d)$ and so we obtain for the overlap of bosonic Wigner functions

$$d\sum_{\mu m} W_1(\mu,m)W_2(\mu,m)$$

$$=\sum_{\mu\nu} \langle\mu-\nu|\hat{F}_1|\mu+\nu\rangle\langle\mu+\nu|\hat{F}_2|\mu-\nu\rangle$$

$$=\sum_{\mu_1\mu_2} \langle\mu_1|\hat{F}_1|\mu_2\rangle\langle\mu_2|\hat{F}_2|\mu_1\rangle$$

$$=\sum_{\mu_1} \langle\mu_1|\hat{F}_1\hat{F}_2|\mu_1\rangle \tag{66}$$

using the completeness of the phase states in the last step. Finally, we arrive at the overlap relation

$$\mathrm{Tr}\{\hat{F}_1\hat{F}_2\}=d\sum_{\mu m} W_1(\mu,m)W_2(\mu,m). \tag{67}$$

In particular, we obtain

$$\mathrm{Tr}\{\hat{\rho}\hat{F}\}=d\sum_{\mu m} W(\mu,m)W_F(\mu,m) \tag{68}$$

for the density operator $\hat{\rho}$ and an arbitrary operator $\hat{F}$. This relation illustrates that also in the discrete case all predictable quantities are filtered projections i.e., quantum shadows, of the discrete Wigner function. The formula (68) provides us with the key for calculating expectation values via Wigner functions or, more generally, for formulating quantum mechanics without probability amplitudes [26]. The overlap relation replaces fully the Weyl ordering of $\hat{\mu}$ and $\hat{m}$ which has been sacrificed from the very beginning. The relation also shows that the Wigner functions $W_1(\mu,m)$ and $W_2(\mu,m)$ of two orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ must be either disjoint or must contain negative elements, since

$$d\sum_{\mu m} W_1(\mu,m)W_2(\mu,m)=\langle\psi_1|\psi_2\rangle=0. \tag{69}$$

Later we will encounter other interesting applications of the fundamental overlap relation.

D. Operations in discrete phase space

1. Displacement

The displacement of discrete Wigner functions in the phase-space lattice was considered by Opatrny *et al.* [12]. For the sake of notation we define a phase-displacement operator

$$\hat{D}_{\varphi}(\mu_0)\equiv\exp\left(-\frac{2\pi i}{d}\,\mu_0\hat{m}\right) \tag{70}$$

and a spin-displacement operator

$$\hat{D}_{m}(m_0)\equiv\exp\left(+\frac{2\pi i}{d}\,m_0\hat{\mu}\right). \tag{71}$$

The effect of the displacements $\hat{D}\hat{\rho}\hat{D}^{\dagger}$ in discrete phase space is obvious,

$$W_{\varphi}(\mu,m)=W(\mu-\mu_0,m) \tag{72}$$

and

$$W_{m}(\mu,m)=W(\mu,m-m_0) \tag{73}$$

i.e., the Wigner functions are simply displaced.

2. Precession

The central intuitive idea of discrete quantum-state tomography is precession tomography. What is the effect of the precession Hamiltonian

$$\hat{H}=\frac{\hat{m}^2}{2} \tag{74}$$

in discrete phase space? Let us consider the precession operator

$$\hat{P}(t)=\exp(-i\hat{H}t), \tag{75}$$

i.e., the time evolution operator corresponding to the Hamiltonian $\hat{H}$, at times

$$t=\frac{4\pi}{d}\,\tau \tag{76}$$

with integer τ [27]. To see the effect of precession in discrete phase space we use the explicit expression (50) for the discrete Wigner function and obtain

$$\begin{aligned}W_P(\mu,m)&=\frac{1}{d}\sum_n \exp\left(-\frac{4\pi i}{d}\,n\mu\right)\langle m-n|\hat{P}\hat{\rho}\hat{P}^\dagger|m+n\rangle\\&=\frac{1}{d}\sum_n \exp\Big[-\frac{2\pi i}{d}\,[2n\mu+\tau(m-n)^2\\&\quad+\tau(m+n)^2]\Big]\langle m-n|\hat{\rho}|m+n\rangle,\end{aligned} \tag{77}$$

yet provided that τ is indeed integer so that we can apply the same modular arithmetic in the exponent as in the notation of the spin states (29). For this subtle reason we obtain

$$W_P(\mu,m)=W(\mu-2m\tau,m) \tag{78}$$

only if τ is integer. Precession in discrete phase space takes place in steps of *two* (in the Bose sector). Or, more precisely, the relation between the precessed and the original Wigner function is less simple for times t which do not match the condition (76). A precessing phase state, for instance, would produce oscillations and negative values in the Wigner function between the times $4\pi\tau d^{-1}$, but as soon as t approaches $4\pi\tau d^{-1}$ the picture becomes clear, and finally the simple precession (78) of the discrete Wigner function is apparent; see Fig. 4. This series of clear pictures during precession resembles the Talbot effect [28] or fractional revivals [29] and it is of course periodic in d when the initial state returns to itself.

E. Tomography

How can we reconstruct the quantum state, i.e., the characteristic or the Wigner function, from observations of a precessing spin? The central idea was already sketched in the remarks on classical precession tomography. Only some subtle modifications are required. First of all, the phase histograms should be observed at the times (76) provided the Hamiltonian (74) is acting. (If the scale of the precession Hamiltonian is different then t should be scaled accordingly.)

