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# Solving some two-body dynamical problems in $\langle\zeta|$-$\langle\eta|$ representation

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We use the common eigenvectors $|\zeta\rangle$ of two particles' center-of-mass position $X_c=\mu_1X_1+\mu_2X_2$ and mass-weighted relative momentum $P_r=\mu_2P_1-\mu_1P_2$, as well as the eigenvectors $|\eta\rangle$ of the other pair commutative operators $X=X_1-X_2$ and $P=P_1+P_2$ to solve the dynamics of two coupled oscillators. The coupling involves both interbond potential and kinetic coupling. This approach seems to be simple, convenient, and natural.

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## I. INTRODUCTION

In Ref. [1] the explicit form of the common eigenvectors $|\eta\rangle$ of the relative position $X=X_1-X_2$ and the total momentum $P=P_1+P_2$ of the two particles, which were first considered by Einstein, Podolsky, and Rosen [2] in their argument that the quantum mechanical state vector is not complete, are constructed in the two-mode Fock space,

$$|\eta\rangle=\exp\{-\tfrac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^* a_2^\dagger+a_1^\dagger a_2^\dagger\}|0,0\rangle, \tag{1}$$

in which $\eta=\eta_1+i\eta_2$ is a complex number. It is remarkable that $\eta$'s real part and imaginary part are the eigenvalues of $X_1-X_2$ and $P_1+P_2$, respectively; e.g.,

$$(X_1-X_2)|\eta\rangle=\sqrt{2}\,\eta_1|\eta\rangle,\quad X_i=(1\sqrt{2})\,(a_i+a_i^\dagger),\quad i=1,2. \tag{2}$$

$$(P_1+P_2)|\eta\rangle=\sqrt{2}\,\eta_2|\eta\rangle,\quad P_i=(1/\sqrt{2}i)\,(a_i-a_i^\dagger), \tag{3}$$

When two particles have different masses, we have to consider the fact that center-of-mass coordinate $X_c$ and mass-weighted relative momentum $P_r$ are permuted, where

$$X_c=\mu_1X_1+\mu_2X_2,\quad P_r=\mu_2P_1-\mu_1P_2, \tag{4}$$

$$\mu_1=m_1/(m_1+m_2),\quad \mu_2=m_2/(m_1+m_2) \tag{5}$$

are reduced masses, with $\mu_1+\mu_2=1$, then in Ref. [3] the explicit form of the common eigenstate of $X_c$ and $P_r$ is derived, which is

$$\begin{aligned}|\zeta\rangle=\exp\{&-\tfrac{1}{2}|\zeta|^2+(1/\sqrt{\lambda})\,[\zeta+(\mu_1-\mu_2)\zeta^*]a_1^\dagger\\&+(1/\sqrt{\lambda})\,[\zeta^*+(\mu_2-\mu_1)\zeta]a_2^\dagger+[(\mu_2-\mu_2)/\lambda]\\&\times(a_1^{\dagger 2}-a_2^{\dagger 2})-(4\mu_1\mu_2/\lambda)\,a_1^\dagger a_2^\dagger\}|0,0\rangle,\end{aligned} \tag{6}$$

where

$$\lambda=2(\mu_1^2+\mu_2^2). \tag{7}$$

$\zeta$'s real part and imaginary part are the eigenvalues of $X_c$ and $P_r$, respectively, e.g.,

$$X_c|\zeta\rangle=\sqrt{\lambda/2}\,\zeta_1|\zeta\rangle,\quad P_r|\zeta\rangle=\sqrt{\lambda/2}\,\zeta_2|\zeta\rangle. \tag{8}$$

Both $|\zeta\rangle$ and $|\eta\rangle$ are eligible to compose a representation because they are orthonormal and complete; e.g.,

$$\langle\zeta|\zeta'\rangle=\pi\delta^{(2)}(\zeta-\zeta'),\quad \langle\eta|\eta'\rangle=\pi\delta^{(2)}(\eta-\eta'), \tag{9}$$

$$\int\frac{d^2\eta}{\pi}|\eta\rangle\langle\eta|=\int\frac{d^2\eta}{\pi}:\exp\{-[\eta-(a_1-a_2^\dagger)]\times[(\eta^*-(a_1^\dagger-a_2)]\}:=1, \tag{10}$$

$$\int\frac{d^2\zeta}{\pi}|\zeta\rangle\langle\zeta|=1, \tag{11}$$

where the integration within an ordered product (IWOP) of operators [4–6] is employed.