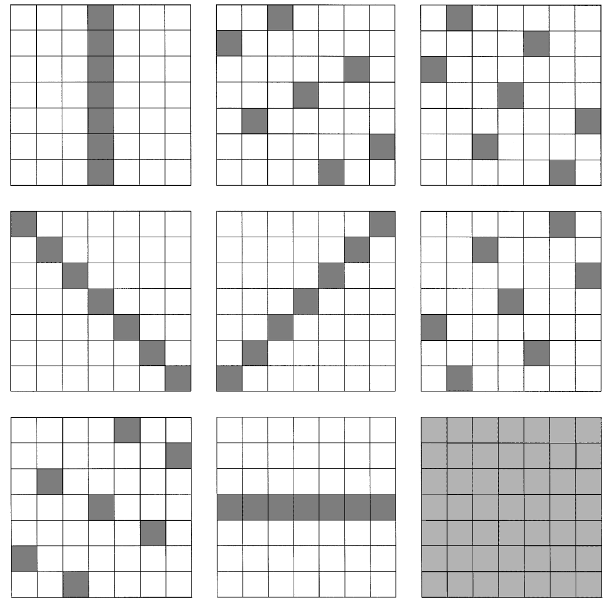

FIG. 4. Schematic diagram to show the precession in discrete phase space for a prime-dimensional system ($d=7$). A precessing phase state $\hat{P}(t)|0\rangle_\varphi$, see Eq. (75), is depicted at the times $4\pi\tau/d$ with integer τ. The picture shows that precession takes place in steps of two. It also illustrates that the precession states together with the spin state $|0\rangle_m$ cover the whole phase space completely. This remarkable feature reveals the *multicomplementarity* of the precession states in prime dimension. It indicates that the precession states combined with the spin states represent all complementary aspects of a prime-dimensional quantum system.

Phase states (25) are discrete Fourier transformations and their experimental realization depends, of course, on the particular system [30]. It has been much discussed in the context of quantum computing [31]. Given the set of phase histograms $\mathrm{pr}_\varphi(\mu;\tau)$ at the times (76) we can reconstruct the characteristic function. According to the overlap relation (67) the phase histograms can be seen as

$$\begin{aligned}\mathrm{pr}_\varphi(\mu_0;\tau)&\equiv\langle\mu_0|\hat{P}\hat{\rho}\hat{P}^\dagger|\mu_0\rangle\\&=\sum_{\mu m} W(\mu,m)\,\delta(\mu-\mu_0+2m\tau;d).\end{aligned} \tag{79}$$

Consequently, the Fourier transformation $\widetilde{\mathrm{pr}}_\varphi(\nu;\tau)$ of the histogram $\mathrm{pr}_\varphi(\mu;\tau)$

$$\begin{aligned}\widetilde{\mathrm{pr}}_\varphi(\nu;\tau)&\equiv\sum_{\mu_0} \exp\left(-\frac{4\pi i}{d}\,\mu_0\nu\right)\mathrm{pr}_\varphi(\mu_0;\tau)\\&=\sum_{\mu m} \exp\left[-\frac{4\pi i}{d}\,\nu(\mu+2m\tau)\right]W(\mu,m)\end{aligned} \tag{80}$$

is the characteristic function at the points $(\nu,2\tau\nu)$,

$$\widetilde{\mathrm{pr}}_\varphi(\nu;\tau)=\widetilde{W}(\nu,2\tau\nu). \tag{81}$$

Like in classical precession tomography we cannot infer the characteristic function from phase observations only. The conjugate quantity, i.e., the angular-momentum component must be measured as well. In fact, the Fourier-transformed spin distribution $\mathrm{pr}_m(m)$ yields

$$\widetilde{\mathrm{pr}}_m(n) \equiv \sum_{m_0} \exp\left(-\frac{4\pi i}{d} m_0 n\right) \mathrm{pr}_m(m_0)$$
$$= \sum_{m_0} \exp\left(-\frac{4\pi i}{d} m_0 n\right) \sum_{\mu} W(\mu,m)$$
$$= \widetilde{W}(0,n). \tag{82}$$

If the dimension d is prime we can represent every lattice point (ν,n) of the characteristic function in terms of $(\nu_0,2\tau\nu_0)$ or $(0,\mu)$ *modulo* d, since primes define a *field* [32]. This means that we can reconstruct every point of the characteristic function from observations of the precessed phase and of the angular-momentum component. However, this is not true for composite dimensions. Then we can represent (ν,n) as $(\nu_0,2\tau\nu_0)$ *modulo* d only if ν_0 and the dimension have no common factor a. Otherwise, if ν_0 contained the factor a, $2\tau\nu_0$ would be proportional to the number a as well, and points (ν,n) which do not meet this condition are missed in the reconstruction of the characteristic function. There is a simple reason for this: Precession shears layers with equal angular-momentum components and we observe the phase distribution. This (Schrödinger-like) picture is equivalent to letting the phase states precess and leaving the system state unchanged (Heisenberg-like picture). If the dimension is prime, the precessing phase states and the angular-momentum states will cover the complete phase space; see Fig. 4. If the dimension is composite, some points in discrete phase space are never reached by the precessing spin states and these regions remain unexplored; see Fig. 5. However, precessing broken-line states fill these gaps. Suppose we observe the probability distribution

$$\mathrm{pr}(\mu_0,m_0;\tau)$$
$$\equiv \left\langle \mu_0,0 \left| \hat{P}\left(\frac{t}{a}\right) \hat{D}_m^\dagger(m_0)\hat{\rho}\hat{D}_m(m_0)\hat{P}^\dagger\left(\frac{t}{a}\right)\right| 0,\mu_0 \right\rangle$$
$$= \sum_{\mu m} W(\mu,m)\,\delta\left(\mu-\mu_0+2(m-m_0)\frac{\tau}{a};b\right)\delta(m-m_0;a). \tag{83}$$