A question thus naturally arises: Can we find more applications of the new $\langle\zeta|$ and $\langle\eta|$ representation? Recall that when tackling a system composed of two particles in quantum mechanics, it is frequently useful to convert from individual particle coordinates to center-of-mass coordinates; thus we expect the $\langle\zeta|$-$\langle\eta|$ representations to play some role in solving some dynamical problems. In this work we show that the $\langle\zeta|$-$\langle\eta|$ representation provides a simple and intuitive frame for establishing some differential equations for the wave function of two-particle systems. These equations are solvable even though the interaction between two particles includes both interbond potential and kinetic coupling; such cases often occur in the theory of molecule dynamics.

## II. THE OVERLAP OF $\langle\eta|\zeta\rangle$

By using the center-of-mass coordinate and momentum $(X_c,P)$ and relative coordinate and momentum $(X,P_r)$, we can transform the general form of the two-particle Hamiltonian

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$$H=\frac{1}{2m_1}P_1^2+\frac{1}{2m_2}P_2^2+kP_1P_2+V(X_1-X_2), \quad (12)$$

here the potential depends only on the distance between two particles, and $P_1P_2$ represents the kinetic coupling (such terms are often used to describe interbond potential and kinetic coupling in molecule dynamics [7]), into

$$H=(1/2M+k\mu_1\mu_2)P^2+(1/2\mu-k)P_r^2 + k(\mu_2-\mu_1)PP_r+V(X), \quad (13)$$

where $M=m_1+m_2$ is the total mass and $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass.

From this form we see that it is hard to separate the motion of center of mass, since the coupling $PP_r$ is also included in (13). Therefore we turn to $\langle\eta|$ representation.

Sandwiching $H$ between $\langle\eta|$ and $H$'s eigenstate $|E_n\rangle$ we obtain

$$E_n\langle\eta|E_n\rangle=[(1/M+2k\mu_1\mu_2)\eta_2^2+V(\sqrt{2}\eta_1)]\langle\eta|E_n\rangle + (1/2\mu-k)\langle\eta|P_r^2|E_n\rangle + k(\mu_2-\mu_1)\sqrt{2}\eta_2\langle\eta|P_r|E_n\rangle. \quad (14)$$

Because of Eqs. (2), (3), and (11), we can put Eq. (14) into

$$E_n\langle\eta|E_n\rangle=[(1/M+2k\mu_1\mu_2)\eta_2^2+V(\sqrt{2}\eta_1)]\langle\eta|E_n\rangle + (1/2\mu-k)\left\langle\eta\left|P_r^2\int\frac{d^2\zeta}{\pi}\right|\zeta\right\rangle\langle\zeta|E_n\rangle + k(\mu_2-\mu_1)\sqrt{2}\eta_2\left\langle\eta\left|P_r\int\frac{d^2\zeta}{\pi}\right|\zeta\right\rangle\langle\zeta|E_n\rangle, \quad (15)$$

so it seems necessary to calculate the overlap $\langle\eta|\zeta\rangle$. This can be achieved by using coherent states' overcompleteness [8] and the IWOP technique,

$$\langle\eta|\zeta\rangle=\left\langle 0,0\left|\exp\{-\tfrac{1}{2}|\eta|^2+\eta^*a_1-\eta a_2+a_1a_2\}\int\frac{d^2Z_1}{\pi}\frac{d^2Z_2}{\pi}\right|Z_1Z_2\right\rangle\left\langle Z_1Z_2\left|\exp\left\{-\tfrac{1}{2}|\zeta|^2+\frac{1}{\sqrt{\lambda}}[\zeta+(\mu_1-\mu_2)\zeta^*]a_1^\dagger + \frac{1}{\sqrt{\lambda}}[\zeta^*+(\mu_2-\mu_1)\zeta]a_2^\dagger+\frac{\mu_2-\mu_1}{\lambda}(a_1^{\dagger 2}-a_2^{\dagger 2})-\frac{4\mu_1\mu_2}{\lambda}a_1^\dagger a_2^\dagger\right\}\right|0,0\right\rangle$$
$$=\int\frac{d^2Z_1}{\pi}\frac{d^2Z_2}{\pi}\exp\left\{-|Z_1|^2-|Z_2|^2-\tfrac{1}{2}|\eta|^2-\tfrac{1}{2}|\zeta|^2+\eta^*Z_1-\eta Z_2+\frac{1}{\sqrt{\lambda}}[\zeta+(\mu_1-\mu_2)\zeta^*]Z_1^*+Z_1Z_2 + \frac{1}{\sqrt{\lambda}}[\zeta^*+(\mu_2-\mu_1)\zeta]Z_2^*+\frac{\mu_2-\mu_1}{\lambda}(Z_1^{*2}-Z_2^{*2})-\frac{4\mu_1\mu_2}{\lambda}Z_1^*Z_2^*\right\}. \quad (16)$$

In terms of the mathematical formulas

$$\int\frac{d^2Z}{\pi}\exp\{s|Z|^2+fZ^2+gZ^{*2}+\alpha Z+\beta Z^*\}=1/\sqrt{s^2-4gf}\,\exp\{(-s\alpha\beta+f\beta^2+g\alpha^2)/(s^2-4fg)\}, \quad (17)$$

where the convergence condition is $\mathrm{Re}(s+f+g)<0$, $\mathrm{Re}[(s^2-4fg)/(s+f+g)]<0$, or $\mathrm{Re}(s-f-g)<0$, $\mathrm{Re}[(s^2-4fg)/s-f-g)]<0$, we perform the integration in (16). After a straightforward but tedious calculation we finally obtain

$$\langle\eta|\zeta\rangle=\int\frac{d^2Z_2}{\pi}\exp\{-\tfrac{1}{2}(|\eta|^2+|\zeta|^2)\}\exp\{-|Z_2|^2-\eta Z_2+(1/\sqrt{\lambda})[\zeta^*+(\mu_2-\mu_1)\zeta]Z_2^*+[(\mu_2-\mu_1)/\lambda](Z_2+\eta^*)^2 - [(\mu_2-\mu_1)/\lambda]Z_2^{*2}+1/\sqrt{\lambda}[\zeta+(\mu_1-\mu_2)\zeta^*-(4\mu_1\mu_2/\sqrt{\lambda})Z_2^*](Z_2+\eta^*)\}$$
$$=\sqrt{\lambda/4}\,\exp\{i[(\mu_2-\mu_1)(\zeta_1\zeta_2-\eta_1\eta_2)+\sqrt{\lambda}(\zeta_2\eta_1-\zeta_1\eta_2)]\}, \quad (18)$$

which leads to, for the $\mu_1=\mu_2$ case,

$$\langle\eta|\zeta\rangle=\tfrac{1}{2}\exp\{i(\zeta_2\eta_1-\zeta_1\eta_2)\}. \quad (19)$$

### III. THE REPRESENTATIONS OF $(P_r, X_c)$ IN $\langle\eta|$ BASES AND $(P,X)$ IN $\langle\zeta|$ BASES

Using Eqs. (11) and (8) we can deduce

$$\begin{aligned}\langle\eta|P_r &= \left\langle \eta \middle| P_r \int \frac{d^2\zeta}{\pi} \middle| \zeta \right\rangle \langle\zeta| = \left\langle \eta \middle| \int \frac{d^2\zeta}{\pi} \sqrt{\frac{\lambda}{2}}\zeta_2 \middle| \zeta \right\rangle \langle\zeta| \\ &= \int \frac{d^2\zeta}{\pi}\left(-\sqrt{\tfrac{1}{2}}i\right)\left[\frac{\partial}{\partial\eta_1} - i(\mu_1-\mu_2)\eta_2\right]\langle\eta|\zeta\rangle\langle\zeta| \\ &= \left\{-\sqrt{\tfrac{1}{2}}i\frac{\partial}{\partial\eta_1} - \frac{1}{\sqrt{2}}(\mu_1-\mu_2)\eta_2\right\}\langle\eta|, \end{aligned} \tag{20}$$