Note that we have divided the time t defined in (76) by a. Even in this case the discrete Wigner function of broken-line states is transformed according to the classical-like law (78) as is easily verified. The histogram $\mathrm{pr}(\mu_0,m_0;\tau)$ is a-periodic in m_0 and b-periodic in μ_0 and τ. Performing a discrete Fourier transformation of the broken-line distribution

$$\widetilde{\mathrm{pr}}(\mu_0,m_0;\tau)$$
$$\equiv \frac{1}{a}\sum_{\mu_0 m_0} \exp\left[-\frac{4\pi i}{d}\nu(a\mu_0+\tau m_0)\right]\mathrm{pr}(\mu_0,m_0;\tau)$$
$$= \frac{1}{a}\sum_{m_0}\sum_{\mu m} \exp\left[-\frac{4\pi i}{d}\nu(a\mu+\tau m)\right]$$
$$\times W(\mu,m)\,\delta(m-m_0;a)$$
$$= \sum_{\mu m} \exp\left[-\frac{4\pi i}{d}\nu(a\mu+\tau m)\right] W(\mu,m)$$
$$= \widetilde{W}(a\nu,2\tau\nu) \tag{84}$$

reveals the missing elements of the discrete characteristic function, and finally, using the relations (81), (82), and (84) the quantum state can be reconstructed.

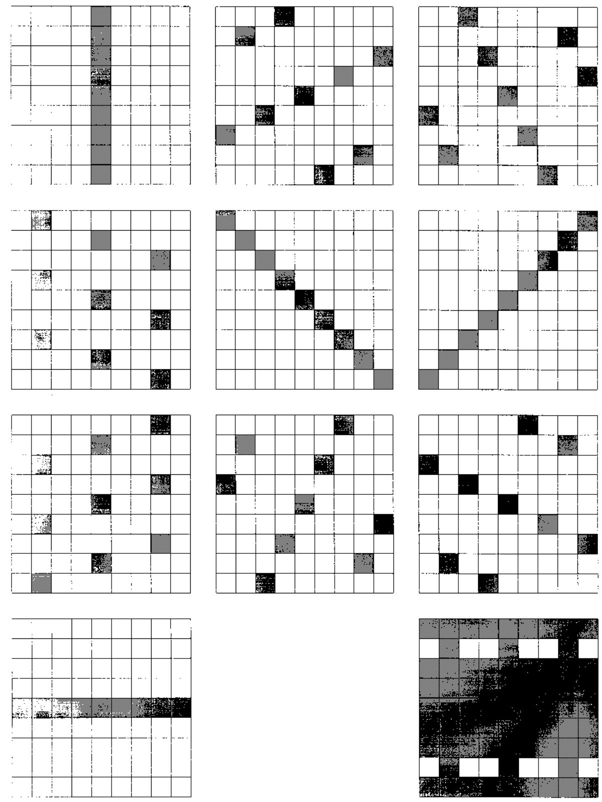

FIG. 5. Schematic diagram to show the precession in discrete phase space for a composite-dimensional system ($d=9$). The precessing phase states leave gaps in the discrete phase space which are to be closed by broken-line states to gain sufficient information for state reconstruction. The picture also illustrates that the precession states may overlap in three points or may have zero overlap in accordance with their lack of *multicomplementarity*, see Eq. (91).

F. Primes and multicomplementarity

There is another way of understanding the distinction of prime dimensions in precessing tomography [34]. The precession states are *multicomplementary*. Let us denote these states by

$$|\mu;\tau\rangle \equiv \hat{P}(t)|\mu\rangle. \tag{85}$$

Note that τ and t are related according to the formula (76). The precession states have the remarkable property [35] that

$$|\langle \mu_2;\tau_2|\mu_1;\tau_1\rangle|^2 = d^{-1} \quad \text{for } \tau_1 \neq \tau_2 \pmod d,$$
$$|\langle m|\mu;\tau\rangle|^2 = d^{-1}, \tag{86}$$

if and only if the dimension d is prime. What does this property (86) mean? The probability of finding the state $|\mu_2;\tau_2\rangle$ in $|\mu_1;\tau_1\rangle$ is completely uniform, so that the $|\mu_2;\tau_2\rangle$ state contains no information at all about the $|\mu_1;\tau_1\rangle$ state and vice versa, a phenomenon familiar from position and momentum eigenstates. The $d+1$ sets of states $|\mu;\tau\rangle$ and $|m\rangle$ are mutually *complementary*. They cover the complete phase space (see Fig. 4), and they represent all complementary aspects [36] of the quantum system. The knowledge of the distributions of all complementary features is equivalent to the knowledge of the quantum state.

To verify the property (86) for precession states in prime dimensions we recall that according to the overlap relation (67) the transition probability between $|\mu_2;\tau_2\rangle$ and $|\mu_1;\tau_1\rangle$ is given by the overlap of the corresponding Wigner functions. A glance at Fig. 4 shows that the precession states have exactly one element in phase space in common and the overlap of the two Wigner functions multiplied with d yields the value d^{-1} of the transition probability (86).

What happens if the dimension is not prime? Obviously, the scalar product $\langle\mu_2;\tau_2|\mu_1;\tau_1\rangle$ equals $\langle\mu_2-\mu_1|\hat{P}^\dagger(t_2-t_1)|0\rangle_\varphi$, and it is sufficient to consider only

$$|\langle\mu|\hat{P}^\dagger(t)|0\rangle_\varphi|^2=\frac{1}{d^2}\left|\sum_m \exp\left[\frac{2\pi i}{d}(\tau m^2+\mu m)\right]\right|^2. \tag{87}$$

There are two cases to distinguish. First, τ and d may have no common factor. Then we can always represent μ as $2\tau\mu'$ *modulo* d and obtain

$$\begin{aligned}|\langle\mu|\hat{P}^\dagger(t)|0\rangle_\varphi|^2&=\frac{1}{d^2}\left|\sum_m \exp\left[\frac{2\pi i}{d}\tau(m+\mu')^2\right]\right|^2\\&=\frac{1}{d^2}\left|\sum_m \exp\left[\frac{2\pi i}{d}\tau m^2\right]\right|^2\\&=d^{-1},\end{aligned} \tag{88}$$

using in the last line a result found by Gauβ [37]. Second, suppose τ and d have the greatest common factor a, i.e.,

$$\tau=a\tau', \quad d=ab. \tag{89}$$

We represent m as $bm'+m''$ *modulo* d and obtain from Eq. (87)

$$\begin{aligned}&|\langle\mu|\hat{P}^\dagger(t)|0\rangle_\varphi|^2\\&\quad=\frac{1}{d^2}\left|\sum_{m'm''} \exp\left[\frac{2\pi i}{d}(a\tau' m''^2+\mu m''+\mu b m')\right]\right|^2\\&\quad=\frac{1}{b^2}\left|\sum_{m''} \exp\left[\frac{2\pi i}{d}(a\tau' m''^2+\mu m'')\right]\delta(\mu;a)\right|^2\end{aligned} \tag{90}$$

and using the same arguments as in the previous case

$$|\langle\mu|\hat{P}^\dagger(t)|0\rangle_\varphi|^2=b^{-1}\delta(\mu;a). \tag{91}$$

The overlap vanishes if μ is not divisible by a. It is accordingly larger than d^{-1} if μ contains the factor a. Figure 5 illustrates that precession states may not overlap at all or they may overlap in more than just one element in phase space. The precession states contain mutual information about each other and so they are not strictly complementary. Consequently, there is more than the precessing phases and angular momenta to be measured for inferring the quantum state of a system with composite dimension.

IV. FERMI SECTOR

A. Problems

Why is the definition (48) of the discrete Wigner function not suitable for even-dimensional systems? What is odd with even dimensions? This is most easily seen by considering the marginal distribution

$$\sum_m W(\mu,m)=\sum_\nu \delta(2\nu;d)\langle\mu-\nu|\hat{\rho}|\mu+\nu\rangle; \tag{92}$$

cf. Eq. (62). If the dimension d is even there are two cases, $\nu=0$ and $\nu=d/2$, where $\delta(2\nu;d)$ is different from zero and we obtain

$$\begin{aligned}\sum_m W(\mu,m)&=\langle\mu|\hat{\rho}|\mu\rangle+\langle\mu-d/2|\hat{\rho}|\mu+d/2\rangle\\&=\langle\mu|\hat{\rho}|\mu\rangle+\langle\mu+d/2|\hat{\rho}|\mu+d/2\rangle\end{aligned} \tag{93}$$

because of the periodicity (28) of the phase states. This is not the desired result (63). It indicates already that spins require more thought in defining a discrete Wigner function than angular momenta. There is another reason: Wigner functions expressed as (49) or (50) depend entirely on matrix elements with even index difference $(\mu+\nu)-(\mu-\nu)=2\nu$ *modulo* the dimension d. This matters if d is even because then 2ν (mod d) is an absolute even number. For even dimensions, Wigner functions (49) and (50) lack the information about density-matrix elements with odd index differences and so the Wigner functions do not represent uniquely quantum states anymore. There is another way of seeing this: If the dimension d is even the characteristic function (42) is periodic with respect to $d/2$

$$\widetilde{W}(\nu+d/2,n)=\widetilde{W}(\nu,n+d/2)=\widetilde{W}(\nu,n), \tag{94}$$

provided of course that ν and n are integer. The characteristic function contains redundant information. It seems likely that these oddities of the even-dimensional Wigner function have a deeper topological reason. (Spins require a 4π rotation to return to the initial state while 2π is sufficient for angular momenta.)