$$\begin{aligned}\langle\eta|X_c &= \left\langle \eta \middle| X_c \int \frac{d^2\zeta}{\pi} \middle| \zeta \right\rangle \langle\zeta| = \left\langle \eta \middle| \int \frac{d^2\zeta}{\pi} \sqrt{\frac{\lambda}{2}}\zeta_1 \middle| \zeta \right\rangle \langle\zeta| \\ &= \int \frac{d^2\zeta}{\pi}\frac{i}{\sqrt{2}}\left[\frac{\partial}{\partial\eta_2} - i(\mu_1-\mu_2)\eta_1\right]\langle\eta|\zeta\rangle\langle\zeta| \\ &= \left\{\frac{1}{\sqrt{2}}i\frac{\partial}{\partial\eta_2} + \frac{1}{\sqrt{2}}(\mu_1-\mu_2)\eta_1\right\}\langle\eta|. \end{aligned} \tag{21}$$

Similar to the discussion above, we can get

$$\begin{aligned}\langle\zeta|P &= \left\langle \zeta \middle| P \int \frac{d^2\eta}{\pi} \middle| \eta \right\rangle \langle\eta| = \left\langle \zeta \middle| \int \frac{d^2\eta}{\pi} \sqrt{2}\eta_2 \middle| \eta \right\rangle \langle\eta| \\ &= -i\sqrt{\frac{2}{\lambda}}\left[\frac{\partial}{\partial\zeta_1} - i(\mu_1-\mu_2)\zeta_2\right]\left\langle \zeta \middle| \int \frac{d^2\eta}{\pi} \middle| \eta \right\rangle \langle\eta| \\ &= \left\{-\sqrt{\frac{2}{\lambda}}i\frac{\partial}{\partial\zeta_1} - \sqrt{\frac{2}{\lambda}}(\mu_1-\mu_2)\zeta_2\right\}\langle\zeta|, \end{aligned} \tag{22}$$

$$\begin{aligned}\langle\zeta|X &= \left\langle \zeta \middle| X \int \frac{d^2\eta}{\pi} \middle| \eta \right\rangle \langle\eta| = \left\langle \zeta \middle| \int \frac{d^2\eta}{\pi} \sqrt{2}\eta_1 \middle| \eta \right\rangle \langle\eta| \\ &= \sqrt{\frac{2}{\lambda}}i\left[\frac{\partial}{\partial\zeta_2} - i(\mu_1-\mu_2)\zeta_1\right]\left\langle \zeta \middle| \int \frac{d^2\eta}{\pi} \middle| \eta \right\rangle \langle\eta| \\ &= \left\{\sqrt{\frac{2}{\lambda}}i\frac{\partial}{\partial\zeta_2} + \sqrt{\frac{2}{\lambda}}(\mu_1-\mu_2)\zeta_1\right\}\langle\zeta|. \end{aligned} \tag{23}$$

These representations, as we will see shortly, will be of great use in solving the dynamics of two-body system.

### IV. DIFFERENTIAL EQUATIONS FOR ENERGY EIGENFUNCTIONS IN $\langle\eta|$ REPRESENTATION

Based on the $\langle\eta|$ representation of $P_r$, we are able to transform the operator equation (15) as

$$\begin{aligned}&\{[(1/M+2k\mu_1\mu_2)\eta_2^2 + V(\sqrt{2}\eta_1)] - E_n - \tfrac{1}{2}(1/2\mu - k) \\ &\times[\partial/\partial\eta_1 - i(\mu_1-\mu_2)\eta_2]^2 - ik(\mu_2-\mu_1)\eta_2 \\ &\times[\partial/\partial\eta_1 - i(\mu_1-\mu_2)\eta_2]\}\langle\eta|E_n\rangle = 0. \end{aligned} \tag{24}$$