B. Two dimensions

For two-dimensional systems (spin $\frac{1}{2}$) the characteristic function would contain just one independent element $\widetilde{W}(0,0)$ if ν and n are assumed to be integer. What about half-integer arguments? This changes the situation significantly because then we obtain from the explicit expressions (46) and (47)

$$\widetilde{W}(0,0)=\mathrm{Tr}\{\hat{\rho}\}, \tag{95}$$

$$\widetilde{W}(0,\tfrac{1}{2})=\mathrm{Tr}\{\hat{\rho}\sigma_x^\varphi\}=\mathrm{Tr}\{\hat{\rho}\sigma_z^m\},$$

$$\widetilde{W}(\tfrac{1}{2},0)=\mathrm{Tr}\{\hat{\rho}\sigma_z^{\varphi}\}=\mathrm{Tr}\{\hat{\rho}\sigma_x^{m}\},$$

$$\widetilde{W}(\tfrac{1}{2},\tfrac{1}{2})=\mathrm{Tr}\{\hat{\rho}\sigma_y^{\varphi}\}=-\mathrm{Tr}\{\hat{\rho}\sigma_y^{m}\},$$

with σ_x^{φ}, σ_y^{φ}, σ_z^{φ} and σ_x^{m}, σ_y^{m}, σ_z^{m} being the Pauli matrices in the phase and spin representation, respectively,

$$\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{96}$$

The characteristic function for half-integer arguments is the Bloch vector. The discrete Wigner function should be the Fourier-transformed characteristic function

$$W(\mu,m)=\frac{1}{4}\sum_{\nu n}\exp[2\pi i(\nu\mu+nm)]\widetilde{W}(\nu,n). \tag{97}$$

Here ν and n take on half-integer values while μ and m should be integer. Explicitly,

$$\begin{aligned}
W(0,0)&=\tfrac{1}{4}[2\langle 0|\hat{\rho}|0\rangle+(1+i)\langle 1|\hat{\rho}|0\rangle+(1-i)\langle 0|\hat{\rho}|1\rangle],\\
W(0,1)&=\tfrac{1}{4}[2\langle 0|\hat{\rho}|0\rangle-(1+i)\langle 1|\hat{\rho}|0\rangle-(1-i)\langle 0|\hat{\rho}|1\rangle],\\
W(1,0)&=\tfrac{1}{4}[2\langle 1|\hat{\rho}|1\rangle+(1-i)\langle 1|\hat{\rho}|0\rangle+(1+i)\langle 0|\hat{\rho}|1\rangle],\\
W(1,1)&=\tfrac{1}{4}[2\langle 1|\hat{\rho}|1\rangle-(1-i)\langle 1|\hat{\rho}|0\rangle-(1+i)\langle 0|\hat{\rho}|1\rangle]
\end{aligned} \tag{98}$$

in phase representation. This is precisely Wootters' expression [15] of a Wigner function for two-dimensional systems. We have seen that it can be derived from the characteristic function (42), (43) by allowing half-integer arguments.

Since the Wigner function is the Fourier transform of the Bloch vector it represents the quantum state uniquely. Using the explicit expression (98) we see that the Wigner function has interesting marginal distributions,

$$\begin{aligned}
W(0,0)+W(0,1)&=\langle 0|\hat{\rho}|0\rangle_{\varphi}=\tfrac{1}{2}(\langle 0|+\langle 1|)\hat{\rho}(|0\rangle+|1\rangle)_m,\\
W(0,1)+W(1,1)&=\langle 1|\hat{\rho}|1\rangle_{\varphi}=\tfrac{1}{2}(\langle 0|-\langle 1|)\hat{\rho}(|0\rangle-|1\rangle)_m.
\end{aligned} \tag{99}$$

The phase marginal distribution yields the phase probabilities or, in other words, the probabilities for the eigenstates of $\sigma_z^{\varphi}=\sigma_x^{m}$. Furthermore,

$$\begin{aligned}
W(0,0)+W(1,0)&=\tfrac{1}{2}(\langle 0|+\langle 1|)\hat{\rho}(|0\rangle+|1\rangle)_{\varphi}=\langle 0|\hat{\rho}|0\rangle_m,\\
W(0,1)+W(1,1)&=\tfrac{1}{2}(\langle 0|-\langle 1|)\hat{\rho}(|0\rangle-|1\rangle)_{\varphi}=\langle 1|\hat{\rho}|1\rangle_m.
\end{aligned} \tag{100}$$

The spin marginal distribution yields the spin probabilities, i.e., the probabilities for the eigenstates $\sigma_x^{\varphi}=\sigma_z^{m}$. Finally,

$$\begin{aligned}
W(0,1)+W(1,0)&=\tfrac{1}{2}(\langle 0|-i\langle 1|)\hat{\rho}(|0\rangle+i|1\rangle)_{\varphi}\\
&=\tfrac{1}{2}(\langle 0|+i\langle 1|)\hat{\rho}(|0\rangle-i|1\rangle)_m,\\
W(0,0)+W(1,1)&=\tfrac{1}{2}(\langle 0|+i\langle 1|)\hat{\rho}(|0\rangle-i|1\rangle)_{\varphi}\\
&=\tfrac{1}{2}(\langle 0|-i\langle 1|)\hat{\rho}(|0\rangle+i|1\rangle)_m.
\end{aligned} \tag{101}$$

The diagonal sums of the Wigner function $W(\mu,m)$ yield the probabilities for the eigenstates of $\sigma_y^{\varphi}=-\sigma_y^{m}$. We have seen that the two-dimensional Wigner function $W(\mu,m)$ has properties of a magic square, i.e., the sums across lines in $W(\mu,m)$ give certain physically meaningful results.