To solve this differential equation, we make the ansatz

$$\langle\eta|E_n\rangle = \exp\{i(\mu_1-\mu_2)\eta_1\eta_2\}\psi_n. \tag{25}$$

After substituting (25) into (24), we obtain the following equation for $\psi_n$:

$$\begin{aligned}&\left\{\left[\left(\frac{1}{M}+2k\mu_1\mu_2\right)\eta_2^2 + V(\sqrt{2}\eta_1)\right] - E_n - \frac{1}{2}\left(\frac{1}{2\mu}-k\right)\frac{\partial^2}{\partial\eta_1^2} \right.\\ &\left. - ik(\mu_2-\mu_1)\eta_2\frac{\partial}{\partial\eta_1}\right\}\psi_n = 0. \end{aligned} \tag{26}$$

Further, we introduce a wave function $\varphi_n$ through the relation

$$\psi_n = \exp\{i(\mu_1-\mu_2)[2\mu k/(1-2\mu k)]\,\eta_1\eta_2\}\varphi_n \tag{27}$$

and find that $\varphi_n$ satisfies the following equation:

$$\left\{\frac{1}{M}\left(\frac{1-\mu M k^2}{1-2\mu k}\right)\eta_2^2 + V(\sqrt{2}\eta_1) - E_n - \frac{1}{2}\left(\frac{1}{2\mu}-k\right)\frac{\partial^2}{\partial\eta_1^2}\right\}\varphi_n = 0. \tag{28}$$

Thus we see that once we have worked in $\langle\eta|$ representations, the complicated dynamical problem of two-coupled particles can be simplified as a one-variable differential equation, including another variable as a parameter.

### V. ENERGY VALUES OF THE HAMILTONIAN FOR SOME SPECIAL POTENTIALS

First, when $V(X)=\frac{1}{2}\mu\omega^2X^2$, the parabolic potential, Eq. (28) becomes

$$\left\{-\frac{1}{2}\left(\frac{1}{2\mu}-k\right)\frac{\partial^2}{\partial\eta_1^2} + \mu\omega^2\eta_1^2 - E_n + \frac{1}{M}\left(\frac{1-\mu M k^2}{1-2\mu k}\right)\eta_2^2\right\}\varphi_n = 0. \tag{29}$$

Comparing with the Hermite equation representing a harmonic oscillator

$$\left\{-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}m\omega^2x^2 - E_n\right\}\psi_n = 0, \quad E_n = (n+\tfrac{1}{2})\omega, \tag{30}$$

we obtain the energy level for Eq. (29),

$$E_n = (n+\tfrac{1}{2})\sqrt{1-2k\mu}\,\omega + \frac{1}{M}\frac{1-\mu M k^2}{1-2\mu k}\eta_2^2. \tag{31}$$

Because the total momentum commutes with the Hamiltonian, $[P,H]=0, \sqrt{2}\eta_2$ is the eigenvalue of $P$, thus Eq. (31) can be rewritten as [Eq. (39)]

$$E_n = (n+\tfrac{1}{2})\sqrt{1-2k\mu}\,\omega + \frac{1-\mu M k^2}{1-2\mu k}\frac{p^2}{2M}. \tag{32}$$

Second, when $V(X_1-X_2) = -V_0\delta(X_1-X_2)$, the $\delta$-function potential, Eq. (28) turns to

$$\{-\tfrac{1}{2}\,(1/2\mu-k)\,(\partial^2/\partial\eta_1^2)-V_0\delta(\sqrt{2}\,\eta_1)-E_n + 1/M\,[(1-\mu Mk^2)/(1-2\mu k)]\,\eta_2^2\}\varphi_n=0. \quad (33)$$

According to the standard procedures for deducing the bound-state energy of the $\delta$-function potential, we can obtain $E_n$ for Eq. (33), which is

$$E_n=-\frac{\mu V_0^2}{2(1-2\mu k)}+\frac{1}{M}\,\frac{1-\mu Mk^2}{1-2\mu k}\,\eta_2^2 =-\frac{\mu V_0^2}{2(1-2\mu k)}+\frac{1-\mu Mk^2}{1-2\mu k}\,\frac{p^2}{2M}. \quad (34)$$

Third, when $V(\vec{X}_1-\vec{X}_2)=-e^2/(|\vec{X}_1-\vec{X}_2|)$, the three dimensional (3D) Coulomb potential, by introducing 3D $\langle\vec{\eta}|$ representation, which is the common eigenfunction of the total momentum vector $\vec{P}$ and the relative coordinate vector $\vec{X}$, $\vec{P}|\vec{\eta}\rangle=\sqrt{2}\,\vec{\eta}_2|\vec{\eta}\rangle$, $\vec{X}|\vec{\eta}\rangle=\sqrt{2}\,\vec{\eta}_1|\vec{\eta}\rangle$, the corresponding energy eigenvector equation in $\langle\vec{\eta}|$ representation is

$$\left\{\left(\frac{1}{M}+2k\mu_1\mu_2\right)|\vec{\eta}_2|^2-\frac{e^2}{|\sqrt{2}\,\vec{\eta}_1|}-E_n-\tfrac{1}{2}\left(\frac{1}{2\mu}-k\right)\times\left[\frac{\partial}{\partial\vec{\eta}_1}-i(\mu_1-\mu_2)\vec{\eta}_2\right]^2-ik(\mu_2-\mu_1)\vec{\eta}_2\times\left[\frac{\partial}{\partial\vec{\eta}_1}-i(\mu_1-\mu_2)\vec{\eta}_2\right]\right\}\langle\vec{\eta}|E_n\rangle=0. \quad (35)$$

Following the same procedure as in deriving (28), we obtain

$$\{-\tfrac{1}{2}(1/2\mu-k)\,\partial^2/\partial\vec{\eta}_1^{\,2}-e^2/|\sqrt{2}\,\vec{\eta}_1|-E_n + 1/M\,[(1-\mu Mk^2)/(1-2\mu k)]|\vec{\eta}_2|^2\}\varphi_n=0. \quad (36)$$

where $\varphi_n$ is defined as

$$\varphi_n=\exp\{i(\mu_2-\mu_1)[1/(1-2\mu k)]\,\vec{\eta}_1\cdot\vec{\eta}_2\}\langle\vec{\eta}|E_n\rangle. \quad (37)$$

From (36) the energy level can be directly derived,

$$E_n-|\vec{\eta}_2|^2(1-\mu Mk^2)/M(1-2\mu k) =-\mu e^4/2n^2(1-2\mu k),\quad n=1,2,3,\ldots, \quad (38)$$

which differs from the well-known energy of Coulomb potential by $1/(1-2\mu k)$, stemming from the $kP_1P_2$ term. However, from Eq. (28) we see that $\varphi_n$ can only be determined up to a function of $\eta_2$, hence we turn to $\langle\zeta|$ to establish some other equations to specify it.

## VI. DIFFERENTIAL EQUATIONS IN THE $\langle\zeta|$ REPRESENTATION

Since $P$ is conservative, we can introduce common eigenstates of $P$ and $H$,

$$H|p,E_n\rangle=E_n|p,E_n\rangle,\quad P|p,E_n\rangle=p|p,E_n\rangle. \quad (39)$$

In $\langle\zeta|$ representation, the wave functions of $H|p,E_n\rangle$ and $P|p,E_n\rangle$ are given by, respectively,

$$E_n\langle\zeta|p,E_n\rangle=\left[\left(\frac{1}{2M}+k\mu_1\mu_2\right)p^2+k(\mu_2-\mu_1)\left(\frac{\lambda}{2}\right)^{1/2}\zeta_2 p+\left(\frac{1}{2\mu}-k\right)\frac{\lambda}{2}\zeta_2^2\right]\langle\zeta|p,E_n\rangle+\langle\zeta|V(X)|p,E_n\rangle, \quad (40)$$

$$p\langle\zeta|p,E_n\rangle=\langle\zeta|P|p,E_n\rangle. \quad (41)$$

Because of Eqs. (22) and (23), we can put Eqs. (40) and (41) into

$$\left\{V\left[i\left(\frac{2}{\lambda}\right)^{1/2}\frac{\partial}{\partial\zeta_2}+\left(\frac{2}{\lambda}\right)^{1/2}(\mu_1-\mu_2)\zeta_1\right]+\left(\frac{1}{2M}+k\mu_1\mu_2\right)p^2+k(\mu_2-\mu_1)\left(\frac{\lambda}{2}\right)^{1/2}\zeta_2 p+\left(\frac{1}{2\mu}-k\right)\frac{\lambda}{2}\zeta_2^2-E_n\right\}\langle\zeta|p,E_n\rangle=0, \quad (42)$$