C. Even dimensions

It turns out that a second step is necessary to define a discrete Wigner function in arbitrary even dimensions. We introduce half odds between the integer elements of the discrete phase space for both the characteristic and the Wigner function, i.e.,

$$W(\mu,m)\equiv\frac{1}{2d}\sum_{\nu n}\exp\left[\frac{4\pi i}{d}(\nu\mu+nm)\right]\widetilde{W}(\nu,n), \tag{102}$$

where ν and n run from $-d/2+\frac{1}{2}$ to $d/2$ in steps of $\frac{1}{2}$ and μ and m take on half-integer values (integers and half odds). Explicitly,

$$W(\mu,m)=\frac{1}{2d}\sum_{\nu}\exp\left[+\frac{4\pi i}{d}\nu m\right]\langle\mu-\nu|\hat{\rho}|\mu+\nu\rangle \tag{103}$$

$$=\frac{1}{2d}\sum_{n}\exp\left[-\frac{4\pi i}{d}n\mu\right]\langle m-n|\hat{\rho}|m+n\rangle \tag{104}$$

with the convention that the matrix elements with half-odd arguments should vanish. The definition (102) and the explicit expressions (103) and (104) include of course the previously considered Wigner functions for spin $\frac{1}{2}$. However, half-odd elements like $W(0,\frac{1}{2})$, $W(\frac{1}{2},0)$,... have been supplemented.

D. Properties

1. Marginals

Does this procedure of supplementing the discrete phase space with half-odd elements solve the problem (93) with the marginal distributions? We consider

$$\sum_m W(\mu,m)=\frac{1}{2d}\sum_{m\nu}\exp\left(\frac{4\pi i}{d}\nu m\right)\langle\mu-\nu|\hat{\rho}|\mu+\nu\rangle \tag{105}$$

and see that since

$$\begin{aligned}
\sum_m \exp\left(\frac{4\pi i}{d}\nu m\right)&=\sum_{k=-d+1}^{d}\exp\left(\frac{2\pi i}{2d}2\nu k\right)\\
&=2d\ \delta(2\nu;2d)
\end{aligned} \tag{106}$$

with $m=k/2$ and integer k, the index ν must be zero in the summation (105). Consequently,

$$\sum_m W(\mu,m)=\langle\mu|\hat{\rho}|\mu\rangle \tag{107}$$

if μ is integer, and because of the matrix-element convention in the explicit expression (103)

$$\sum_m W(\mu,m)=0 \tag{108}$$

for half-odd μ. Using the expression (104) we obtain the equivalent result for the spin distribution. Although the half-odd elements do their duty in the mathematical construction of fermionic Wigner functions, they do not appear as physical observables. They are ghost variables of zero probability for observation.

2. Overlap relation

We can also prove that fermionic Wigner functions obey the same overlap relation (67) as the Wigner functions in the Bose sector. We use the explicit expression (103) for the Wigner functions W_1 and W_2 of the arbitrary operators $\hat{F}_1$ and $\hat{F}_2$ to obtain

$$\begin{aligned}d\sum_{\mu m} W_1(\mu,m)W_2(\mu,m) &= \frac{1}{4d}\sum_{\mu m}\sum_{\nu_1\nu_2}\exp\left[\frac{4\pi i}{d}m(\nu_1+\nu_2)\right]\langle\mu-\nu_1|\hat{F}_1|\mu+\nu_1\rangle \\ &\quad\times\langle\mu-\nu_2|\hat{F}_2|\mu+\nu_2\rangle \\ &= \frac{1}{2}\sum_{\mu\nu}\langle\mu-\nu|\hat{F}_1|\mu+\nu\rangle\langle\mu+\nu|\hat{F}_2|\mu-\nu\rangle.\end{aligned} \tag{109}$$

We represent $\mu+\nu$ by μ_1 and $\mu-\nu$ by μ_2 and note that μ_1 and μ_2 must be integer for nonvanishing matrix elements. If μ_1 is integer, μ_2 must be integer as well, and we obtain

$$\begin{aligned}d\sum_{\mu m} W_1(\mu,m)W_2(\mu,m) &= \sum_{\mu_1\mu_2}\langle\mu_1|\hat{F}_1|\mu_2\rangle\langle\mu_2|\hat{F}_2|\mu_1\rangle \\ &= \mathrm{Tr}\{\hat{F}_1\hat{F}_2\}.\end{aligned} \tag{110}$$

We have seen that fermionic Wigner functions obey the same overlap relation (67) as bosonic ones [38].

E. Examples

1. Line states

What is the fermionic Wigner function for phase states $|\mu_0\rangle$?

$$W(\mu,m)=\frac{1}{2d}\sum_\nu \exp\left(\frac{4\pi i}{d}m\nu\right)\langle\mu-\nu|\mu_0\rangle\langle\mu_0|\mu+\nu\rangle. \tag{111}$$

There are two possibilities so that both $\langle\mu-\nu|\mu_0\rangle$ and $\langle\mu_0|\mu+\nu\rangle$ do not vanish. Either $\nu=0$ and simultaneously $\mu=\mu_0$ or $\nu=d/2$ and $\mu=\mu_0+d/2$. Consequently,

$$\begin{aligned}W(\mu,m) &= \frac{1}{2d}\delta(\mu-\mu_0;d)+\frac{1}{2d}(-1)^{2m} \\ &\quad\times\delta\left(\mu-\mu_0-\frac{d}{2};d\right).\end{aligned} \tag{112}$$

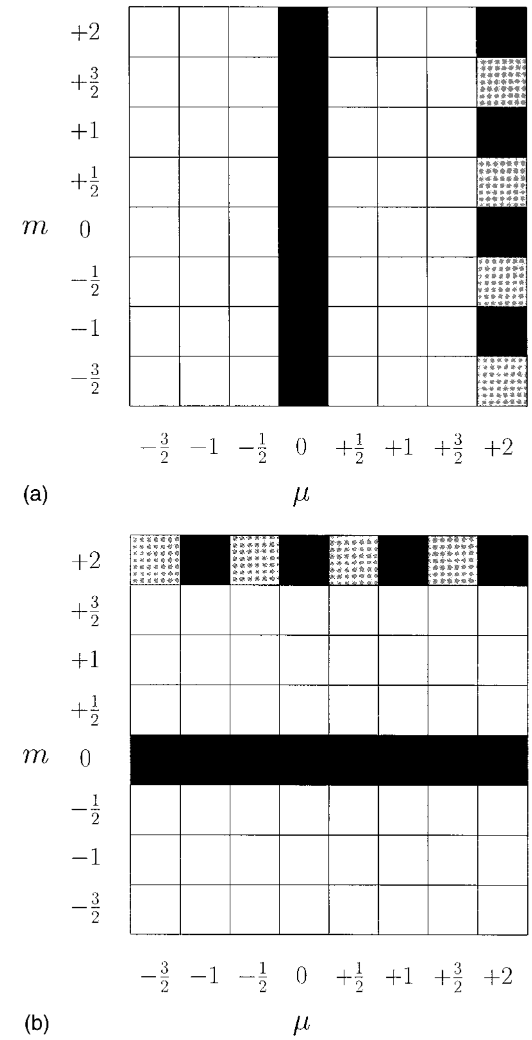

FIG. 6. Example of discrete Wigner functions (112) for line states in even dimensions. Here the dimension $d=4$. In (a) a phase state $|0\rangle_\varphi$ and in (b) a spin state $|0\rangle_m$ is depicted. Note that gray color is used to indicate negative values.

The Wigner function for a phase state appears as a phase line at $\mu=\mu_0$ accompanied by a ghost line at $\mu=\mu_0+d/2$, see Fig. 6. The ghost line exhibits positive values for integer elements m and negative values for half-odd spin elements. That this odd behavior is indeed reasonable is seen on the marginal distributions. Summing $W(\mu,m)$ with respect to m yields zero apart from $\mu=\mu_0$ (mod d) since either the values of $W(\mu,m)$ are zero or the values on the ghost line cancel each other,

$$\sum_m W(\mu,m)=\delta(\mu-\mu_0;d). \tag{113}$$

This is what we would expect for a phase state. On the other hand, summing $W(\mu,m)$ with respect to μ yields d^{-1} for integer m and zero for half-odd m,

$$\sum_\mu W(\mu,m)=d^{-1}\delta(m;1). \tag{114}$$

Phase states have uniform spin distributions and the half-odd ghost variables do not appear as physical quantities. Again, this is accomplished by the ghost line in the fermionic Wigner function of a phase state. In the same way we obtain the equivalent Wigner function for a spin state,

$$W(\mu,m)=\frac{1}{2d}\,\delta(m-m_0;d)+\frac{1}{2d}\,(-1)^{2\mu}\times\delta\left(m-m_0-\frac{d}{2};d\right). \tag{115}$$

2. Broken-line states

If the dimension d is composite, i.e., $d=ab$ with integers a and b, we can introduce broken-line states (54), (56). Their fermionic Wigner function is obtained from the explicit expression (103)

$$W(\mu,m)=\frac{1}{2da}\sum_{\nu}\sum_{\mu_1\mu_2}\times\exp\left[\frac{2\pi i}{d}\,[2\nu m+bm_0(\mu_1'-\mu_2')]\right]\times\delta(\mu-\nu-\mu_0-b\mu_1';d)\times\delta(\mu+\nu-\mu_0-b\mu_2';d). \tag{116}$$

As for line states, there are two possibilities to obtain nonzero matrix elements: Either $\nu=0$ and $\mu=\mu_0$ (mod b) (which means that $\mu_1'-\mu_2'$ must be even) or $\nu=b/2$ and $\mu=\mu_0+b/2$ (mod b) (which implies that $\mu_1'-\mu_2'$ must be odd). Consequently, we obtain

$$W(\mu,m)=\frac{1}{2d}\,(-1)^{4(\mu-\mu_0)(m-m_0)/d}\,\delta(2\mu-2\mu_0;b)\times\delta(2m-2m_0;a). \tag{117}$$

We see that fermionic broken-line states exhibit in addition to the familiar lattice in phase space a ghost lattice which does its duty to guarantee the right marginal properties.

F. Precession tomography

How is the idea of precession tomography modified for spins? Let us consider the precession operator

$$\hat{P}(t)=\exp\left(-i\,\frac{\hat{m}^2}{2}\,t\right) \tag{118}$$

at *half* of the precession times (76) as for angular momenta (in the Bose sector)

$$t=\frac{2\pi}{d}\,\tau \tag{119}$$

with integer τ. How is the Wigner function transformed?

$$W(\mu,m;t)=\frac{1}{d}\sum_n\exp\left(-\frac{4\pi i}{d}\,n\mu\right)\langle m-n|\hat{P}\hat{\rho}\hat{P}^{\dagger}|m+n\rangle \tag{120}$$

$$=\frac{1}{d}\sum_n\exp\left[-\frac{2\pi i}{d}\left(2n\mu+\frac{\tau}{2}\,(m-n)^2-\frac{\tau}{2}\,(m+n)^2\right)\right]\langle m-n|\hat{\rho}|m+n\rangle. \tag{121}$$