$$\left\{-i\left(\frac{2}{\lambda}\right)^{1/2}\frac{\partial}{\partial\zeta_1}+\left(\frac{2}{\lambda}\right)^{1/2}(\mu_2-\mu_1)\zeta_2-p\right\}\langle\zeta|p,E_n\rangle=0. \quad (43)$$

By making the ansatz

$$\Psi_n=\exp\{i(\mu_2-\mu_1)\zeta_1\zeta_2\}\langle\zeta|p,E_n\rangle \quad (44)$$

and letting

$$\zeta_0=-\sqrt{2}\mu k(\mu_2-\mu_1)p/(1-2\mu k)\sqrt{\lambda}, \quad (45)$$

$$T=\left(\frac{1}{2M}+k\mu_1\mu_2\right)p^2-\left(\frac{1}{2\mu}-k\right)\frac{\lambda}{2}\zeta_0^2=\frac{1-\mu Mk^2}{1-2\mu k}\,\frac{p^2}{2M}, \quad (46)$$

we obtain two simpler equations for $\Psi_n$,

$$\left\{V\left[i\left(\frac{2}{\lambda}\right)^{1/2}\frac{\partial}{\partial\zeta_2}\right]+\left(\frac{1}{2\mu}-k\right)\frac{\lambda}{2}(\zeta_2-\zeta_0)^2+T-E_n\right\}\Psi_n=0, \quad (47)$$

$$\left\{-i\left(\frac{2}{\lambda}\right)^{1/2}\frac{\partial}{\partial\zeta_1}-p\right\}\Psi_n=0. \quad (48)$$

Especially when $V(x)=\frac{1}{2}\mu\omega^2X^2$, $\Psi_n$ can be solved analytically. In fact, from Eq.(48) we know that $\Psi_n$ should take the form

$$\Psi_n=\exp\{i\,(\lambda/2)^{1/2}p\zeta_1\}\chi_n. \quad (49)$$

Here $\chi_n$ is independent of $\zeta_1$ and can be derived by substituting Eq. (49) into Eq. (47),

$$\left\{-\frac{1}{\lambda}\mu\omega^2\frac{\partial^2}{\partial\zeta_2^2}+\frac{\lambda}{2}\left(\frac{1}{2\mu}-k\right)(\zeta_2-\zeta_0)^2+T-E_n\right\}\chi_n=0. \tag{50}$$

The result is

$$\chi_n=N_n\exp\left\{-\frac{\lambda\sqrt{(1-2\mu k)}}{4\mu\omega}(\zeta_2-\zeta_0)^2\right\}\times H_n\left\{\left[\frac{\lambda^2(1-2\mu k)}{4\mu^2\omega^2}\right]^{1/4}(\zeta_2-\zeta_0)\right\}, \tag{51}$$

where $H_n$ is the Hermite polynomial and $N_n$ is the normalization constant. Combining Eqs. (44), (49) and (51), we have the wave function

$$\langle\zeta|p,E_n\rangle=N_n\exp\left\{i\left(\frac{\lambda}{2}\right)^{1/2}p\zeta_1+i(\mu_1-\mu_2)\zeta_1\zeta_2-\frac{\lambda\sqrt{(1-2\mu k)}}{4\mu\omega}(\zeta_2-\zeta_0)^2\right\}\times H_n\left\{\left[\frac{\lambda^2(1-2\mu k)}{4\mu^2\omega^2}\right]^{1/4}(\zeta_2-\zeta_0)\right\}.$$

Also, from Eq. (50) we see that the energy level is $E_n=(n+\frac{1}{2})\sqrt{1-2\mu k}\omega+T$, just the same as Eq. (32), because of Eq. (46). In conclusion, by establishing the $\langle\zeta|$-$\langle\eta|$ representation, we provided a convenient approach for solving some dynamical problems of two-body systems.

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