There is a subtle reason why we can perform the last mathematical transformation for even-dimensional systems and not necessarily in the odd-dimensional case. We notice that the spin-state index $m+n$ in the summation (120) might be out of the original parameter range $\{-d/2+\frac{1}{2},+d/2\}$. Then the spin-periodicity condition (29) is applied to define a meaningful state, i.e., $m+n$ is replaced by $m+n-d$ to lie again in the original parameter range. In this case we obtain

$$\begin{aligned}\hat{P}^{\dagger}|m+n\rangle&=\exp\left[\frac{\pi i}{d}\,\tau(m+n-d)^2\right]|m+n\rangle\\&=\exp\left[\frac{\pi i}{d}\,\tau[(m+n)^2-2d(m+n)+d^2]\right]|m+n\rangle\\&=\exp\left[\frac{\pi i}{d}\,\tau(m+n)^2\right]|m+n\rangle\end{aligned} \tag{122}$$

since d is even. Of course, the same might happen if $m-n$ is out of the original parameter range. For this subtle reason the discrete Wigner function of spins precesses in steps of *one* and not in steps of *two* as in the bosonic case,

$$W(\mu,m;t)=W(\mu-\tau m,m). \tag{123}$$

All other features of the precession tomography require only minor modifications. We should perform discrete Fourier transformations of the measured phase and spin distributions and see again that

$$\widetilde{\mathrm{pr}}_{\varphi}\left(\frac{\nu}{2};\tau\right)=\widetilde{W}\left(\frac{\nu}{2},\tau\,\frac{\nu}{2}\right),$$

$$\widetilde{\mathrm{pr}}_m\left(\frac{n}{2}\right)=\widetilde{W}\left(0,\,\frac{n}{2}\right)$$

or

$$\widetilde{\mathrm{pr}}\left(\frac{\nu}{2},\,\frac{n}{2};\tau\right)=\widetilde{W}\left(\frac{\nu}{2}\,a,\tau\,\frac{\nu}{2}\right) \tag{124}$$

for the observation of broken-line states. The half-integer arguments in the discrete Fourier transformation are necessary because the characteristic function $\widetilde{W}(\nu,n)$ should be reconstructed for half-integer arguments. Otherwise, insufficient information is gained because of the redundancy

$$\widetilde{W}\left(\nu+\frac{d}{2},n\right)=(-1)^{2n}\,\widetilde{W}\,(\nu,n),$$

$$\widetilde{W}\left(\nu,n+\frac{d}{2}\right)=(-1)^{2\nu}\,\widetilde{W}\,(\nu,n) \tag{125}$$

of the characteristic function for spins.

V. SUMMARY

Quantum objects cannot be seen *as they are*. We may observe only particular aspects of a quantum system in a single experimental setting. Quantum-state tomography is a method for building up a complete picture of a quantum object from a set of distinct observations, i.e., by seeing the object from different points of view. In precession tomography the phase distribution of a precessing angular momentum or spin is measured to reconstruct the state. This is possible because precession mixes the state space sufficiently so that phase observations explore almost all aspects of the system. If in addition the angular momentum or spin components are measured the complete picture of the object can be inferred. In this paper a fully quantum version of precession tomography is elaborated. As the basic theoretical tool a discrete Wigner formalism is developed. The discrete Wigner function represents the complete picture of an angular momentum or spin or, more generally, of a finite-dimensional quantum system. This picture is to be reconstructed in quantum-state tomography. Seen from a more abstract point of view, the Wigner function serves as a guideline for finding a complete set of observables which is sufficient for state determination. These observables correspond to a set of basis systems in state space, the set of precession states. It turns out that in precession tomography prime-dimensional systems are favored in the sense that the precession states are mutually complementary (in the precise sense defined in this paper). The precession states together with the angular-momentum eigenstates represent indeed all complementary aspects of the system. If the dimension of the state space is composite some features are missing. They can be inferred by observing the precessing angular momentum in terms of the broken-line states defined in this paper. It is not clear, however, whether this distinction of prime-dimensional systems is generic or just an artificial feature of the method presented. Possible relations between number theory and quantum mechanics are certainly a challenge for much future work. In quantum computing [39], for instance, quantum mechanics is used for solving number-theoretical puzzles like the factorization of large numbers in a computationally efficient way [40]. In this paper odd- and even-dimensional systems (angular momenta and spins, bosons, and fermions) are distinguished. The Wigner function for odd-dimensional quantum objects is easily guessed (and has been given before [15,16]), being transcribed from Wigner's original formula (1) for discrete variables. The Wigner function for even-dimensional systems requires more thought. In this paper the discrete phase space is supplemented by ghost variables to solve the problems of fermionic Wigner functions. Like in Wigner's paper [3] the discrete quasiprobabilities were ''chosen from all possible expressions because [they seem] to be the simples.'' It is not clear which properties define them uniquely. Finally, almost needless to say, discrete Wigner functions might find also other applications apart from the originally intended purpose.

ACKNOWLEDGMENTS

I would like to thank U. Bandelow, A. Barenco, V. Bužek, R. Hvass Hansen, U. Janicke, I. Jex, I. Leonhardt, C. Müller, T. Opatrný, H. Paul, M. G. Raymer, W. P. Schleich, S. Stenholm, J. A. Vaccaro, A. Vourdas, M. Wilkens, and P. Zoller for their encouragement and help. My work at the University of Oregon was supported by the Max Planck Society.

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